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COMPLEX ANALYSIS HOMEWORK PROBLEMS
WINTER QUARTER 2010

Please provide plenty of details! Pix are definitely kewl ($\ddot\smile$).

(1) Please be sure to read Ahlfors and look at (and work) the suggested problems; see last quarter's web page. For this quarter, for now, start reading Chapter 4.

(2) You can also look at Palka's book. For starters, his §1 of Chapter 4 is a great discussion of paths. The rest of his Chapter 4 is good too, as are his problems on pp.136-139.

Recall that when f is complex differentiable at a, $T_{f,a}(z) := f(a) + f'(a)(z-a)$ is the complex linear first-order approximation for f near $z = a$. It gives the 'best' complex linear approximation for f in the sense given in the next problem. Note that we well understand the geometry of the complex linear map $z \mapsto T_{f,a}(z)$ (right?), and this knowledge is useful in studying the geometry of the map f, at least near a when f is differentiable at a.

(3) Suppose that f is differentiable at a. Let $T := T_{f,a}$. Demonstrate that for any complex linear map L, there is a $\delta > 0$ such that for all $z \in D(a;\delta)$,

$$|f(z) - L(z)| \geq |f(z) - T(z)| .$$

(4) Suppose $\mathbb{C} \xrightarrow{\rho} \mathbb{C}$ is given by $\rho(z) := c\bar{z} + d$ for some $c, d \in \mathbb{C}$. Determine when ρ is reflection in some line, and when so, find the line.

(5) Suppose $\hat{\mathbb{C}} \xrightarrow{\rho} \hat{\mathbb{C}}$ is given by $\rho(z) := (a\bar{z}+b)/(c\bar{z}+d)$ for some $a, b, c, d \in \mathbb{C}$. Determine when ρ is reflection in some circle, and when so, find the circle.

(6) (a) Prove that

$$\forall\, z, w \in \mathbb{C}\,, \quad \chi(z,w) \leq 2|z-w|\,.$$

Deduce that $(\mathbb{C}, |\cdot|) \xrightarrow{\text{id}} (\mathbb{C}, \chi)$ is continuous (in fact, Lipschitz continuous).

(b) Prove that for each $R > 0$,

$$\forall\, z, w \in \mathbb{C}\,, \quad |z|, |w| \leq R \implies \chi(z,w) \geq 2|z-w|/(1+R^2)\,.$$

Deduce that $(\mathbb{C}, \chi) \xrightarrow{\text{id}} (\mathbb{C}, |\cdot|)$ is continuous.

(c) Explain why $(\mathbb{C}, |\cdot|) \xrightarrow{\text{id}} (\mathbb{C}, \chi)$ is a homeomorphism.

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(7) Let $\mathbb{C} \xrightarrow{L} \mathbb{C}$ be a non-constant complex linear map. Define $\hat{\mathbb{C}} \xrightarrow{\hat{L}} \hat{\mathbb{C}}$ by

$$\hat{L}(z) := \begin{cases} L(z) & \text{when } z \in \mathbb{C}\,, \\ \infty & \text{when } z = \infty\,. \end{cases}$$

Prove that $\hat{L}$ is continuous at ∞. (Suggestions: Start by showing that

$$\chi(z,\infty) < \delta < 1 \implies |z| > 1/\delta \quad\text{and}\quad |w| > R \implies \chi(w,\infty) < 2/R\,.$$

Assume $L(z) = az + b$. Given $\varepsilon > 0$, show $\delta := \min\{1, \frac{1}{2}\frac{|a|}{|b|+1}, \frac{|a|\varepsilon}{4}\}$ "works".)

Verify that $\hat{L}$ is a self-homeomorphism of $\hat{\mathbb{C}}$ onto itself.

(8) Let $\mathbb{C} \xrightarrow{P} \mathbb{C}$ be a non-constant complex polynomial, say

$$P(z) := a_0 + a_1 z + \cdots + a_{n-1}z^{n-1} + z^n\,.$$

Define $\hat{\mathbb{C}} \xrightarrow{\hat{p}} \hat{\mathbb{C}}$ by $\hat{P}(z) := \begin{cases} P(z) & \text{when } z \in \mathbb{C} \\ \infty & \text{when } z = \infty\,. \end{cases}$ Prove that $\hat{P}$ is continuous at ∞.

(Suggestions: Use the estimates from the previous problem. Note that

$$|z| > 2n\max\{|a_0|, |a_1|, \ldots, |a_{n-1}|, 1\} \implies \left|1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right| \geq \frac{1}{2}\,.)$$

Explain why $\hat{\mathbb{C}} \xrightarrow{\hat{P}} \hat{\mathbb{C}}$ is continuous. Do you see any way to show that $\hat{P}$ is surjective? Why is this last question easy to answer for polynomials $\mathbb{R} \to \mathbb{R}$?

(9) Repeat the above problem for a complex rational function, say $R := P/Q$ where P and Q are complex polynomials. You must determine how to define $\hat{R}$ at ∞ and at each zero of q in order to obtain a continuous map $\hat{R} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

(10) Let $\mathbb{D} \xrightarrow{f} \mathbb{C}$ be a continuous map. Given $z = x + iy \in \mathbb{D}$, let $\gamma_z := [0,x] + [x,z]$; thus γ_z is the piecewise smooth path in $\mathbb{D}$ with trajectory $[0,x] \cup [x,z]$—the union of a horizontal line segment and a vertical line segment (of course either of these may reduce to a single point, e.g., if $x = 0$ or $y = 0$). For $z \in \mathbb{D}$, define

$$F(z) := \int_{\gamma_z} f(\zeta)\, d\zeta\,.$$

(a) Demonstrate that F is continuous in $\mathbb{D}$.

