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COMPLEX ANALYSIS HOMEWORK PROBLEMS
AUTUMN QUARTER 2009

Please provide plenty of details! Pix are definitely kewl ($\ddot\smile$).

(1) Read chapter one in Ahlfors. Please be sure to look at (and work) the suggested problems from Ahlfors; these are listed on the web page.

(2) Verify the 'parallelogram law' for complex numbers z, w:

$$|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2.$$

(3) Given complex numbers z, w, prove that $|z+w| \le |z| + |w|$ and that equality holds for $z \neq 0 \neq w$ if and only if $w = tz$ for some $t > 0$.

(4) Let $c \in \mathbb{D}$. Demonstrate that $|z+c| \le |1 + \bar{c}z|$ if and only if $|z| \le 1$, with equality holding if and only if $|z| = 1$.

(5) Provide geometric descriptions for the following subsets of $\mathbb{C}$:
(a) $\{z : |z-1| = \mathfrak{Re}(z)\}$. (b) $\{z : |z-i| + |z+i| = 4\}$. (c) $\{z : |z-i|^2 + |z+i|^2 = 4\}$.
Give both a written/verbal description and a pictorial description.

(6) Consider the equation

$$r|z|^2 + cz + \bar{c}\bar{z} + s = 0 \quad \text{where } r, s \in \mathbb{R},\ c \in \mathbb{C} \text{ and } |c|^2 > rs\,.$$

Demonstrate that this is the equation of a line when $r = 0$ and a circle when $r \neq 0$. Give the standard equations in each case; especially, when this equation describes a circle, what are its center and radius? Also, verify that every line or circle can be described by such an equation.

(7) Consider the equation $a\,z + b\,\bar{z} + c = 0$ where $a, b, c \in \mathbb{C}$. We want to understand the possible solution set for such an equation.
(a) What are the 'trivial' cases?
(b) Determine when this equation has a unique solution z, and give z.
(c) Determine when this equation represents a straight line, and find a 'standard' equation for this line.
(d) What can you say about all other cases?

(8) Verify the formulas

$$\arg(1/z) = -\arg(z) \quad , \quad \arg(zw) = \arg(z) + \arg(w)$$

and explain precisely what these actually mean. Find z, w that satisfy

$$\operatorname{Arg}(1/z) \neq -\operatorname{Arg}(z) \quad \text{and} \quad \operatorname{Arg}(zw) \neq \operatorname{Arg}(z) + \operatorname{Arg}(w)\,.$$

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Can you determine when we do have

$$\operatorname{Arg}(1/z) = -\operatorname{Arg}(z) \quad \text{or} \quad \operatorname{Arg}(zw) = \operatorname{Arg}(z) + \operatorname{Arg}(w)\,?$$

(9) Prove that there does not exist a continuous branch of the argument function in $\mathbb{C}_*$.

(10) Show that when $|z| = 1$ and $z \neq -1$, $\operatorname{Arg}(z) = 2\operatorname{Arg}(z+1)$.
(Hint: What is $\{z + 1 : |z| = 1\}$?)

(11) Give examples to illustrate that, in general, $\sqrt[\text{pv}]{zw} \neq \sqrt[\text{pv}]{z}\,\sqrt[\text{pv}]{w}$, where $\sqrt[\text{pv}]{z}$ stands for the principal value of the square root of z. Confirm that equality does hold provided either z or w is a positive real number.

(12) (a) Clearly, for any $z \in \mathbb{C}$, $\sqrt[\text{pv}]{z^2}$ is either z or $-z$. For which z is which true?

(b) For which complex numbers is it true that $\sqrt[\text{pv}]{z/\bar{z}} = z/|z|$?

(13) Prove Proposition 1.3 in the notes.

(14) (a) Prove that the map $T(z) := (1 - z)/(1 + z)$ is a bijection between $\mathbb{D}$ and $\mathbb{H}$.

(b) Find a formula for the inverse map T^{-1}.

(15) Determine the images of the sets $\mathbb{R}$, $i\mathbb{R}$, $\{x + y = 1\}$, $\mathbb{D}$, $\mathbb{H}$ under the complex linear map $L(z) := (1 + i\sqrt{3})z + 2$.

(16) Recall that z is a *fixed point* of the map f if $f(z) = z$. Let $L(z) := az + b$.

(a) Prove that L has no fixed points if and only if $a = 1$ and $b \neq 0$.

(b) Suppose L has a fixed point. Describe the set of all fixed points of L.

(17) Suppose the complex linear map $L(z) := az + b$ has a unique fixed point c. Confirm that L can be expressed as $L(z) = c + a(z - c)$. Use this representation to explain the geometric effect of the mapping L; you should be able to describe L in terms of standard dilations and rotations but now with respect to the fixed point (instead of with respect to the origin).

(18) Let K and K' be two lines in $\mathbb{C}$. Does there necessarily exist a complex linear map L with $L(K) = K'$? If so, is L unique? If L is not unique, what else can you say? What if K, K' are two circles?

(19) Explain why and how each complex linear map L induces maps $\mathcal{L} \xrightarrow{\Phi} \mathcal{L}$ and $\mathcal{C} \xrightarrow{\Psi} \mathcal{C}$. Are these maps injective or surjective? (See the end of §1.3 in the notes for the notation.)

(20) Let L be a complex linear map. Verify that L transforms every polygon $\Pi \subset \mathbb{C}$ into a polygon $\Pi' := L(\Pi)$ which is similar to Π. What can you say about the image, under L, of a conic section (i.e., an ellipse, parabola, or hyperbola)?

(21) Show that given any points $z_1, w_1 \in \mathbb{C}$, there are many complex linear maps L with $L(z_1) = w_1$. What can you say about n pairs of points $z_1, \ldots, z_n$ and $w_1, \ldots, w_n$ in $\mathbb{C}$: When does there exist a complex linear map L with $L(z_i) = w_i$ for all $1 \leq i \leq n$? And when is such an L unique?

(22) Let **Lin** denote the family of all non-constant complex linear maps $\mathbb{C} \to \mathbb{C}$.
(a) Confirm that **Lin** is a group (with product given by composition). Is it abelian?
(b) Find two elements of **Lin**, one with finite order and one with infinite order.
(c) Check that the family **Tran** of pure translations is a normal subgroup of **Lin**.
(d) Identify the quotient group **Lin**/**Tran**.
(e) Is the family **Rotn** of pure rotations a subgroup of **Lin**? (A *pure rotation about* $a \in \mathbb{C}$ is a map of the form $z \mapsto c(z-a)+a$ where $c \in \mathbb{T}$.)

