MidTerm Exam Solutions

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#① $z^i = 1 \iff i \operatorname{log} z = \log 1 = 2k\pi i$

$\therefore$ the set of all $z \in \mathbb{C}$ st $z^i = 1$ is $\{e^{2k\pi} : k \in \mathbb{Z}\}$.

#② $\mathbb{C} \supset \Omega \xrightarrow{f} \mathbb{C}$, $a \in \Omega$ <u>Claim</u> f diff'l at $a \Rightarrow f$ cts at a.

<u>Proof</u> Let $\varepsilon > 0$ be given. Pick $0 < \delta < \varepsilon/(1+|f'(a)|)$ st $D(a;\delta) \subseteq \Omega$ and st

$$0 < |z-a| < \delta \Rightarrow \left| \frac{f(z)-f(a)}{z-a} - f'(a) \right| < 1$$

(such a δ exists because Ω is open & f is diff'l at a).

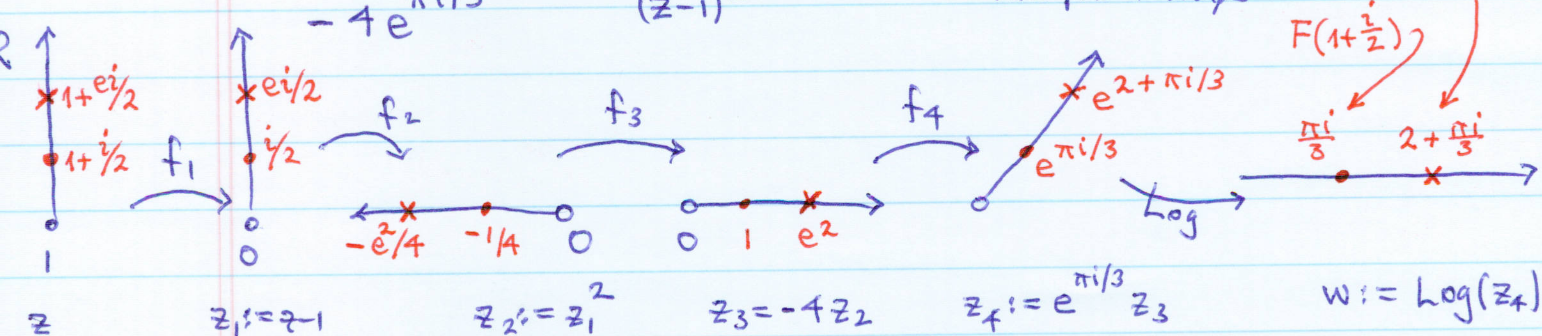
Let $z \in D_*(a;\delta)$. Then (multiplying thru by $z-a$) see that

$$|z-a| > \left| \left(f(z)-f(a)\right) - f'(a)\left(z-a\right) \right| \geq |f(z)-f(a)| - |f'(a)|\,|z-a|$$

So

$$|f(z)-f(a)| < |z-a| + |f'(a)|\,|z-a| = \left(1+|f'(a)|\right)|z-a| < \left(1+|f'(a)|\right)\delta < \varepsilon. \quad \square$$

#③ $$F(z) := \operatorname{Log}\Big[\underbrace{-2(1+\sqrt{3}\,i)}_{-4e^{\pi i/3}}\,\underbrace{(z^2-2z+1)}_{(z-1)^2}\Big] = \operatorname{Log} \circ \underbrace{f_4 \circ f_3 \circ f_2}_{\text{the pic'd maps}} \circ f_1\,(z)$$



See that $F\left(\{1+iy \mid y>0\}\right) = \{x + \frac{\pi i}{3} \mid x \in \mathbb{R}\}$.

#④ <u>Claim</u> $g(z) := \begin{cases} 0 & z=0 \\ z^5/|z|^4 & z \neq 0 \end{cases}$ satisfies CRE at origin, but is <u>not</u> diff'l at origin

<u>Proof</u>

$$\frac{g(x)-g(0)}{x} = \frac{x^5}{x|x|^4} = \frac{x^4}{|x|^4} = 1 \Rightarrow \exists\, g_x(0) = 1$$

$$\frac{g(iy)-g(0)}{y} = \frac{(iy)^5}{y|y|^4} = i^5\frac{y^4}{|y|^4} = i \Rightarrow \exists\, g_y(0) = i$$

So $g_x(0) = -i\,g_y(0)$ & CRE hold.

But for $z = te^{\pi i/4}$ (e.g.) w/ $t \downarrow 0$ see that

$$\frac{g(z)-g(0)}{z} = \frac{z^5}{z|z|^4} = \frac{(te^{\pi i/4})^4}{t^4} = e^{\pi i} = -1 \neq 1 \text{ so } \nexists \lim_{z\to 0} \frac{g(z)-g(0)}{z}.$$