(b) Calculate the partial derivative $\dfrac{\partial F}{\partial y}$ with your 'bare hands'. That is, find

$$\lim_{h\to 0} \frac{F(z+ih) - F(z)}{h} \qquad \text{where } z \in \mathbb{D} \text{ and } h \in \mathbb{R}\,.$$

(Except for the Fundamental Theorem of Calculus, please prove all so-called Fundamental Theorems that you use.)

(c) Is F holomorphic in $\mathbb{D}$? Prove, or disprove, your answer.

(11) Derive the Complex Green's Theorem from the 'usual' Green's Theorem (both for rectangles): Suppose $f \in \mathcal{C}^1(\Omega)$ and a standard rectangle R lies in Ω. Then

$$\int_{\partial R} f(z)\, dz = 2i \iint_R \frac{\partial f}{\partial \bar{z}}\, dxdy\,.$$

(12) Let p, q be $\mathcal{C}^1$ functions in some rectangle $\Omega := \{(x,y) : |x-a| < r, |y-b| < s\}$. Suppose that

$$\forall\, z \in \Omega\,, \quad \frac{\partial p}{\partial y}(z) = \frac{\partial q}{\partial x}(z)\,.$$

Prove that there is an $F \in \mathcal{C}^2(\Omega)$ with $\nabla F = (p,q)$ (i.e., $F_x = p$ and $F_y = q$) in Ω. (Hint: You are asked to produce an F with $dF = p\,dx + q\,dy$. For $(x,y) \in \Omega$ define

$$F(x,y) := \int_a^x p(t,b)\,dt + \int_b^y q(x,s)\,ds = \int_{\beta(x,y)} p\,dx + q\,dy.$$

for the broken-line-segment path $\beta(x,y) := [(a,b),(x,b)] + [(x,b),(x,y)]$. What does Green's Theorem say about $\int_{\partial R} p\,dx + q\,dy$ for any standard rectangle $R \subset \Omega$, and what does this say about $F(x,y)$?)

(13) Explain why the conclusion of the above problem is also valid if Ω is an open disk or an open ellipse or an open half-plane. What is the crucial feature needed to make the proof work?

(14) Let Ω be an open rectangle (or an open disk or ellipse or half-plane). Suppose that f is holomorphic and $\mathcal{C}^1$ in Ω. Prove that f has a holomorphic anti-derivative in Ω. (Hint: Use HW#(12) with $p\,dx + q\,dy := f\,dz$.) Do not use Cauchy's Theorem!

(15) Suppose that $\mathbb{C} \xrightarrow{f} \mathbb{C}$ is continuous and is such that for any piecewise $\mathcal{C}^1$ path γ in $\mathbb{C}$,

$$\int_\gamma f(\zeta)\,d\zeta \qquad \text{depends only on the endpoints of } \gamma\,.$$

Define a function $\mathbb{C} \xrightarrow{F} \mathbb{C}$ as follows: for $z \in \mathbb{C}$ put $F(z) := \displaystyle\int_{[0,z]} f(\zeta)\,d\zeta\,.$

(Recall that the line segment $[0,z]$ from the origin to z can be parameterized via $[0,1] \ni t \mapsto tz$.) Prove that F is entire and that $F' = f$. Do not use Cauchy's Theorem!

(16) Let Γ be the loop given by $\Gamma(t) := 2\cos(t) + i\sin(t)$ for $0 \le t \le 2\pi$. Evaluate $\displaystyle\int_\Gamma \frac{dz}{z}\,.$

(Hint: Compare to $\int_{\mathbb{T}} dz/z$.)

(17) Calculate $\displaystyle\int_{|z|=2} \frac{dz}{z^2-1}\,.$

(Hint: Where does $(z^2-1)^{-1} = \dfrac{1/2}{z-1} - \dfrac{1/2}{z+1}$ have a holomorphic anti-derivative?)

(18) Confirm that $\displaystyle\int_{C(a;r)} \frac{d\zeta}{\zeta - z} = \begin{cases} 2\pi i & \text{if } |z-a| < r\,, \\ 0 & \text{if } |z-a| > r\,. \end{cases}$

(Hint: Find a region in which $\dfrac{1}{\zeta - z} - \dfrac{1}{\zeta - a}$ has a holomorphic anti-derivative.)

(19) Let $f \in \mathcal{H}(\Omega) \cap \mathcal{C}^1(\Omega)$ with $|f-1| < 1$. Prove that for every piecewise smooth loop Γ in Ω,

$$\int_\Gamma \frac{f'(z)}{f(z)}\,dz = 0\,.$$

(20) Let $f \in \mathcal{H}(\Omega) \cap \mathcal{C}^1(\Omega)$. Show that for each piecewise smooth loop Γ in Ω,

$$\int_\Gamma \overline{f(z)} f'(z)\, dz$$

is purely imaginary.

(21) Let P be a complex polynomial. Evaluate $\displaystyle\int_{C(a;r)} P(z)\, d\bar{z}$.

(22) Let $f(z) = z^2$. Calculate $\displaystyle\int_0^{2\pi} f(2 + e^{i\theta})\, d\theta$ (and confirm that it is non-zero). Doesn't Cauchy's Theorem say that $\displaystyle\int_{|z-2|=1} f(z)\, dz = 0$?? Explain!

(23) Evaluate $\displaystyle\int_{\mathbb{T}} |z-1|\, |dz|$.