Hint: Problems #(16, 17) might prove useful here.

Recall that a map f is an *isometry (* or *similarity)* if it preserves (dilates) all distances; i.e.,

$$\forall z, w : \quad |f(z) - f(w)| = |z - w| \text{ (or } |f(z) - f(w)| = \sigma |z - w| \text{ for some constant } \sigma > 0) .$$

(23) Prove that the only isometry of $\mathbb{C}$ that fixes three non-collinear points is the identity map. (Hints: What can you say about a if there are distinct z, z' with $|z-a| = |z'-a|$? Deduce that if $a, b, c \in \mathbb{C}$ are non-collinear, then each $z \in \mathbb{C}$ is uniquely determined by the numbers $|z-a|, |z-b|, |z-c|$.)

(24) (a) Suppose $\mathbb{C} \xrightarrow{f} \mathbb{C}$ is an isometry with $f(0) = 0$. Demonstrate that there exist $c \in \mathbb{T}$ such that f can be expressed as $f(z) = cz$ or $f(z) = c\bar{z}$. (Hint: What if $f(1) = 1$?)
(b) Determine the general form of an isometry $\mathbb{C} \to \mathbb{C}$.
(c) Determine the general form of a similarity $\mathbb{C} \to \mathbb{C}$.
(Hint: Look at $g := [f - f(0)]/[f(1) - f(0)]$.)

(25) Consider the map $z \mapsto a\bar{z} + b$ where $a \in \mathbb{C}_*$ and $b \in \mathbb{C}$ are constants. Which of the many properties for complex linear maps continue to hold for such a map? Which properties fail to hold?

(26) Describe the geometric effect of the map $z \mapsto a\bar{z} + b$, where $a \in \mathbb{C}_*$ and $b \in \mathbb{C}$ are constants.

(27) Determine the fixed points of the map $z \mapsto a\bar{z} + b$, where $a \in \mathbb{C}_*$ and $b \in \mathbb{C}$ are constants. (Hint: Recall problem #(7).)

Two points z and z^* are said to be *symmetric with respect to a line* K if K is the perpendicular bisector of the segment $[z, z^*]$.

(28) (a) Show that z and $\bar{z}$ are symmetric with respect to $\mathbb{R}$.
(b) Verify that if a and b are symmetric with respect to a line K, then

$$\operatorname{dist}(a, K) = \frac{1}{2}|a-b| = \operatorname{dist}(b, K) .$$

(c) Demonstrate that a and b are symmetric with respect to a line K if and only if

$$\forall\, z \in K , \quad |z-a| = |z-b| .$$

(29) Let K be a line in $\mathbb{C}$. Suppose L is a complex linear map with $L(K) = \mathbb{R}$. Verify that for each $z \in \mathbb{C}$, $z^* := L^{-1}(\overline{L(z)})$ enjoys the property that z and z^* are symmetric with respect to K.

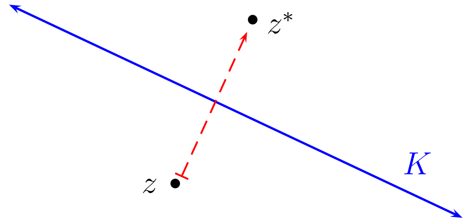

FIGURE 1. Reflection across K

(30) Let K be a line in $\mathbb{C}$. Prove that for each $z \in \mathbb{C}$ there is a unique $z^* \in \mathbb{C}$ such that z and z^* are symmetric with respect to K. **Provide** a formula for z^*. (Suggestion: Start with a standard equation for K such as $\mathfrak{Re}((z-a)\bar{\nu}) = 0$ or $cz + \bar{c}\bar{z} + s = 0$.)

Given a line K, the map $\mathbb{C} \xrightarrow{\rho_K} \mathbb{C}$ defined by $z \mapsto \rho_K(z) := z^*$, where z^* is the unique point such that z and z^* are symmetric with respect to K, is called *reflection across* the line K. See Figure 1.

(31) (a) Prove that complex linear maps preserve symmetric points. That is, if z, z^* are symmetric with respect to a line K, and w, w^*, K' are the images of z, z^*, K under a complex linear map, then w, w^* are symmetric with respect to K'.

(b) Let K be a line in $\mathbb{C}$ and $\mathbb{C} \xrightarrow{L} \mathbb{C}$ be a complex linear map. Put

$$K' := L(K)\,, \quad \rho := \rho_K\,, \quad \rho' := \rho_{K'}\,;$$

so K' is another line and ρ and ρ' are reflections across K and K' respectively. Deduce that $L \circ \rho = \rho' \circ L$, so the pictured diagram commutes.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\ \rho\ } & \mathbb{C} \\ \big\downarrow L & & \big\downarrow L \\ \mathbb{C} & \xrightarrow{\ \rho'\ } & \mathbb{C} \end{array}$$

(Hint: All of the above actually hold for any similarity L, right?)

(32) Find the image of $\mathbb{T}$ under the map $z \mapsto w := az + b\bar{z}$ where $a, b \in \mathbb{C}$ and $|a| > |b|$. (Suggestions: Start by looking at, e.g., $2z + \bar{z}$ or—even better—$r\,z + s\,\bar{z}$ where $r > s > 0$. What are $\max|w|$ and $\min|w|$ where z varies over $\mathbb{T}$? Consider a change of variables $Z := e^{i\varphi}z$, $W := e^{i\psi}w$ (for appropriate φ, ψ) and the map $Z \mapsto W$.)

(33) Let H be one of the open half-planes determined by some line passing thru the origin. Confirm that $f(z) := z^2$ is injective on H. What is $f(H)$?

(34) Determine all preimages of horizontal and vertical lines under the map $f(z) := z^2$.

(35) Using cartesian coordinates (instead of polar coordinates), analyze the map $f(z) := z^2$ and find:

(a) $f(\mathbb{R})$ and $f(i\mathbb{R})$.

(b) $f(\{x + imx : x \in \mathbb{R}\})$ where $m \in \mathbb{R}$. What role does m play?

