Solutions to MidTerm

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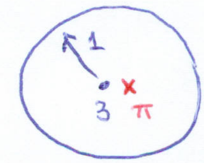
#① $f(z) := \dfrac{z^2 e^{-z}}{z^3 - 8}$ is holom in $D(0;2)$, so by CTD $I = \int_{\Pi} f(z)\,dz = 0$.



$g(z) := \cos(z)$ has $g'(z) = -\sin(z)$, $g''(z) = -\cos(z)$ so

$$J = \int_{C(3;1)} \frac{g(z)}{(z-\pi)^3}\,dz = \frac{2\pi i}{2}\, g''(\pi) = \pi i.$$

#② Recalling that $|e^w| = e^{Re(w)}$ see that for $z \in \mathbb{C}$ & $t \in [0,1]$, $Re(t^2 z) = t^2 Re(z)$.
Thus when $Re(z) \geq 0$, $Re(t^2 z) \leq Re(z)$ and when $Re(z) < 0$, $Re(t^2 z) < 0$.
We conclude that $|e^{t^2 z}| \leq \max\{1, |e^z|\}$ and therefore

$$\left| \int_0^1 t^2 e^{t^2 z}\,dt \right| \leq \max\{1, |e^z|\} \int_0^1 t^2\,dt = \frac{1}{3}\max\{1, |e^z|\}.$$

Now with $F(z) := \int_0^1 e^{t^2 z}\,dz$, we claim that $F'(z) := \int_0^1 t^2 e^{t^2 z}\,dz$. We calculate

$$\frac{F(z+h) - F(z)}{h} - \int_0^1 t^2 e^{t^2 z}\,dt = \int_0^1 \left[\frac{e^{t^2(z+h)} - e^{t^2 z}}{h} - t^2 e^{t^2 z} \right] dt =$$

$$= \int_0^1 t^2 e^{t^2 z} \left[\frac{e^{t^2 h} - 1}{t^2 h} - 1 \right] dt.$$

Let $\varepsilon > 0$ be given. Since $\lim_{w \to 0} \frac{e^w - 1}{w} = 1$, $\exists$ $\delta > 0$ st $\forall$ $0 < |w| < \delta$, $\left| \frac{e^w - 1}{w} - 1 \right| < \frac{\varepsilon}{M}$ where $M := \max\{1, |e^z|\}$ (Here z is fixed)

Thus for $0 < |h| < \delta$,

$$\left| \frac{F(z+h) - F(z)}{h} - \int_0^1 t^2 e^{t^2 z}\,dt \right| \leq \frac{\varepsilon}{M} \left| \int_0^1 t^2 e^{t^2 z}\,dz \right| \leq \frac{\varepsilon}{M} \cdot \frac{1}{3} M < \varepsilon. \quad \square$$

#③ To see that $\exists$ $c \in \mathbb{C}$ (in fact $c \in \overline{\mathbb{D}}$) st $f = cP$, it suffices to prove that $F := f/P$ has a holom extension to all of $\mathbb{C}$. For then we have F entire w/ $|F| \leq 1$, so Liouville's Thm gives $F \equiv c$.

Clearly F is holom in $\mathbb{C} \setminus Z$ where $Z := P^{-1}(0)$ is a finite set.

We claim that Riemann's Extension Theorem can be applied to remove each of the singularities in Z. To do this requires us to demonstrate that

$$\forall\, a \in Z, \quad \lim_{z \to a} (z-a)\, F(z) = 0.$$

Evidently, $\forall\, z \in \mathbb{C} \setminus Z$, $|F(z)| \leq 1$, so $|(z-a)F(z)| \leq |z-a| \to 0$ as $z \to a$.

(A4) $\overline{\mathbb{D}} \xrightarrow{f} \mathbb{C}$ cts and holom in $\mathbb{D}$. $\Rightarrow$

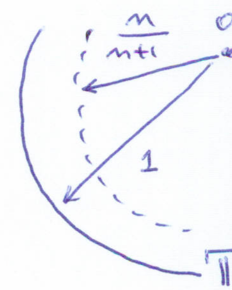
$$(\star) \quad \forall z \in \mathbb{D}, \quad f(z) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z}\,d\zeta .$$

Proof

1st/ For each $n \in \mathbb{N}$, define $f_n(z) := f\left(\frac{n}{n+1} z\right)$. Then $f_n \in \mathcal{H}(D_n)$ where $D_n := \frac{n+1}{n}\mathbb{D} := D(0; R_n)$ w/ $R_n := \frac{n+1}{n} > 1$. Thus by CIFC (applied to f_n and $\mathbb{T} = \partial \mathbb{D}$ w/ $\overline{\mathbb{D}} \subseteq D_n$), $(\star)$ holds for f_n for each n.

2nd/ Claim: $(f_n)_1^\infty$ cvgs uniformly to f on $\mathbb{T}$. (In fact even on $\overline{\mathbb{D}}$) This is bcuz f is uniformly cts on $\overline{\mathbb{D}}$. To prove the claim:

Let $\varepsilon > 0$ be given. Since f is uniformly cts on $\overline{\mathbb{D}}$, we can – and do – pick a $\delta > 0$ (with $\delta < 1$ too) so that



$$\forall z, z' \in \overline{\mathbb{D}} : |z - z'| < \delta \Rightarrow |f(z) - f(z')| < \varepsilon$$

Now select $N \in \mathbb{N}$ w/ $N > 1/\delta$. Then $\forall n \geq N$, $1 - \frac{n}{n+1} = \frac{1}{n+1} < \delta$.

Thus $\forall n \geq N$, $\forall \zeta \in \mathbb{T}$:

$$\left|\frac{n}{n+1}\zeta - \zeta\right| = \left|\frac{n}{n+1} - 1\right| |\zeta| = \frac{1}{n+1} < \delta, \text{ so}$$

$$|f_n(\zeta) - f(\zeta)| = \left|f\left(\frac{n}{n+1}\zeta\right) - f(\zeta)\right| < \varepsilon .$$

3rd/ Claim: (For each $z \in \mathbb{D}$) $\lim_{n\to\infty} \int_{\mathbb{T}} \frac{f_n(\zeta)}{\zeta - z}\,d\zeta = \int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z}\,d\zeta$. (This holds because $(f_n)_1^\infty$ cvgs unifly to f on $\mathbb{T}$, but we prove it below.)

We are done once we have this, because then we get

$$\forall z \in \mathbb{D}, \quad f(z) = \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_n(\zeta)}{\zeta - z}\,d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z}\,d\zeta .$$

bcuz $(\star)$ holds for f_n $\forall n$

Thus it remains to establish the above claim.

Fix a pt $z \in \mathbb{D}$. Let $\varepsilon > 0$ be given. Since $f_n \to f$ unifly on $\mathbb{T}$, we can pick $N \in \mathbb{N}$ st $\forall n \geq N$, $\forall \zeta \in \mathbb{T}$: $|f_n(\zeta) - f(\zeta)| < \varepsilon \cdot (1 - |z|)$.* Then $\forall n \geq N$ we have

$$\left|\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f_n(\zeta)}{\zeta - z}\,d\zeta - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{f(\zeta)}{\zeta - z}\,d\zeta\right| \leq \frac{1}{2\pi}\int_{\mathbb{T}} \frac{|f_n(\zeta) - f(\zeta)|}{|\zeta - z|}\,|d\zeta| \leq$$

$$\leq \frac{1}{2\pi} \cdot \frac{\varepsilon(1 - |z|)}{1 - |z|} \cdot 2\pi = \varepsilon . \quad \boxtimes$$

* z is fixed.