(24) Let γ be a piecewise smooth path and $|\gamma| \xrightarrow{f} \mathbb{C}$ be continuous. Prove that

$$\left| \int_\gamma f(z)\, dz \right| \le \max_{z \in |\gamma|} |f(z)| \cdot \ell(\gamma).$$

(25) Let $[0, \pi] \xrightarrow{\gamma} \mathbb{C}$ be the path $\gamma(t) := \exp(1 + it)$. Prove that

$$\left| \int_\gamma \frac{dz}{\operatorname{Log} z} \right| \le e \operatorname{Log}(\pi + \sqrt{\pi^2 + 1}).$$

(26) Find a function f that has the following properties:
- f is $\mathcal{C}^1$ in some open set Ω with $\bar{\mathbb{D}} \subset \Omega$.
- $\displaystyle\int_{\mathbb{T}} f(z)\, dz = 0$.
- f fails to be holomorphic at any point of Ω.

(27) Suppose that f is continuous in $\mathbb{D}$ and for all $0 < r < 1$, $\displaystyle\int_{|z|=r} f(z)\, dz = 0$.
Must f be holomorphic in $\mathbb{D}$? Must either f or $\bar{f}$ be holomorphic in $\mathbb{D}$?

(28) Suppose $f \in \mathcal{H}(\mathbb{C}_*)$. Let S be the unit square $S = \{z : |\mathfrak{Re}(z)| \le 1, |\mathfrak{Im}(z)| \le 1\}$. Use Cauchy's Theorem (in a disk—or in several disks) to demonstrate that

$$\int_{\partial S} f(z)\, dz = \int_{\mathbb{T}} f(z)\, dz.$$

(29) Here are two more ways to calculate the integral in HW#(18) (so don't use your solution from there!). First, as in the problem directly above, use Cauchy's Theorem (in a disk—or in several disks). This is easy when $|z - a| > r$. For the case $|z - a| < r$, explain why

$$\int_{C(a;r)} \frac{d\zeta}{\zeta - z} = \int_{C(z; r - |z-a|)} \frac{d\zeta}{\zeta - z} = 2\pi i.$$

Second, define $\mathbb{C} \setminus C(a;r) \xrightarrow{I} \mathbb{C}$ by $I(z) := \displaystyle\int_{C(a;r)} \frac{d\zeta}{\zeta - z}$.

Examine the difference quotient

$$\frac{I(z+h)-I(z)}{h} \quad \text{and verify that}$$

$$\lim_{h\to 0}\left|\frac{I(z+h)-I(z)}{h} - \int_{C(a;r)} \frac{d\zeta}{(\zeta-z)^2}\right| = 0\,.$$

(Be sure to give <u>all</u> the details!) Explain why I is holomorphic in $\mathbb{C}\setminus C(a;r)$ and give $I'(z)$. Using your formula for I', show that I' is some constant. Use this information to calculate $I(z)$. (What are $I(a)$ and $\lim_{z\to\infty} I(z)$?)

(30) Suppose f is holomorphic in some open set Ω that contains the closed disk $D[a;r]$. Let z be a fixed (but arbitrary) point in the open disk $D(a;r)$. Using Cauchy's Integral Formula (for the circle $C(a;r)$), calculate

$$\frac{f(z+h)-f(z)}{h} \quad \text{and verify that}$$

$$\lim_{h\to 0}\left|\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_{C(a;r)} \frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta\right| = 0\,.$$

(Be sure to give <u>all</u> the details!) What can you now conclude?

(31) Let γ be a piecewise smooth path in the plane $\mathbb{C}$. Suppose $|\gamma| \xrightarrow{\varphi} \mathbb{C}$ is continuous. Define $\Phi : \mathbb{C}\setminus|\gamma| \to \mathbb{C}$ by

$$\Phi(z) := \int_\gamma \frac{\varphi(\zeta)}{\zeta - z}\,d\zeta \qquad \text{for } z \in \mathbb{C}\setminus|\gamma|\,.$$

In class we proved a fancy Proposition about Φ. Here we'll do less, but with our 'bare hands'. By looking at a difference quotient (the same argument as for the above problem), it is straightforward to show that $\Phi \in \mathcal{H}(\mathbb{C}\setminus|\gamma|)$ with

$$\Phi'(z) = \int_\gamma \frac{\varphi(\zeta)}{\zeta - z}^2\,d\zeta \qquad \text{for } z \in \mathbb{C}\setminus|\gamma|\,.$$

In a similar way, prove that $\Phi' \in \mathcal{H}(\mathbb{C}\setminus|\gamma|)$ with

$$\Phi''(z) = 2\int_\gamma \frac{\varphi(\zeta)}{(\zeta - z)^3}\,d\zeta \qquad \text{for } z \in \mathbb{C}\setminus|\gamma|\,.$$

(32) Calculate the following integrals (using HW#(18) and partial fractions):

$$\int_{C(0;4)} \frac{d\zeta}{(\zeta-1)(\zeta-2i)}\,, \quad \int_{C(1;5)} \frac{\zeta^2+\zeta}{(\zeta-2i)(\zeta+3)}\,d\zeta\,.$$

(33) Let $f \in \mathcal{H}(\Omega)$ and suppose $D[a;r] \subset \Omega$. Confirm that

$$f(a) = \frac{1}{2\pi}\int_0^{2\pi} f(a+re^{i\theta})\,d\theta\,.$$

Derive a similar formula for $f^{(n)}(a)$.