(36) Let K be any line in $\mathbb{C}$ that does not pass through the origin. Determine the image of K under the squaring map $z \mapsto z^2$. (Hint: Notice that $e^{-2i\varphi}(e^{i\varphi}z)^2 = z^2$. Consider a change of variables $Z := e^{i\varphi}z$, $W := e^{i\psi}w$ (for appropriate φ, ψ) and the map $Z \mapsto W$.)

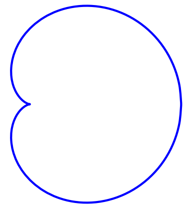

FIGURE 2. A standard cardioid

(37) Recall the picture for the 'standard' cardioid; this is the plane curve given by the polar equation $r = 1 + \cos\theta$. See Figure 2. Let's call any plane curve a *cardioid* if it is similar to the 'standard' cardioid. Prove the the image of any circle in $\mathbb{C}$ that passes through the origin under the squaring map $z \mapsto z^2$ is a cardioid. (Hint: Start with the circle $C(1;1)$. Next, notice that the circle with center a that passes through the origin is $C(a;|a|) = \{a + ae^{it} \mid -\pi/2 < t \le \pi/2\}$. Consider a change of variables $Z := e^{i\varphi}z$, $W := e^{i\psi}w$ (for appropriate φ, ψ) and the map $Z \mapsto W$.)

(38) Determine the images of the following lines under the map $w = \sqrt{z}$.
   (a) A horizontal line.
   (b) A vertical line.
   (c) The line with equation $y = \sqrt{3}\,x$.

(39) Consider the map $f(z) := z + 1/z$ for $z \in \mathbb{C}_*$. Find:
   (a) $f((0,1])$, $f([-1,0))$, $f((0,i])$ and $f((0,-i])$.
   (b) $f(\{re^{i\theta_0} : r > 1\})$ where $\theta_0 \in (0, \pi/2)$ is fixed.
   (c) How does your answer to (b) change if you replace θ_0 by $-\theta_0$, $\pi - \theta_0$, or $\theta_0 - \pi$?.
   (d) How does your answer to (b) change if you replace $r > 1$ with $0 < r < 1$?

(40) Prove that for all $z, w \in \mathbb{C}$: $e^z e^w = e^{z+w}$, $e^{-z} = 1/e^z$, $\forall n \in \mathbb{N} : (e^z)^n = e^{nz}$.

(41) Find the image of the line $\{x + imx : x \in \mathbb{R}\}$ (where $m \in \mathbb{R}$) under the map $w = e^z$. What role does m play?

(42) Determine the real and imaginary parts of $\exp(e^z)$.

(43) Find examples to illustrate that in general, $\mathrm{Log}(ab) \ne \mathrm{Log}(a) + \mathrm{Log}(b)$. For a given $a \in \mathbb{C}_*$, determine the set of all z with $\mathrm{Log}(az) = \mathrm{Log}(a) + \mathrm{Log}(z)$.

(44) Prove that for all $z \in \mathbb{D}$, $\mathrm{Log}(1 - z^2) = \mathrm{Log}(1 - z) + \mathrm{Log}(1 + z)$. What can you say about $\mathrm{Log}[(1 - z)/(1 + z)]$ for $z \in \mathbb{D}$? (Hint: Recall problem #(14).)

(45) Put $\Omega := \mathbb{C} \setminus S$ where S is the spiral $S = \{e^{(1+i)t} : t \in \mathbb{R}\}$. Let $\lambda(z)$ be the branch of $\log(z)$ defined in Ω that satisfies $\lambda(e) = 1$. Calculate $\lambda(e^6)$, $\lambda(-e^{-8})$, $\lambda(ie^{\pi k})$ where k is an integer. Also, determine the range of λ.

(46) Establish the following facts concerning complex powers a^b where $a \in \mathbb{C}_*$ and $b \in \mathbb{C}$.
   (a) If $b \in \mathbb{Z}$, then a^b is a single value.
   (b) If $b = p/q \in \mathbb{Q}$ is rational, say with $p \in \mathbb{Z}$, $q \in \mathbb{N}$ and p, q having no common factors, then there are q distinct complex values of a^b and these are symmetrically located around the circle $|w| = |a|^b$.
   (c) If $b \in \mathbb{R} \setminus \mathbb{Q}$ is irrational, then a^b consists of infinitely many complex values located around the circle $|w| = |a|^b$.

Everywhere above $|a|^b$ stands for the principle value of the b^{th} power of $|a|$; that is, $|a|^b := \exp(b \operatorname{Log}|a|)$.

(47) Find an example to illustrate that, in general, the 'law of exponents' $a^b a^c = a^{b+c}$ fails to hold, even when this is considered as a set equality.

(48) When is it true that $|a^b| = |a|^b$? (Hint: Consider when a or b is real.)

(49) Determine the real and imaginary parts of z^z.

(50) Prove that a subset of $\mathbb{C}$ is open if and only if it can be written as a union of open sets if and only if it can be written as a union of open disks. What can you say about the above statement if every occurrence of the word *open* is replaced by *closed*?

(51) Recall that the *distance from a point z to a set $A \subset \mathbb{C}$* is defined by

$$\operatorname{dist}(z, A) := \inf_{a \in A} |z - a|.$$

Prove that the following are equivalent.

(a) The set A is closed.

(b) For all $z \notin A$, $\operatorname{dist}(z, A) > 0$.

(c) For each convergent sequence $(a_n)_{n=1}^{\infty}$ with $a_n \in A$ for all n, $\lim_{n\to\infty} a_n \in A$.

(52) We can also define the *distance between two sets $A, B \subset \mathbb{C}$* by

$$\operatorname{dist}(A, B) := \inf\{|a - b| : a \in A, b \in B\}.$$

Demonstrate that for $C \subset \mathbb{C}$ closed and $K \subset \mathbb{C}$ compact,

$$C \cap K = \emptyset \iff \operatorname{dist}(C, K) > 0.$$

Is this result still true if we merely assume that C and K are both closed?

(53) Let $E \subset \mathbb{C}_*$ be connected. Suppose λ and λ' are (single-valued continuous) branches of the logarithm in E. Prove that there is a constant $k \in \mathbb{Z}$ such that for all $z \in E$, $\lambda(z) = \lambda'(z) + 2k\pi i$. (Hint: What can you say about a continuous map from a connected set into $\mathbb{Z}$?)

