Random Dynamics of distance expanding maps
Cincinnati and Indiana (2009)

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joint work with

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Deterministic dynamics: studies the iterates $f^n$ of some map $f$, for example of a rational map $f : \hat{C} \to \hat{C}$ or a smooth map of some Riemannian manifold.

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Z_0 \xrightarrow{f} Z_1 \xrightarrow{f} \ldots \xrightarrow{f} Z_n \xrightarrow{f} \ldots
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$\mathcal{F}_f = \{ z \in \hat{\mathbb{C}} ; \ (f^n)_n \text{ normal on some neighborhood } U \text{ of } z. \}$

$\mathcal{J}_f = \hat{\mathbb{C}} \setminus \mathcal{F}_f.$

These sets are invariant: $f(\mathcal{J}_f) = f^{-1}(\mathcal{J}_f) = \mathcal{J}_f.$
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$$f : \hat{C} \to \hat{C} \quad \text{for example of a rational map}$$

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Define similarly $\mathcal{F}_0$ and $\mathcal{J}_0$. Then

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\mathcal{J}_0 &\xrightarrow{f_0} \mathcal{J}_1 & \xrightarrow{f_1} \ldots & \xrightarrow{f_{n-1}} \mathcal{J}_n & \xrightarrow{f_n} \ldots
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\]

where $\mathcal{J}_n = f_n \circ f_{n-1} \circ \ldots \circ f_0(\mathcal{J}_0)$ is the "Julia set" of the family $(f_N \circ f_{N-1} \circ \ldots \circ f_n)_{N \geq n}$.
Some comments on quadratic polynomials \( p_c(z) = z^2 + c \)

For the Julia set \( \mathcal{J}_c \) of \( p_c(z) = z^2 + c \) where \( |c| < 1/4 \) we have the following

1. Bowen’s Formula: \( \delta_c = Hdim(\mathcal{J}_c) \) is the only zero of the "pressure function".
2. \( \delta_c > 1 \) excepted when \( c = 0 \).
3. \( 0 < HM^{\delta_c}(\mathcal{J}_c) < \infty \).
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- There is a Bowen's Formula (Kifer, Crauel and Flandoni, Bodenschütz and Ochs, Rugh, our paper,...).
- Brück and Bürger asked if (2) holds in the random setting "a.e."
- Bodenschütz and Ochs conjectured that (3) holds in the random setting.
Randomness modeled by a \((X, \mathcal{F}, m, \theta)\) measure preserving dynamical system where

- \((X, \mathcal{F}, m)\) a probability space,
- \(\theta : X \to X\) is a invertible and ergodic "base" map s.t. \(m\) is invariant.
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Example: $X = \mathbb{D}(0, 1/4)^\mathbb{Z}$, $m = m_0^\mathbb{Z}$ with $m_0$ any (e.g. Lebesgue) probability measure on $\mathbb{D}(0, 1/4)$ and $\theta$ the shift map.

This model corresponds exactly to iid choices of parameters $c \in \mathbb{D}(0, 1/4)$. 
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Random System: Let \((\mathcal{J}_x, \varrho_x), x \in X\), be compact metric spaces normalized in size by \(\text{diam}_{\varrho_x}(\mathcal{J}_x) \leq 1\). Let

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\mathcal{J} = \bigcup_{x \in X} \{x\} \times \mathcal{J}_x
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(1)

and let

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T_x : \mathcal{J}_x \rightarrow \mathcal{J}_{\theta(x)} , \ x \in X ,
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For every \(n \geq 0\) we denote \(T^\ast_x := T_{\theta^{n-1}(x)} \circ \ldots \circ T_x : \mathcal{J}_x \rightarrow \mathcal{J}_{\theta^{n}(x)}\).
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For every \(n \geq 0\) we denote \(T^n_x := T_{\theta^{n-1}(x)} \circ \ldots \circ T_x : J_x \rightarrow J_{\theta^n(x)}.\)

There is an associated skew-product: \(T : J \rightarrow J\) defined by
\[
T(x, z) = (\theta(x), T_x(z)).
\]  
(3)
With this notation one has \(T^n(x, y) = (\theta^n(x), T^n_x(y)).\)
Distance expanding random maps.