(34) Calculate the following integrals:

$$\int_{\mathbb{T}} \frac{e^z}{z}\,dz\,,\ \int_{\mathbb{T}} \frac{e^z}{z^n}\,dz\,,\ \int_{C(0;2)} \frac{dz}{z^2+1}\,,\ \int_{C(0;2)} z^n(1-z)^m\,dz\ (m,n\in\mathbb{Z})\,.$$

(35) Calculate the following integrals, assuming that $r \neq |a|$:

$$\int_{C(0;r)} \frac{|dz|}{|z-a|^2}\,, \quad \int_{C(0;r)} \frac{|dz|}{|z-a|^4}\,.$$

(36) Let K be any line or circle in $\mathbb{C}$. Suppose that $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ is continuous with $f \in \mathcal{H}(\Omega \setminus K)$. Prove that $f \in \mathcal{H}(\Omega)$. (Thus lines and circles are removable sets for holomorphicity.)

(37) Let f be holomorphic in $D[0;R]$ with $|f| \leq M$ on $C(0;R)$. Fix $r \in (0,R)$. Give a uniform upper bound for $|f^{(n)}(z)|$ that is valid for all $z \in D[0;r]$.

(38) Let f be holomorphic in $\mathbb{D}$. Suppose that for all $z \in \mathbb{D}$, $|f(z)| \leq 1/(1-|z|)$. Find the best upper bound for $|f^{(n)}(0)|$ that is obtainable by using a Cauchy estimate.

(39) Let $f \in \mathcal{H}(\Omega)$. Fix a point $a \in \Omega$. Prove that the derivatives $f^{(n)}$ cannot satisfy $|f^{(n)}(a)| > n!\,n^n$ (for all $n \in \mathbb{N}$). Formulate, and prove, a sharper theorem with a similar result.

(40) Prove that an entire function whose real part is always non-negative must be a constant. Conclude that any entire function that maps the plane into some half-plane must be a constant.

(41) Suppose $\mathbb{C} \xrightarrow{f} \mathbb{C}$ is a non-constant entire function. Demonstrate that $f(\mathbb{C})$ is dense in $\mathbb{C}$. (That is, $\overline{f(\mathbb{C})} = \mathbb{C}$; i.e., if $U \subset \mathbb{C}$ is any non-empty open set, then $U \cap f(\mathbb{C}) \neq \emptyset$.)

(42) Let f be an entire function. Suppose there are constants $k > 0$ and $R > 0$ and some $n \in \mathbb{N}$ such that for all $z \in \mathbb{C}$, $|z| \geq R \implies |f(z)| \leq k|z|^n$. Prove that f must be a polynomial. What is the maximum possible degree of f?

(43) Prove that a complex polynomial P of degree n can be factored in the form

$$P(z) := b\,(z-a_1)(z-a_2)\dots(z-a_n) \quad \text{for some } b, a_1, a_2, \dots, a_n \in \mathbb{C}\,.$$

Deduce that a complex polynomial of degree n has exactly n zeroes (provided the zeroes are counted according to multiplicity).

(44) Let P be a (complex) polynomial. Fix $a \in \mathbb{C}$, $r > 0$ and assume that for all $z \in C(a;r)$, $P(z) \neq 0$. Calculate

$$\int_{C(a;r)} \frac{P'(z)}{P(z)}\,dz\,.$$

(45) Let R be a (complex) rational function. Fix $a \in \mathbb{C}$, $r > 0$ and assume that for all $z \in C(a;r)$, $R(z)$ is defined with $R(z) \neq 0$. Calculate

$$\int_{C(a;r)} \frac{R'(z)}{R(z)}\,dz\,.$$

(46) In Ahlfors, do the following problems on p.37: #'s 1-5.

(47) Prove that for each $z \in \mathbb{D}$, $\lim_{n\to\infty} z^n = 0$. Demonstrate that, however, the function sequence $(z^n)_1^\infty$ does <u>not</u> converge uniformly to 0 in $\mathbb{D}$. What can you say about its convergence on a compact subset of $\mathbb{D}$?

(48) Demonstrate that a uniform limit of continuous functions is continuous.

(49) Let γ be a piecewise smooth path in $\mathbb{C}$. Let $(f_n)_1^\infty$ be a sequence of continuous functions $f_n : |\gamma| \to \mathbb{C}$. Suppose that $(f_n)_1^\infty$ converges uniformly on $|\gamma|$. Prove that

$$\lim_{n\to\infty} \int_\gamma f_n(z)\, dz = \int_\gamma \Big(\lim_{n\to\infty} f_n(z) \Big)\, dz\,.$$

(50) Prove the *Limit Comparison Test*:
Let $(a_n)_1^\infty, (b_n)_1^\infty$ be sequences of positive real numbers. Define

$$L := \limsup_{n\to\infty} \frac{a_n}{b_n} \quad \text{and} \quad \ell := \liminf_{n\to\infty} \frac{a_n}{b_n}\,.$$

Then:
- (a) If $L < \infty$ and $\sum_1^\infty b_n$ converges, then so does $\sum_1^\infty a_n$.
- (b) If $\ell > 0$ and $\sum_1^\infty a_n$ converges, then so does $\sum_1^\infty b_n$.
- (c) Deduce that when $0 < \ell$ and $L < \infty$, the series $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ either both converge or both diverge.

Here's an easier version to 'warm-up' with. Suppose that $L := \lim_{n\to\infty}(a_n/b_n)$ exists in $[0, +\infty]$. Show that when $L < \infty$, convergence of $\sum_1^\infty b_n$ implies convergence of $\sum_1^\infty a_n$; and, when $L > 0$, convergence of $\sum_1^\infty a_n$ implies convergence of $\sum_1^\infty b_n$. Deduce that $0 < L < \infty$ implies that the series $\sum_1^\infty a_n$ and $\sum_1^\infty b_n$ either both converge or both diverge.