(54) Let $\mathbb{C} \xrightarrow{f} \mathbb{C}$ be continuous. Suppose $K \subset \mathbb{C}$ is compact.

(a) Explain why there exists a point $a \in K$ such that for all $z \in K$, $|f(z)| \le |f(a)|$. Is this necessarily true when K is not compact?

(b) Assume that for all $z \in K$, $f(z) \neq 0$. Confirm that there exists $\varepsilon > 0$ such that for all $z \in K$, $|f(z)| \ge \varepsilon$. Is this necessarily true when K is not compact?

(55) Demonstrate that a function f is continuous if and only if both $\mathfrak{Re}(f)$ and $\mathfrak{Im}(f)$ are continuous. Establish a similar result involving continuity of $|f|$ and $\arg(f)$.

(56) Let $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ with Ω open. Suppose that f is (complex) differentiable at $a \in \Omega$. Prove that f is continuous at a.

(57) Let $\Omega \subset \mathbb{C}$ be open, $a \in \Omega$, and $\Omega \xrightarrow{f} \mathbb{C}$. Demonstrate that the following are equivalent.

(a) f is (complex) differentiable at the point a.

(b) $\exists\, c \in \mathbb{C}$ and $\Omega \xrightarrow{\varepsilon} \mathbb{C}$ such that $\lim_{z\to a} \dfrac{\varepsilon(z)}{z-a} = 0$ and
$$\forall\, z \in \Omega\,, \quad f(z) = f(a) + c\,(z-a) + \varepsilon(z)\,.$$
(c) $\exists\, \Omega \xrightarrow{\varphi} \mathbb{C}$ that is continuous at $z=a$ and such that
$$\forall\, z \in \Omega\,, \quad f(z) = f(a) + (z-a)\varphi(z)\,.$$
In (b), $L(z) := f(a) + c\,(z-a)$ is called the *(complex) linear* (or *first-order*) *approximation of f near $z=a$*. In (c): What is $\varphi(a)$?

(58) Let $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ with Ω open. Suppose $f \in \mathcal{C}^1(\Omega)$. Show that
$$\frac{\partial f}{\partial r} = D_{e^{i\theta}} f = \cos\theta\, \frac{\partial f}{\partial x} + \sin\theta\, \frac{\partial f}{\partial y}$$
and
$$\frac{\partial f}{\partial \theta} = D_{i\,e^{i\theta}} f = -r\,\sin\theta\, \frac{\partial f}{\partial x} + r\,\cos\theta\, \frac{\partial f}{\partial y}$$

(59) Let $\Omega \subset \mathbb{C}$ be open and $\Omega \xrightarrow{f} \mathbb{C}$. Suppose that f is (complex) differentiable at the point $a \in \Omega$. Write $f = u + iv$, $z = x + iy$ and examine the difference quotients
$$\frac{f(z)-f(a)}{z-a} = \frac{u(z)-u(a)}{z-a} + i\,\frac{v(z)-v(a)}{z-a}\,.$$
(a) Keeping $y = \mathfrak{Im}(z) = \mathfrak{Im}(a)$ fixed and letting $x = \mathfrak{Re}(z)$ vary, what do you get when you take a limit as $x \to \mathfrak{Re}(a)$?
(b) Keeping $x = \mathfrak{Re}(z) = \mathfrak{Re}(a)$ fixed and letting $y = \mathfrak{Im}(z)$ vary, what do you get when you take a limit as $y \to \mathfrak{Im}(a)$?
What do you conclude?

(60) Let $\mathbb{C} \supset \Omega \xrightarrow{u} \mathbb{R}$ with Ω open and connected. Suppose u has the property that for each disk $\Delta \subset \Omega$, $u|_\Delta$ is a constant. Prove that u is a constant function.

(61) Let $\mathbb{C} \supset \Omega \xrightarrow{u} \mathbb{R}$ with Ω open and connected. Assume that u has first-order partial derivatives at each point of Ω. Suppose that for all $z \in \Omega$, $\nabla u(z) = 0$. Prove that u is a constant function.

(62) Let $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ be holomorphic with Ω a domain (i.e., open and connected). Suppose that for all $z \in \Omega$, $f'(z) = 0$. Prove that f is a constant map.

(63) Let f be holomorphic in some domain $\Omega \subset \mathbb{C}$ with $|f|$ constant. Confirm that f is a constant map.

(64) Let f be holomorphic in some open disk D. Suppose that $f(D)$ is contained in some line or some circle. Demonstrate that f must be a constant map.

(65) Find a polynomial function of x and y that is complex differentiable at each point of the parabola with equation $y = x^2$, but at no other point.

(66) When is the map $z \mapsto az + b\bar{z}$ complex differentiable?

(67) Let $\mathbb{C} \xrightarrow{f} \mathbb{C}$ and define $\mathbb{C} \xrightarrow{g} \mathbb{C}$ by $g(z) := \overline{f(\bar{z})}$. Prove that f is holomorphic if and only if g is holomorphic.

(68) Let $f := u + iv$ where $z := x + iy$, $u(z) := x^3 + axy^2$, and $v(z) := bx^2y + cy^3 + 1$. Determine $a, b, c \in \mathbb{R}$ so that f is an entire function. Then express f as a complex polynomial; that is, as a function involving only powers of z.

(69) Find the entire function $f := u+iv$ satisfying $f(0) = 2$ and such that for $z := x+iy$, $v(z) := (x \sin y + \sin y + y \cos y)e^x$. Try to express f as a function of z—instead of x and y—but note that this may require some clever factoring!

(70) A function $\mathbb{C} \supset \Omega \xrightarrow{u} \mathbb{R}$ (with Ω open) is said to be *harmonic* in Ω if $u \in \mathcal{C}^2(\Omega; \mathbb{R})$ (i.e., u has continuous second-order partial derivatives) and satisfies *Laplace's equation*

$$\forall z \in \Omega : \qquad \Delta u(z) = \frac{\partial^2 u}{\partial x^2}(z) + \frac{\partial^2 u}{\partial y^2}(z) = 0\,.$$

Prove that the real and imaginary parts of every holomorphic map are harmonic. (Assume you know that holomorphic maps are $\mathcal{C}^2$.)