- Deterministic distance expanding maps have been introduced for the first time in Ruelle’s monograph "Thermodynamical Formalism", 1978.
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One of the main features of this class is that their definition does not require any differentiability or smoothness condition.

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One of the main features of this class is that their definition does not require any differentiability or smoothness condition.

It is a very general class comprising symbol systems and expanding maps of smooth manifolds but goes far beyond.

We only suppose these maps to be measurable expanding in the sense their expanding constant is measurable and

$$\gamma_x > 1 \quad \text{a.e.}$$

or

$$\int_X \log \gamma_x \, dm(x) > 0.$$
Expanding Random Maps

A map $T : \mathcal{J} \rightarrow \mathcal{J}$ is called expanding random map if

- the mappings $T_x : \mathcal{J}_x \rightarrow \mathcal{J}_{\theta(x)}$ are continuous, open and surjective.

There exist a function $\eta : X \rightarrow \mathbb{R}^+, x \mapsto \eta_x$, and a real number $\xi > 0$ such that following conditions hold.

**Uniform Openness.** $T_x(B_x(z, \eta_x)) \supset B_{\theta(x)}(T_x(z), \xi)$ for every $(x, z) \in \mathcal{J}$.

**Measurably Expanding.** There exists a measurable function $\gamma : X \rightarrow (1, +\infty)$, $x \mapsto \gamma_x$ such that $\varrho_{\theta(x)}(T_x(z_1), T_x(z_2)) \geq \gamma_x \varrho_x(z_1, z_2)$ whenever $\varrho(z_1, z_2) < \eta_x$, $z_1, z_2 \in \mathcal{J}_x$ holds $m$-a.e.

**Measurability of the Degree.** The map $x \mapsto \text{deg}(T_x) := \sup_{y \in \mathcal{J}_{\theta(x)}} \# T^{-1}_x\{y\}$ is measurable.

**Topological Exactness.** There exists a measurable function $x \mapsto n_\xi(x)$ such that $T^{n_\xi(x)}_x(B_x(z, \xi)) = \mathcal{J}^{n_\xi(x)}_{\theta(x)}(z)$ for every $z \in \mathcal{J}_x$ and a.e. $x \in X$. 

Thermodynamical Formalism

In order to determine the fractal structure of the RDS, we first develop the TF. This theory relies on the behavior of the

**Transfer operator** \( \mathcal{L}_x = \mathcal{L}_{\varphi,x} : C(\mathcal{I}_x) \rightarrow C(\mathcal{I}_{\theta(x)}) \) defined by

\[
\mathcal{L}_x g_x(w) = \sum_{T_x(z) = w} g_x(z) e^{\varphi_x(z)}, \quad w \in \mathcal{I}_{\theta(x)}.
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Transfer operator $L_x = L_{\varphi, x} : C(J_x) \rightarrow C(J_{\theta(x)})$ defined by

$$L_x g_x(w) = \sum_{T_x(z) = w} g_x(z)e^{\varphi_x(z)}, \ w \in J_{\theta(x)}.$$

Potential $\varphi \in \mathcal{H}^\alpha(J)$: mainly this means that $\int_X \|\varphi_x\|_\infty dm(x) < \infty$

and $\varphi_x : J_x \rightarrow \mathbb{R}$ $\alpha$–Hölder function.

Later on (conformal maps) we will use "geometric potentials", i.e. $-t \log |f'_x|$. 