(51) Prove the *Geometric Series Test*:
The infinite series $\displaystyle\sum_0^\infty z^n$:
- (a) converges to $(1-z)^{-1}$ for each $z \in \mathbb{D}$;
- (b) converges, in $\hat{\mathbb{C}}$, to ∞ for each $z \in \mathbb{C} \setminus \bar{\mathbb{D}}$;
- (c) does not converge for any $z \in \mathbb{T}$.

(52) Prove that, in its disk of convergence, a power series is continuous. (Do not use the fact that power series are holomorphic in their disk of convergence!)

(53) In Ahlfors, do the following problems on p.41: #'s 1-9.

(54) Determine the set of all $z \in \mathbb{C}$ with $\displaystyle\sum_{n=1}^\infty \frac{z^n}{1+z^{2n}}$ convergent.
Discuss where we do, or do not, get absolute and/or uniform convergence.

(55) Let $R := P/Q$ be a complex rational function with P and Q complex polynomials having no common zeroes. Let $Z := \{z \in \mathbb{C} : Q(z) = 0\}$ and put $\Omega := \mathbb{C} \setminus Z$. Prove that for each $a \in \Omega$, the radius of convergence for the Taylor series for R centered at the point a is precisely $\operatorname{dist}(a, Z)$. (Suggestion: Look first at the special case $R(z) = 1/(z-b)$ where b is a point in Z with $|a-b| = \operatorname{dist}(a, Z)$.) Please do not use Taylor's Theorem here, in any way.

(56) Let h be any holomorphic branch of the logarithm function in $\Omega := \mathbb{C} \setminus [0, +\infty)$. Find the Taylor series for h with center $a := 1+i$. Prove that the radius of convergence R for this power series satisfies $R > \operatorname{dist}(a, \partial\Omega)$. Find a point $z \in \Omega \cap D(a; R)$ such

that at z the series converges to something different from $h(z)$. Can you determine exactly what the series does converge to in $D(a;R)$?

(57) Let $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ be continuous. Let $[0,1] \xrightarrow{\gamma_n} \Omega$ be piecewise smooth paths that converge uniformly to γ. Assume that γ is also a piecewise smooth path and that $\dot{\gamma_n} \to \dot{\gamma}$ uniformly. (How much of these latter hypotheses are actually needed? Check out http://www.mathcs.org/analysis/reals/funseq/uconv.html) Prove that

$$\lim_{n\to\infty} \int_{\gamma_n} f(z)\,dz = \int_\gamma f(z)\,dz\,.$$

(58) Let $(\varphi_n)_1^\infty$ be a sequence of continuous maps defined on a compact set $K \subset \mathbb{C}$. Suppose that for each $z \in K$, $(|\varphi_n(z)|)_1^\infty$ is a decreasing sequence that converges to zero. Prove that $(\varphi_n)_1^\infty$ converges uniformly to zero on K. (Hint: Given $\varepsilon > 0$, consider the sets $K_n := \{z \in K \mid |\varphi_n(z)| \geq \varepsilon\}$. Explain why $\bigcap_n K_n = \emptyset$, and use this to assert the existence of an $N \in \mathbb{N}$ with $K_N = \emptyset$. Recall that a decreasing sequence of non-empty compact sets has a non-empty intersection, right?) Can you find an example of a sequence $(\varphi_n)_1^\infty$ of continuous maps on a compact set $K \subset \mathbb{C}$ that converges to zero but not uniformly?

(59) Let $(f_n)_1^\infty$ be a sequence of continuous maps defined on an open set $\Omega \subset \mathbb{C}$. Suppose that $\sum_1^\infty |f_n|$ converges pointwise in Ω to a continuous function g. Prove that $\sum_1^\infty f_n$ converges normally in Ω. (Hint: Use the previous problem with $\varphi_n := g - \sum_1^n |f_k|$.)

(60) Prove that normal convergence in an open set Ω is equivalent to uniform convergence on every closed disk in Ω.

(61) Let $(f_n)_1^\infty$ be a sequence of continuous functions defined in some open set Ω, and let γ be a piecewise smooth path in Ω. Suppose that $(f_n)_1^\infty$ converges normally in Ω. Demonstrate that $\lim_{n\to\infty} f_n$ is continuous in Ω and that

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma \lim_{n\to\infty} f_n(z)\,dz\,.$$

(62) Let $(f_n)_1^\infty$ be a sequence of continuous functions defined in some open set Ω, and let γ be a piecewise smooth path in Ω. Suppose that $\sum_{n=1}^\infty f_n$ converges normally in Ω. Demonstrate that $\sum_{n=1}^\infty f_n$ is continuous in Ω and that

$$\int_\gamma \sum_{n=1}^\infty f_n(z)\,dz = \sum_{n=1}^\infty \int_\gamma f_n(z)\,dz\,.$$

(63) Let $(f_n)_{n=1}^\infty$ be a sequence of functions that are continuous in $\Omega \subset \mathbb{C}$. Assume that $(f_n)_{n=1}^\infty$ converges normally in Ω to some function f. Suppose $(z_n)_{n=1}^\infty$ is a sequence of points in Ω that converges to a point $a \in \Omega$. Prove that the sequence $(f_n(z_n))_{n=1}^\infty$ converges to $f(a)$. Suppose that in addition, each $f_n \in \mathcal{H}(\Omega)$. Demonstrate that for each $k \in \mathbb{N}$, $(f_n^{(k)})_1^\infty$ converges to $f^{(k)}(a)$.