(71) A pair of harmonic functions $\mathbb{C} \supset \Omega \xrightarrow{u,v} \mathbb{R}$ are said to be *harmonic conjugates* if they satisfy the Cauchy Riemann equations in Ω; that is,

$$\forall z \in \Omega : \ \frac{\partial u}{\partial x}(z) = \frac{\partial v}{\partial y}(z) \quad \text{and} \quad \frac{\partial u}{\partial y}(z) = -\frac{\partial v}{\partial x}(z)\,.$$

Prove that if u, v are harmonic conjugates in Ω, then $f := u + iv$ is holomorphic in Ω.

(72) Let $f \in \mathcal{H}(\Omega)$ and assume $f \in \mathcal{C}^2(\Omega)$. Demonstrate that $f' \in \mathcal{H}(\Omega)$. Conclude that each function $f \in \mathcal{H}(\Omega) \cap \mathcal{C}^\infty(\Omega)$ is infinitely complex differentiable in Ω; that is, for each $n \in \mathbb{N}$, $f^{(n)}$ exists and moreover is holomorphic in Ω.

(73) Suppose there is an $n \in \mathbb{N}$ such that $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ is n-times complex differentiable in the open and connected set Ω. Suppose that for all $z \in \Omega$, $f^{(n)}(z) = 0$. Prove that f is a complex polynomial of degree at most $n - 1$. (Suggestion: First, look at functions f with $f' = 0$ [and recall problem #(62)]. Next, consider f with $f'' = 0$. Etc.)

(74) Suppose $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ is a polynomial in x and y. That is, suppose there are constants $c_{ij} \in \mathbb{C}$ such that, when $z := x + iy$, we have

$$f(z) = c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2 + \cdots + c_{n0}x^n + \cdots + c_{0n}y^n.$$

Assuming that f is holomorphic, confirm that f must be a complex polynomial. What is the degree of f?

(75) (a) Show that $f(z) := \sqrt{z}$ (the principal value of the square root) is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$ and calculate its derivative. (Hint: First show that f is continuous in this region. Then use the standard calculus trick.)

(b) Now find a holomorphic branch of the square root function in $\mathbb{C} \setminus [0, +\infty)$.

(76) (a) Exhibit all branches of the cube root function in the domain $\mathbb{C} \setminus (-\infty, 0]$.

(b) Let $\Delta := \mathbb{C} \setminus [0, +\infty)$ and define $f : \Delta \to \mathbb{C}$ by $f(z) := \sqrt[3]{z}$ for $\Im\mathfrak{m}(z) \geq 0$ and $f(z) := \omega\sqrt[3]{z}$ for $\Im\mathfrak{m}(z) < 0$, where $\omega := (-1 + i\sqrt{3})/2$. Verify that f is indeed a branch of the cube root in Δ and calculate f'. Identify all other branches of the cube root function in Δ.

(77) (a)Define a function g which is holomorphic in the slit plane $\mathbb{C}_{\text{slit}} := \mathbb{C} \setminus (-\infty, 0]$ and satisfies both $g(1) = i$ and $g(z)^4 = z$ for all $z \in \mathbb{C}_{\text{slit}}$. Find $g(\mathbb{C}_{\text{slit}})$.

(b) Find a similar function h satisfying $h(1) = -i$.

(c) What if we replace the domain $\mathbb{C}_{\text{slit}}$ with $\mathbb{C} \setminus [0, +\infty)$? (Of course we need, e.g., $g(-1) = e^{i\pi/4}$.)

(78) Verify that $f(z) := \cos(\sqrt{z})$ defines an entire function whereas $\sin(\sqrt{z})$ does not. Here $\sqrt{z}$ denotes the principle value of the square root of $z \in \mathbb{C}$. Note that $\sqrt{z}$ is not even continuous in $\mathbb{C}$, so you really have to prove something here! (Suggestion: You can try to show that f_x and f_y exist and are continuous in $\mathbb{C}$, but a more clever approach is much easier. First explain why f is holomorphic in $\mathbb{C}_*$. Then demonstrate that f is differentiable at the origin. A helpful observation [that requires proof] might be that when $h : \mathbb{C} \to \mathbb{C}$ is continuous and even [i.e., $h(-z) = h(z)$], the function $k(z) := h(\sqrt{z})$ is well-defined and continuous.)

(79) Let $\Delta := \mathbb{C} \setminus [0, +\infty)$ and define $\vartheta : \Delta \to \mathbb{R}$ by $\vartheta(z) := \operatorname{Arg}(-z) + \pi$. Confirm that ϑ is a branch of the argument function in Δ. Find $\vartheta(\Delta)$. Find formulas for the branch of $\log(z)$ in Δ corresponding to ϑ as well as the branch of the complex c-power function $z \mapsto z^c$.

(80) Suppose that g is a (single-valued) branch of the inverse tangent function in some domain Δ. State precisely what this means. Prove that g is holomorphic in Δ with $g'(z) = (1 + z^2)^{-1}$ for all $z \in \Delta$. Demonstrate that if h is any other branch of the inverse tangent function in Δ, then there is some $k \in \mathbb{Z}$ such that for all $z \in \Delta$, $h(z) = g(z) + k\pi$.

Recall that the complex differential operators are defined by

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

(81) Let $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ be a $\mathcal{C}^1$ mapping. Define $\Omega \xrightarrow{\bar{f}} \mathbb{C}$ by $\bar{f}(z) := \overline{f(z)}$. Prove that

$$\frac{\partial \bar{f}}{\partial \bar{z}} = \overline{\frac{\partial f}{\partial z}} \quad \text{and} \quad \frac{\partial \bar{f}}{\partial z} = \overline{\frac{\partial f}{\partial \bar{z}}}.$$

(82) State and prove the Chain Rules for the partial differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$. (What are $(g \circ f)_z$ and $(g \circ f)_{\bar{z}}$?)

(83) Express the partial differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ in polar notation.
Answer:

$$\frac{\partial}{\partial z} = \frac{e^{-i\theta}}{2}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial \theta}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}} = \frac{e^{i\theta}}{2}\left(\frac{\partial}{\partial r} + \frac{i}{r}\frac{\partial}{\partial \theta}\right).$$

What are the Cauchy-Riemann equations in polar coordinates?

(84) Assume that f is a $\mathcal{C}^1$ mapping. Use the partial differential operators $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$ to demonstrate that f is holomorphic if and only if g is holomorphic where $g(z) := \overline{f(\bar{z})}$. (See problem #(67).)