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Global operator \( \mathcal{L} : \mathcal{C}(\mathcal{J}) \to \mathcal{C}(\mathcal{J}) = \{ g : \mathcal{J} \to \mathbb{R} ; \ g_x = g|_{\mathcal{J}_x} \in \mathcal{C}(\mathcal{J}_x) \ \text{a.e.} \} \)

defined by \( (\mathcal{L}g)_x(w) = \mathcal{L}_{\theta^{-1}(x)}g_{\theta^{-1}(x)}(w) \).
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Transfer operator $L_x = L_{\varphi,x} : C(\mathcal{J}_x) \to C(\mathcal{J}_{\theta(x)})$ defined by

$$L_x g_x(w) = \sum_{T_x(z) = w} g_x(z) e^{\varphi_x(z)}, \quad w \in \mathcal{J}_{\theta(x)}.$$

Potential $\varphi \in H^\alpha(\mathcal{J})$: mainly this means that $\int_{\mathcal{X}} \|\varphi_x\|_{\infty} dm(x) < \infty$ and $\varphi_x : \mathcal{J}_x \to \mathbb{R}$ $\alpha$–Hölder function.

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Global operator $L : C(\mathcal{J}) \to C(\mathcal{J}) = \{g : \mathcal{J} \to \mathbb{R}; \; g_x = g|_{\mathcal{J}_x} \in C(\mathcal{J}_x) \text{ a.e.}\}$ defined by $(Lg)_x(w) = L_{\theta^{-1}(x)}g_{\theta^{-1}(x)}(w)$.

Denote $L^n_x := L_{\theta^{n-1}(x)} \circ \ldots \circ L_x : C(\mathcal{J}_x) \to C(\mathcal{J}_{\theta^n(x)})$. Note that

$$L^n_x g_x(w) = \sum_{z \in T_x^{-n}(w)} g_x(z) e^{S_n \varphi_x(z)}, \quad w \in \mathcal{J}_{\theta^n(x)},$$

where $S_n \varphi_x(z) = \sum_{j=0}^{n-1} \varphi_x \circ T_x^j(z)$.
Ruelle Perron Frobenius Theorem

We first establish a RPF-theorem without any measurable structure on $\mathcal{J}$. 
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**Theorem**

Let $\varphi \in \mathcal{H}^\alpha (\mathcal{J})$ and let $\mathcal{L} = \mathcal{L}_\varphi$ be the associated transfer operator. Then the following holds.

1. There exists a unique family of probability measures $\nu_x \in \mathcal{M}(\mathcal{J}_x)$ s.t.
   \[ \mathcal{L}_x^* \nu_{\theta(x)} = \lambda_x \nu_x \quad \text{where} \quad \lambda_x = \nu_{\theta(x)} (\mathcal{L}_x \mathbb{1}) \quad m \text{- a.e.} \]

2. There exists a unique function $q \in C^0(\mathcal{J})$ such that $m$-a.e.
   \[ \mathcal{L}_x q_x = \lambda_x q_{\theta(x)} \quad \text{and} \quad \nu_x (q_x) = 1. \]

Moreover, $q_x \in \mathcal{H}^\alpha (\mathcal{J}_x)$ for a.e. $x \in X$.

3. The family of measures $\{\mu_x := q_x \nu_x\}_{x \in X}$ is $T$-invariant.

- $\mathcal{M}^1(\mathcal{J}_x)$ is the set of all Borel probability measures on $\mathcal{J}_x$.
- A family of measures $\{\mu_x\}_{x \in X}$ such that $\mu_x \in \mathcal{M}^1(\mathcal{J}_x)$ is called $T$-invariant if $\mu_x \circ T_x^{-1} = \mu_{\theta(x)}$ for a.e. $x \in X$. 
The Gibbs states $\nu_x$ are obtained in a pointwise manner using a fixed point method (Kifer).