(64) Let $(f_n)_{n=1}^\infty$ be a sequence of functions that are holomorphic in $D := D(a;r)$ and continuous in $\bar{D}$. Suppose that φ is continuous on $C := \partial D = C(a;r)$, that $(f_n)_{n=1}^\infty$

converges pointwise to φ on C, and that $\int_C |f_n(\zeta) - \varphi(\zeta)||d\zeta| \to 0$ as $n \to \infty$. Prove that $(f_n)_{n=1}^{\infty}$ converges normally in D to the function f given by

$$f(z) := \frac{1}{2\pi i} \int_C \frac{\varphi(\zeta)}{\zeta - z}\, d\zeta\,.$$

(65) Prove that $\sum_{n=0}^{\infty} \cos(nz)/n!$ converges normally in $\mathbb{C}$, and find the entire function given by this sum.

(66) Given $t > 0$, verify that the series $\sum_{n=0}^{\infty}(1-z)^n(1+z)^{-n}$ converges absolutely and uniformly on the set $A_t := \{z \in \mathbb{C} \mid \mathfrak{Re}(z) \geq t, |z| \leq 1/t\}$. Conclude that the series converges absolutely and normally in the right half-plane $\mathbb{H}$, and find its sum there. Show that the series fails to converge at every other $z \in \mathbb{C} \setminus (\mathbb{H} \cup \{-1\})$.

(67) Given $\lambda \in \mathbb{C}$, consider the doubly infinite power series $\sum_{-\infty}^{+\infty} \lambda^{|n|} z^n$. Find the set Λ of all λ for which this series has a non-empty annulus of convergence. For each $\lambda \in \Lambda$, identify the function represented by the associated series.

(68) Let $f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\bar{\Omega})$. Fix $a, b \in \mathbb{C}$ and $R > 0$. Assume that $\Omega \cap D(0;R) \neq \emptyset$. Prove that there is a point $\zeta \in \partial\Omega \cap D[0;r]$ such that $|a\,f(\zeta) + b| = \sup_{z \in \Omega \cap D(0;R)} |a\,f(z) + b|$.

(69) Let Ω be a bounded plane domain. Suppose that $f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ with $|f(z)| = 1$ for each $z \in \partial\Omega$. Prove that either f is a constant function or there exists a point $a \in \Omega$ such that $f(a) = 0$

(70) Let Ω be a bounded plane domain. Suppose that $f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ with $|f(z)| = 1$ for each $z \in \partial\Omega$. Prove that either f is a constant function or that $f(\Omega) = \mathbb{D}$. (Hint: Given $b \in \mathbb{D}$, consider $g := T \circ f$ where $T(w) = (w-b)/(1-\bar{b}w)$.)

(71) Let $f \in \mathcal{H}(\mathbb{D}) \cap \mathcal{C}(\bar{\mathbb{D}})$. Suppose there are $A, B \geq 0$ such that for all $z \in \mathbb{T}$,

$$\mathfrak{Im}(z) \geq 0 \implies |f(z)| \leq A \quad \text{and} \quad \mathfrak{Im}(z) \leq 0 \implies |f(z)| \leq B\,.$$

Demonstrate that $|f(0)| \leq \sqrt{AB}$. (Suggestion: consider also $g(z) := f(-z)$.)

(72) Let $f \in \mathcal{H}(\Omega) \cap \mathcal{C}(\bar{\Omega})$ with Ω unbounded. Suppose that

$$(\star) \qquad \lim_{\Omega \ni z \to \infty} f(z) = 0\,.$$

Determine whether or not there exists a point $\zeta \in \partial\Omega$ such that $|f(\zeta)| = \sup_{z\in\Omega} |f(z)|$. (Either prove that this is so, or provide a counter-example.) What happens if we replace the condition $(\star)$ with $\lim_{\Omega \ni z \to \infty} |f(z)| = L$ for some $L > 0$? What are the corresponding results when we replace "sup" with "inf"?

(73) Let $f \in \mathcal{H}(\mathbb{D})$ with $|f'| \leq 1$ and $f(0) = f'(0) = 0$. Demonstrate that for all $z \in \mathbb{D}$, $|f(z)| \leq |z|^2/2$. For which functions does equality hold at some z ? What is the generalization of this when $f(0) = f'(0) = f''(0) = \cdots = f^{(n)}(0) = 0$ and $|f^{(n)}| \leq 1$?

(74) Let f be holomorphic in a region Ω with $f(a) = 0$ for some point $a \in \Omega$. Suppose $D(a;d) \subset \Omega$ and $f \not\equiv 0$ in $D(a;d)$. Prove that there is a unique $m \in \mathbb{N}$ and a unique $F \in \mathcal{H}(\Omega)$ with $F(a) \neq 0$ and such that for all $z \in \Omega$, $f(z) = (z-a)^m F(z)$. Conclude that there exists $0 < r < d$ such that for all $0 < |z-a| < r$, $f(z) \neq 0$.

(75) Read pp.134-136 in Ahlfors and do problems #'d 1-5 on p.136.

(76) State and prove a Schwarz Lemma type result for holomorphic $\mathbb{D} \xrightarrow{f} \bar{\mathbb{H}}$ with $f(0) = 1$.

(77) Let $f \in \mathcal{H}(\mathbb{D}) \cap \mathcal{C}(\bar{\mathbb{D}})$. Suppose that for all $z \in \mathbb{T}$ with $\Im\mathfrak{m}(z) \geq 0$, $f(z) = 0$. Prove that $f \equiv 0$. (Hint: Consider $f(-z)$ along with $f(z)$. Use HW#(84).)