(85) Assuming that $\mathbb{C} \xrightarrow{u} \mathbb{R}$ is a $\mathcal{C}^2$ function, confirm that $\Delta u = 4\,\dfrac{\partial^2 u}{\partial z \partial \bar{z}}$. Thus the Laplace differential operator is

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\,\frac{\partial^2}{\partial z \partial \bar{z}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\,.$$

(86) Assume that f is a $\mathcal{C}^1$ mapping. Prove that the Jacobian of f is given by

$$Jf = |f_z|^2 - |f_{\bar{z}}|^2\,.$$

(87) Let $f := u + iv$ where $u(x+iy) = x^2 + y^2 + \dfrac{x^2 - y^2}{x^2 + y^2}$ and $v(x+iy) = \dfrac{2xy}{x^2+y^2}$. Determine precisely where f is (complex) differentiable. (Suggestion: Write f in terms of z and $\bar{z}$ and compute $f_{\bar{z}}$.)

(88) Let $\mathbb{C} \xrightarrow{F} \mathbb{C}$ be a polynomial in x and y (or equivalently, in z and $\bar{z}$). Suppose that $F_{zz}(z) = 0$ for all $z \in \mathbb{C}$. Prove that there exist complex polynomials (i.e., holomorphic polys) P and Q such that for all $z \in \mathbb{C}$, $F(z, \bar{z}) = zP(\bar{z}) + Q(\bar{z})$.

(89) Let $\mathbb{C} \xrightarrow{F} \mathbb{C}$ be a polynomial in x and y (or equivalently, in z and $\bar{z}$). Suppose that for all $z \in \mathbb{C}$, $(F^2)_{\bar{z}}(z) = 0$. What can you deduce about F? (Hint: F must be holomorphic!)

(90) Recall that the directional derivative of f in the direction $e^{i\theta}$ at $z = a$ is

$$D_{e^{i\theta}} f(a) := \lim_{r \searrow 0} \frac{f(a + re^{i\theta}) - f(a)}{r}\,,$$

provided this limit exists. Here $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$ and $a \in \Omega$. Assume that f is real-differentiable at a. Prove that

$$D_{e^{i\theta}} f(a) = e^{i\theta} f_z(a) + e^{-i\theta} f_{\bar{z}}(a).$$

Conclude that

$$\max_{-\pi < \theta \le \pi} |D_{e^{i\theta}} f(a)| = |f_z(a)| + |f_{\bar{z}}(a)|$$

and that

$$\min_{-\pi < \theta \le \pi} |D_{e^{i\theta}} f(a)| = ||f_z(a)| - |f_{\bar{z}}(a)||.$$

Can you determine where these extreme values occur? (See problem #(32).)

(91) Suppose that $\Omega \xrightarrow{f} \mathbb{C}$ is real-differentiable at the point $z = a$ in Ω. Let $Df(a)$ denote the complex real-derivative of f at $z = a$,

$$Df(a)\zeta = f_z(a)\,\zeta + f_{\bar{z}}(a)\,\bar{\zeta}\,.$$

Prove that the following are equivalent:

(a) f is complex differentiable at $z = a$.

(b) $Df(a) : \mathbb{C} \to \mathbb{C}$ is complex linear.

Prove that either of (a) or (b) implies

(c) f satisfies the Cauchy-Riemann equations at $z = a$.

When does (c) imply (a) or (b)?

(92) Let $c, d \in \mathbb{C}$ with $|c| \neq |d|$. Define $f(z) := cz + d\bar{z}$. Verify that f is a $\mathcal{C}^1$-diffeomorphism of the plane $\mathbb{C}$ onto itself. Determine f^{-1}. What conditions on c, d ensure that f is orientation preserving (i.e., that $Jf > 0$)?

(93) Let $\lambda \in \mathbb{R} \setminus \{0\}$ and define $f(z) := |z|^{\lambda-1}z$. Verify that f is a $\mathcal{C}^1$-diffeomorphism of the punctured plane $\mathbb{C}_*$ onto itself. Determine f^{-1}. When will f be orientation preserving (i.e., $Jf > 0$)? (Here $|z|^{\lambda-1}$ stands for the principle value of the $(\lambda-1)^{\text{st}}$ power of $|z|$; that is, $|z|^{\lambda-1} := \exp[(\lambda-1)\operatorname{Log}|z|]$.)

(94) Let $\Omega \xrightarrow{f} \Omega'$ be a $\mathcal{C}^1$-diffeomorphism of domains $\Omega, \Omega' \subset \mathbb{C}$. Prove that f is isogonal (or conformal) at a point $a \in \Omega$ precisely when $Df(a) : \mathbb{C} \to \mathbb{C}$ is an isogonal (or conformal, respectively) real linear transformation.

(95) Let $\Omega \xrightarrow{f} \Omega'$ be a $\mathcal{C}^1$-diffeomorphism. Prove that f is conformal at a point $a \in \Omega$ if and only if f is isogonal at a with $Jf(a) > 0$. (Note: This is the (b)$\Leftrightarrow$(c) part of the 'Main Theorem', so you cannot appeal to this theorem.)

(96) Decide whether or not there exists a holomorphic $\mathcal{C}^1$-diffeomorphism f defined in a domain which contains the line $L = \{x + ix : x \in \mathbb{R}\}$ and satisfies $f(z) = \sqrt{z}$ for all $z \in L$. (Hint: What is the image of L under the map $z \mapsto \sqrt{z}$?)

(97) Suppose that $\Omega \xrightarrow{f} \Omega'$ is a $\mathcal{C}^1$-diffeomorphism of domains $\Omega, \Omega' \subset \mathbb{C}$. Let $a \in \Omega$. Prove that the following are equivalent:

(a) f is anti-conformal at a.

(b) $\bar{f}$ is conformal at a.

(c) $\bar{f}$ is (complex) differentiable at $z = a$.

(d) $f_z(a) = 0$.

(98) Suppose that $\Omega \xrightarrow{f} \Omega'$ is a $\mathcal{C}^1$-diffeomorphism of domains $\Omega, \Omega' \subset \mathbb{C}$. Let $a \in \Omega$. Prove that the following are equivalent:

(a) f is isogonal at a.

(b) Either f is conformal at a or f is anti-conformal at a.