- $\mathcal{O}_{x_0} = \{x_n = \theta^n(x_0); \ n \in \mathbb{Z}\}$ orbit of $x_0 \in X$.
- $\mathcal{P}(\mathcal{O}_{x_0}) = \prod_{x \in \mathcal{O}_{x_0}} \mathcal{M}^1(\mathcal{F}_x)$ compact subset of a locally convex topological space.
- Consider

$$\nu_{\theta(x)} \longmapsto \frac{\mathcal{L}_x^* \nu_{\theta(x)}}{\mathcal{L}_x^* \nu_{\theta(x)}(1)}.$$
In order to obtain the invariant densities $q_x$, we adapt Bowen’s method which uses positive cones of Hölder functions: for $s \geq 1$, set

$$\Lambda^s_x = \left\{ g \in C(J_x) : g \geq 0, \, \nu_x(g) = 1 \text{ and } g(w_1) \leq e^{sQ_x} e^{\alpha(w_1, w_2)} g(w_2) \right\}$$

for all $w_1, w_2 \in J_x$ with $\rho(w_1, w_2) \leq \xi$.

These cones are preserved under $\mathcal{L}$ and we get some contraction.
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for all $w_1, w_2 \in \mathcal{J}_x$ with $\alpha(w_1, w_2) \leq \xi$.

These cones are preserved under $\mathcal{L}$ and we get some contraction. This leads to crucial exponential convergence:

**Proposition**

*Let $s > 1$. There exist $B < 1$ and a measurable function $A : X \to (0, \infty)$ such that for a.e. $x \in X$ for every $N \geq 1$ and $g_{x-N} \in \Lambda_{x-N}^s$ we have*

$$\| (\tilde{\mathcal{L}}^N g)_x - q_x \|_{\infty} = \| \tilde{\mathcal{L}}_{x-N}^N g_{x-N} - q_x \|_{\infty} \leq A(x)B^N$$

*where $\tilde{\mathcal{L}} = \lambda_x^{-1} \mathcal{L}_x$.***
Measurability

We need measurability of \( \nu_x, q_x \) and \( \mu_x \). For example, right now we do not know if the "pointwise Gibbs states" \( \nu_x \) are the disintegration of a "global Gibbs" state \( \nu \) with marginal \( m \) on the fibered space \( J \). Or, the expression

\[
\int_X \left( \int_{\mathcal{J}_x} g_x \, d\nu_x \right) \, dm(x)
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Definition

Let $T : \mathcal{J} \to \mathcal{J}$ be a expanding random map. Define $\pi_X : \mathcal{J} \to X$ by $\pi_X(x, y) = x$. Let $\mathcal{B} := \mathcal{B}_{\mathcal{J}}$ be a $\sigma$-algebra on $\mathcal{J}$ such that

1. $\pi_X$, $T$ and the transfer operator $\mathcal{L}$ are measurable,
2. for every $A \in \mathcal{B}$, $\pi_X(A) \in \mathcal{F}$,
3. $\mathcal{B}|_{\mathcal{J}_x}$ is the Borel $\sigma$-algebra on $\mathcal{J}_x$.

If in addition $\log \| \mathcal{L}_x 1_1 \|_\infty \in L^1(m)$ (ok if $\log(\deg T_x) \in L^1(m)$), then $T$ is called measurable expanding random map.
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In this setting we deduce measurability of all the objects $\lambda_x$, $\nu_x$, $q_x$, $\mu_x$ from the exponential convergence.
Fractal structure of Conformal RDS

Definition

Let $f : (x, z) \mapsto (\theta(x), f_x(z))$ be a measurable expanding random map s.t.

- the fibers $\mathcal{J}_x \subset Y$, a smooth Riemannian manifold,
- $f_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$ can be extended to a neighborhood of $\mathcal{J}_x$ in $Y$ to a conformal $C^{1+\alpha}$ map and
- $\log \|f_x'\|_\infty \in L^1(m)$.

Then we call $f$ conformal expanding random map. If, in addition, $f$ is uniformly expanding then it is called conformal uniformly expanding.
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Example 1: quadratic polynomials. $D = (0, \delta)$ for some $0 < \delta < 1/4$.