(78) Let f be holomorphic and bounded in $S := \{z : |\Re\mathfrak{e}(z)| < 1, |\Im\mathfrak{m}(z)| < 1\}$ (i.e., the open 'unit' square centered at the origin). Let E denote one of the closed edges of ∂S (so E includes its endpoints). Suppose that for all $w \in E$,

$$\lim_{S \ni z \to w} f(z) = 0\,.$$

Prove that $f \equiv 0$. What can you say if f is not assumed to be bounded?

(79) Let $A \subset \mathbb{C}$. Recall that A' denotes the set of accumulation points of A. Verify that

$$z \in A' \iff \exists\, (a_n)_1^\infty \text{ in } A \setminus \{z\} \text{ with } a_n \to z\,.$$

Deduce that A' is always a closed set (regardless of A).

(80) Let $A \subset \Omega$ with Ω an open subset of $\mathbb{C}$. Demonstrate that A is discrete in Ω if and only if for each $z \in \Omega$ there exists an $r > 0$ such that $D(z; r) \cap A \subset \{z\}$.

(81) Let $A \subset \Omega$ with Ω an open subset of $\mathbb{C}$. Suppose that A is discrete in Ω. Prove that:
(a) $\Omega \setminus A$ is open.
(b) If Ω is a domain (i.e., an open connected set), then so is $\Omega \setminus A$.
(c) If K is a compact subset of Ω, then $A \cap K$ is a finite set.

(82) Let $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ and $\mathbb{C} \supset \Omega' \xrightarrow{g} \mathbb{C}$ be non-constant holomorphic maps with $\Omega' \supset \Omega$. Suppose that the multiplicity of f at a is m, and the multiplicity of g at $b := f(a)$ is n. Prove that the multiplicity of $g \circ f$ at a is mn.

(83) Does there exist a non-constant $f \in \mathcal{H}(\mathbb{H})$ with $f(1/n) = 2$ for each $n \in \mathbb{N}$? Either produce an example of such a function, or prove that none exists.

(84) Let f, g be functions that are both holomorphic in some domain Ω. Suppose that for all $z \in \Omega$, $f(z)g(z) = 0$. Verify that either $f \equiv 0$ in Ω or $g \equiv 0$ in Ω.

(85) Let $f \in \mathcal{H}(\mathbb{D}) \cap \mathcal{C}(\bar{\mathbb{D}})$ be non-constant. Suppose that for all $z \in \mathbb{T} = \partial\mathbb{D}$, $|f(z)| = 1$. Demonstrate that f has the form

$$\forall z \in \mathbb{D}\,, \quad f(z) = c \prod_{i=1}^{k} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{m_i}$$

where $c \in \mathbb{T}$, $a_1, \ldots, a_k \in \mathbb{D}$ are distinct, and $m_1, \ldots, m_k \in \mathbb{N}$. (Recall HW#s(69,70).)

(86) Suppose f is entire and non-constant and satisfies $f(\mathbb{T}) \subset \mathbb{T}$. Prove that there exists $c \in \mathbb{T}$ and $m \in \mathbb{N}$ such that for all $z \in \mathbb{C}$, $f(z) = c\, z^m$.

(87) Suppose that the power series $\sum_0^\infty c_n z^n$ has radius of convergence 1. Let f denote the holomorphic map given by this series. Prove that there exists a point $\zeta \in \mathbb{T}$ such that there is no $r > 0$ with the property that f can be extended to a holomorphic map in $\mathbb{D} \cup D(\zeta; r)$.

(88) Read pp.30-32 in Ahlfors and do problems 2,4,5,6 on pp.32-33.

(89) Prove the *Factor Theorem for Poles:* Let f be holomorphic in $\Omega \setminus \{a\}$. Suppose that f has a pole at a. Then there is a unique $m \in \mathbb{N}$ and a unique $F \in \mathcal{H}(\Omega)$ with $F(a) \neq 0$ and such that for all $z \in \Omega$, $f(z) = F(z)/(z-a)^m$.

(90) Read pp.124-129 in Ahlfors and do problems 2-6 on p.130.

(91) Suppose that f and g have poles of orders m and n respectively at the point $z = a$. Provide as much information as you can about the nature of the singularity at $z = a$ for the maps: (i) $f + g$, (ii) fg, (iii) f/g.

(92) Suppose f has a pole of order m at $z = a$. Confirm that f' has a pole of order $m+1$ at $z = a$.

(93) Let $f \in \mathcal{H}(\mathbb{C}_*)$. Suppose there exists an $M > 0$ such that

$$\forall z \in \mathbb{C}_*, \quad |f(z)| \leq M|z||\operatorname{Log}(z)|.$$

Prove that $f = 0$.

(94) Let $f \in \mathcal{H}(D_*(a;r))$ for some $a \in \mathbb{C}$ and $r > 0$. For those $z \in D_*(a;r)$ with $f(z) \neq 0$, define $g(z) := 1/f(z)$. Discuss the holomorphicity of g. (Where is g holomorphic? Does g have any isolated singularities? If so, classify each of them.)

(95) Let f be non-constant and meromorphic in $\mathbb{C}$. Suppose that for all $z \in \mathbb{T} = \partial\mathbb{D}$, $|f(z)| = 1$. Demonstrate that f has the form

$$f(z) = c \prod_{i=1}^{k} \left(\frac{z - a_i}{1 - \bar{a}_i z} \right)^{m_i} \prod_{j=1}^{l} \left(\frac{1 - \bar{b}_j z}{z - b_j} \right)^{n_j}$$

where $c \in \mathbb{T}$, $a_1, \ldots, a_k \in \mathbb{D}$ are the distinct zeroes of f in $\mathbb{D}$ with respective multiplicities $m_1, \ldots, m_k \in \mathbb{N}$, and $b_1, \ldots, b_l \in \mathbb{D}$ are the distinct poles of f in $\mathbb{D}$ with their multiplicities $n_1, \ldots, n_l \in \mathbb{N}$. (If f is zero free in $\mathbb{D}$, then the first product is missing; if f has no poles in $\mathbb{D}$, then the second product is missing.)