(c) There exists $\displaystyle\lim_{z\to a} \frac{|f(z) - f(a)|}{|z - a|}$.

(Hints: That (a)$\Leftrightarrow$(b)$\Rightarrow$(c) is straightforward. To establish (c)$\Rightarrow$(b), use problem #(90).)

(99) Prove that points z and w in $\mathbb{C}$ correspond to diametrically opposite points on the Riemann sphere if and only if $z\bar{w} = -1$.

(100) Suppose that a cube has its vertices on the sphere $\mathbb{S}$ and its edges parallel to the coordinate axes. Determine the stereographic projections of the vertices.

(101) Do the same problem for a regular tetrahedron in regular position.

(102) Let $\mathbb{C} \xrightarrow{L} \mathbb{C}$ be a complex linear map. Extend the definition of L to all of $\hat{\mathbb{C}}$ by setting $L(\infty) := \infty$. Verify that this makes L a continuous bijection, in fact a homeomorphism, from $\hat{\mathbb{C}}$ onto itself, except for certain 'special cases'. What are these 'special cases'? What is the 'correct' extension for L in these 'special cases'?

(103) Let $J(z) := 1/z$ be the complex inversion map. Extend the definition of J to all of $\hat{\mathbb{C}}$ by setting $J(0) := \infty$ and $J(\infty) := 0$. Prove that this makes J a continuous bijection from $\hat{\mathbb{C}}$ onto itself. In fact, show that $J : (\hat{\mathbb{C}}, \chi) \to (\hat{\mathbb{C}}, \chi)$ is an isometry.

(104) Let $S(z) := z + 1/z$ be the complex 'squash' map. Extend the definition of S to all of $\hat{\mathbb{C}}$ by setting $S(0) := \infty$ and $S(\infty) := \infty$. Prove that this makes S a continuous surjection from $\hat{\mathbb{C}}$ onto itself. What are $S(\mathbb{D})$ and $S(\hat{\mathbb{C}} \setminus \bar{\mathbb{D}})$?

(105) Consider the self-maps J and K of $\mathbb{C}_*$ and $\mathbb{C}$ given by $z \mapsto 1/z$ and $z \mapsto \bar{z}$. How should these maps be defined to obtain continuous maps of $\hat{\mathbb{C}}$? What are the corresponding maps of $\mathbb{S}$? That is, exhibit formulas for the maps $\Pi^{-1} \circ J \circ \Pi$ and $\Pi^{-1} \circ K \circ \Pi$.

(106) Determine self-maps of $\hat{\mathbb{C}}$ which correspond to the following self-maps of $\mathbb{S}$:

$$(x_1, x_2, x_3) \mapsto (-x_1, x_2, x_3), \quad (x_1, x_2, x_3) \mapsto (-x_1, -x_2, x_3), \quad (x_1, x_2, x_3) \mapsto -(x_1, x_2, x_3).$$

(107) Let $c \in \hat{\mathbb{C}}$ and $t > 0$. Describe when $D_\chi(c; t) \subset \mathbb{C}$. In this case, find $a \in \mathbb{C}$ and $r > 0$ so that $D_\chi(c; t) = D(a; r)$.

(108) Let $a \in \mathbb{C}$ and $r > 0$. Find $c \in \hat{\mathbb{C}}$ and $t > 0$ so that $D_\chi(c; t) = D(a; r)$.

(109) Let $L \subset \mathbb{C}$ be the line with equation $ax + by = c$ (so $a, b, c \in \mathbb{R}$ and one of a, b is non-zero). Describe the stereographic pre-image of $\hat{L} = L \cup \{\infty\}$. Find $\zeta \in \mathbb{C}$ and $r > 0$ so that $\hat{L} = \{z \in \hat{\mathbb{C}} : \chi(z, \zeta) = r\}$.

(110) Let $\Phi = \Pi^{-1} : \hat{\mathbb{C}} \to \mathbb{S}$ be the inverse of the stereographic projection; thus for $z = x+iy$

$$(x_1, x_2, x_3) = \Phi(z) = \left(\frac{2x}{|z|^2 + 1}, \frac{2y}{|z|^2 + 1}, \frac{|z|^2 - 1}{|z|^2 + 1} \right).$$

Consider the map $F(z) := \Phi(e^z)$, so $F : \hat{\mathbb{C}} \to \mathbb{S} \subset \mathbb{R}^3$. Let Σ be any horizontal strip in $\mathbb{C}$ of width 2π, say $\Sigma = \{z \in \mathbb{C} : |\Im(z) - y_0| < 2\pi\}$ for some fixed $y_0 \in \mathbb{R}$. Confirm that $F|_\Sigma$ is an isogonal diffeomorphism. (The inverse of this map...)

(111) Prove that every Möbius transformation is a self-homeomorphism of the Riemann sphere $\hat{\mathbb{C}}$. (Recall that when $T(z) = \dfrac{az + b}{cz + d}$ and $c \neq 0$, we have defined $T(-d/c) := \infty$ and $T(\infty) := a/c$).

(112) Suppose T is a Möbius transformation that maps the extended real line $\hat{\mathbb{R}}$ into itself. Confirm that there exist $a, b, c, d \in \mathbb{R}$ such that for all $z \in \mathbb{C}$, $T(z) = \dfrac{az + b}{cz + d}$.

(113) Confirm that a map of the form $w = \dfrac{a\bar{z} + b}{c\bar{z} + d}$ (with $a, b, c, d \in \mathbb{C}$ and $ad \neq bc$) is an orientation reversing isogonal diffeomorphism of $\mathbb{C} \setminus \{-\overline{(d/c)}\}$ onto $\mathbb{C} \setminus \{a/c\}$.

(114) Calculate the following cross ratios: (a) $[1, i, -1, -i]$ (b) $[0, a, \infty, b]$ for $a, b \in \mathbb{C}_*$ with $a \neq b$ (c) $[0, c, 1/c, 1]$ for $c \in \mathbb{C}_* \setminus \{\pm 1\}$ (d) $[\infty, z, 1/z, \bar{z}]$ for $z \in \mathbb{C} \setminus (\mathbb{R} \cup \mathbb{T})$

(115) Let $a, b, c, d \in \hat{\mathbb{C}}$ be distinct. Put $z := [a, b, c, d]$. Express, in terms of z, the twenty-four possible values of the cross ratios of a, b, c, d when all possible permutations are considered. (In fact, there are only six different values which arise. Hint: Explain why it suffices to consider the points $b = 0$, $c = \infty$, $d = 1$, and then use this fact to answer the problem.)