(96) Let f be meromorphic in all of $\mathbb{C}$. Suppose there exist circles K and K' on $\hat{\mathbb{C}}$ such that $f(K) \subset K'$. Prove that f is in fact a rational function.

(97) Let $f \in \mathcal{H}(\Omega)$. Suppose that g is a branch of the logarithm of f in Ω. Prove that $g \in \mathcal{H}(\Omega)$.

(98) Let R be a rational function. State and prove necessary and sufficient conditions for there to be a holomorphic branch of the logarithm of R in some domain Ω.

(99) For which of the following sets A does there exist a holomorphic branch of the logarithm of $z^{-2}(z+1)^{-1}(z^2+1)$ in $\Omega := \mathbb{C} \setminus A$? (i) $A := (-\infty, -1] \cup [0, \infty) \cup \{iy : y \in \mathbb{R}, |y| \geq 1\}$, (ii) $A := (-\infty, -1] \cup \{iy :\in \mathbb{R}, |y| \leq 1\}$, (iii) $A := (-\infty, 0] \cup \{e^{it} : -\pi/2 \leq t \leq \pi/2\}$, (iv) $A := \{e^{it} : \pi/2 \leq t \leq \pi\} \cup \{iy : y \leq 0\}$.

Let $p \in \mathbb{N}$ (with $p \geq 2$) and $\Omega \xrightarrow{f} \mathbb{C}$ be continuous. We call $\Omega \xrightarrow{g} \mathbb{C}$ a *branch of the* p^{th}*-root of* f *in* Ω provide g is continuous and for all $z \in \Omega$, $[g(z)]^p = f(z)$ (briefly, $g^p = f$ in Ω).

(100) Let $f \in \mathcal{H}(\Omega)$. Suppose that g is a branch of the p^{th}-root of f in Ω. Prove that $g \in \mathcal{H}(\Omega)$.

(101) Let f be a quadratic polynomial with distinct zeroes a, b.
  (a) Show that the existence of a square root of f in Ω implies that $\{a,b\} \cap \Omega = \emptyset$. What if $a = b$?
  (b) Demonstrate that the hypothesis
$$\forall \;\text{ PSL } \Gamma \text{ in } \Omega\,, \quad n(\Gamma; a) = n(\Gamma; b)$$
guarantees the existence of a square root of f in Ω.
  (c) Prove that the existence of a square root of f in Ω implies that
$$\forall \;\text{ PSL } \Gamma \text{ in } \Omega\,, \quad n(f \circ \Gamma; 0) \in 2\,\mathbb{Z}\,.$$
(Hints for (b): First look at $f(z) = z^2 - 1$. Explain why there is a square root S of T where $T(z) := (z-a)/(z-b)$. Then examine $g(z) = (z-b)S(z)$.)

(102) For which of the following does there exist a holomorphic branch of the p^{th}-root of f in the domain $\Omega := \mathbb{C} \setminus A$? (i) $p = 3$, $f(z) := z(z-1)(z+1)$, $A := (-\infty, -1] \cup [0, 1]$; (ii) $p = 2$, $f(z) = z^2 - 2z$, $A := [0, 2]$; (iii) $p = 4$, $f(z) := z^3 + z$, $A := (-\infty, 0] \cup [1, +\infty) \cup \{e^{it} : -\pi/2 \le t \le \pi/2\}$; (iv) $p = 3$, $f(z) := z^3 + z^2 + z + 1$, $A := \{e^{it} : \pi/2 \le t \le 3\pi/2\}$.

(103) Explain why there is a branch g of the square root of $f(z) := (z-1)(z-2)(z+2)$ in the domain $\Omega := \mathbb{C} \setminus ((-\infty, -2] \cup [1, 2])$ with $g(0) = 2$. Find a formula for g in terms of *elementary* functions. (NB The function $z \mapsto \sqrt{(z-1)(z-2)(z+2)}$ does not meet the requirements, because it is not even continuous in Ω.)

(104) Let $f \in \mathcal{H}(\Omega)$. State a necessary condition for there to be a holomorphic branch of the p^{th}-root of f in domain Ω.

(105) Let f be an entire function. Suppose that $g(z) = \sqrt{f(z)}$ also defines an entire function (where $\sqrt{w}$ denotes the principal value of the square root of w). What can you deduce about f?

(106) Read pp.1-136 in Ahlfors and do problems 1-4 on p.133.

(107) Let Ω be a bounded plane domain containing the origin. Suppose $\Omega \xrightarrow{f} \Omega$ is holomorphic with $f(0) = 0$ and $f'(0) = 1$. Prove that for all $z \in \Omega$, $f(z) = z$. (Hints: It suffices to show that $f(z) = z$ for all $z \in \Delta$ where $\Delta := D(0; r)$ and $0 < r < \text{dist}(0, \partial\Omega)$. Look at the Maclaurin series for f, say $f(z) = z + a_m z^m + \dots$. Use Cauchy estimates to find a bound for $|a_m|$. Assume that $a_m \neq 0$ and examine the Maclaurin series for the k-fold composition $f \circ f \circ \dots \circ f$ and consider 'what happens' as $k \to \infty$.)

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