(116) Prove that four distinct points $a, b, c, d \in \hat{\mathbb{C}}$ lie on a circle in $\hat{\mathbb{C}}$ if and only if $[a, b, c, d]$ is real. Determine a condition that characterizes precisely when the points a, c will separate the points b, d on the circle. (Hint: The desired condition is that $|[a, b, c, d]| + |[a, d, c, b]| = 1$.)

(117) Let A and B be *oriented* circles that intersect. Express $\Theta(A, B)$ in terms of a cross ratio of points on A and B. (Answer: $\Theta(A, B) = \operatorname{Arg}[b, c, d, a]$ where: c, d are the points of $A \cap B$ and $a \in A$, $b \in B$ are chosen so that the orientations of A, B agree with the orderings c, a, d and c, b, d.)

(118) Find the images of each of

$$\hat{\mathbb{R}}, \quad i\hat{\mathbb{R}}, \quad K_1 = \{z : |z - 1| = 1\}, \quad K_2 = \{z : |z - i| = 1\}$$

under the mapping $w = T(z) := \dfrac{z}{z - (1 + i)}$ of $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

(119) Let K be a circle in $\hat{\mathbb{C}}$. Fix distinct points $a, b \in \hat{\mathbb{C}} \setminus K$. Prove that a and b are symmetric with respect to K if and only if every circle in $\hat{\mathbb{C}}$ which passes though both a, b is orthogonal to K. (Suggestion: First consider the case when K is an extended line.)

(120) Let a, b, c be distinct points in $\hat{\mathbb{C}}$. Demonstrate that there exists a unique circle K in $\hat{\mathbb{C}}$ which passes though c and has the property that a and b are symmetric with respect to K.

(121) Find the general form of any Möbius transformation which maps the unit disk $\mathbb{D}$ onto the upper half-plane. Suppose we add the requirement that the origin should be mapped to i: What changes?

(122) Find a Möbius transformation that maps the unit disk $\mathbb{D}$ onto itself and sends the circle $C(1/4; 1/4)$ onto $C(0; r)$ for some $r > 0$.

(123) Let K_1 and K_2 be two arbitrary disjoint circles in $\hat{\mathbb{C}}$. Prove that there exists a unique pair of points a, a^* with the property that these points are symmetric with respect to both circles K_1 and K_2. (Hint: By using a preliminary Möbius transformation, reduce to the case where $K_1 = \mathbb{T}$ and K_2 is some vertical line $\mathfrak{Re}(z) = x_0 > 1$.) What can you say about the images of K_1 and K_2 under the map $T(z) = (z - a)/(z - a^*)$? If A is the region bounded by K_1 and K_2, what is $T(A)$?

(124) Let $\omega := e^{\pi i/3}$ and put $g(z) := \dfrac{z - \omega}{z - \overline{w}}$. Find the image $g(\Omega)$ of

$$\Omega := \{z : |z| < 1, |z - 1| < 1, \mathfrak{Im}(z) > 0\}.$$

(125) Let K and K' be circles in $\hat{\mathbb{C}}$. Suppose $a \in K$, $a' \in K'$, $b \in \hat{\mathbb{C}} \setminus K$, $b' \in \hat{\mathbb{C}} \setminus K'$. Demonstrate that there exists a unique Möbius transformation that maps K, a, b to K', a', b' respectively.

(126) Let $t \in (0, 1)$ and fix two points $a, b \in \mathbb{C}$. Prove that $K := \{z \in \mathbb{C} : |z - a| = t|z - b|\}$ is a circle. Find $c \in \mathbb{C}$ and $r > 0$ so that $K = C(c; r) = \{z \in \mathbb{C} : |z - c| = r\}$. Which of the points a, b is inside K?

(127) Find a conformal map from the quarter plane $\{x + iy : x > 0, y > 0\}$ onto the unit disk $\mathbb{D}$.

(128) Find a conformal mapping from the domain $\Delta := D(1;1) \cap D(i;1)$ onto the unit disk which maps the segment $(0, 1+i)$ onto the real diameter $(-1, 1)$. What is the preimage of the origin?

(129) Here we scrutinize the complex cosine function $\cos(z) := \dfrac{e^{iz} + e^{-iz}}{2}$.

(a) Confirm that the cosine function is periodic with period 2π; i.e., show that

$$\forall z \in \mathbb{C} \ : \cos(z + 2\pi) = \cos(z)\,.$$

(b) Prove that the cosine function $\mathbb{C} \overset{\cos}{\to} \mathbb{C}$ is surjective.

(c) Find the image of the semi-infinite strip $T := \{x + iy : |x| < \pi, y < 0\}$ under the map $z \mapsto w = \cos(z)$. (Hints: First note that the cosine is injective in T. Why is this important? Now start by examining $\zeta = e^{\tau}$ where $\tau = iz$. How can you write $w = \cos(z)$ in terms of ζ?)

(d) Use your work in part (c), together with the fact that the cosine function is even, to calculate the image of the infinite strip $S = \{x + iy : 0 < x < \pi, y \in \mathbb{R}\}$ under the map $z \mapsto w = \cos(z)$.

(e) In Freshman Calculus we define the inverse cosine function for $t \in [-1, 1]$ by

$$\theta := \cos^{-1}(t) = \arccos(t) \iff \theta \in [0, \pi] \text{ and } \cos(\theta) = t\,.$$

In particular, $\arccos(0) = \pi/2$. Let's do a similar thing for the complex-valued cosine function. Prove that there is a domain Δ which contains $(-1, 1)$ and a function g which is a holomorphic branch of the inverse cosine function in Δ and satisfies $g(0) = \pi/2$. What is the largest possible such domain Δ, and what is $g(\Delta)$? Provide an explicit formula for g—you probably came close to doing this in part (b)! (Your answer will involve logarithms and a square root, so be explicit and be careful.) Calculate g'. Finally, what are the other possible branches of the inverse cosine function? For example, what if we want such a function h with $h(0) = 3\pi/2$?

(130) Please be sure to look at all the suggested problems from Ahlfors; these are listed on the web page.

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