$x = (\ldots, c_{-1}, c_0, c_1, \ldots, c_n, \ldots) \in X = D^\mathbb{Z} \xrightarrow{\pi} c_0 \in D,$

$f_x(z) = z^2 + \pi(x) = z^2 + c_0.$
Fractal structure of Conformal RDS

Definition

Let $f : (x, z) \mapsto (\theta(x), f_x(z))$ be a measurable expanding random map s.t.

- the fibers $\mathcal{J}_x \subset Y$, a smooth Riemannian manifold,
- $f_x : \mathcal{J}_x \to \mathcal{J}_{\theta(x)}$ can be extended to a neighborhood of $\mathcal{J}_x$ in $Y$ to a conformal $C^{1+\alpha}$ map and
- $\log ||f_x'||_{\infty} \in L^1(m)$.

Then we call $f$ conformal expanding random map. If, in addition, $f$ is uniformly expanding then it is called conformal uniformly expanding.

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Example 2: Repellers. $V \Subset U$ open subsets of $\mathbb{C}$.

$\mathcal{R}_d(V, U) := \{f : V_f \to U \text{ holo proper s.t. } V_f \subset V \text{ and } \text{deg}(f) \leq d\}.$
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Suppose the repellers \( f_{x_0}, f_{x_1}, \ldots, f_{x_n}, \ldots \) are chosen iid wrt some arbitrary probability space \( (I, \mathcal{F}_0, m_0) \hookrightarrow (X, \mathcal{F}, m) \) with \( X = I^\mathbb{Z} \) and \( m = m_0^\mathbb{Z} \).
\( \hookrightarrow \text{ random repeller } f^n_{x_0} = f_{x_{n-1}} \circ \ldots \circ f_{x_0} \) with associated random Julia set
\[
    J(x_0, x_1, \ldots) = \bigcap_{n \geq 1} f_{x_0}^{-n}(U)
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Expected pressure and Bowen’s Formula:

Consider potentials \( \varphi_t(x, z) = -t \log |f_x'(z)|, \ t \in \mathbb{R}. \)

The associated **topological pressure** is \( P_x(t) = P(\varphi_t) := \log \lambda_x \) and

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Let \( f \) be a conformal expanding random map. Then

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Bowen’s formula has been obtained previously in various settings first by Kifer and then by Crauel and Flandoni, Bogenschütz - Ochs and Rugh.
Quasi-deterministic and essential systems.

Finer fractal properties (e.g. behavior of $HM^h$ and $PM^h$) rely on the asymptotic behavior of

$$P^n_x(h) = P_x(h) + P_{\theta(x)} + \ldots + P_{\theta^{n-1}(x)} , \quad h \text{ Bowen’s parameter,}$$

seen as random variables on the base space $X$. 

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**Definition**

*The random map $f$ is called *essentially random* if*

$$\limsup_{n \to \infty} P_{x}^{n}(h) = +\infty \quad \text{and} \quad \liminf_{n \to \infty} P_{x}^{n}(h) = -\infty \quad m-a.e.,$$

*and *quasi-deterministic* if for $m-a.e. \ x \in X$ there exists $L_{x} > 0$ s.t.

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With this respect, "quasi-deterministic systems are rather exceptional".

These limit theorems do hold for the random repeller examples over the shift space.
Theorem

Suppose $f : J \rightarrow J$ is a conformal uniformly expanding random map.

(a) If the system is essential, then

$$\mathcal{H}^h(J_x) = 0 \quad \text{and} \quad \mathcal{P}^h(J_x) = +\infty$$

for $m$-a.e. $x \in X$. 
Theorem

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(b) On the other hand, if the system is quasi-deterministic then, for every $x \in X$, $\nu^h_x$ is a geometric measure with exponent $h$ and therefore we have:

- $0 < \mathcal{H}^h(\mathcal{J}_x), \mathcal{P}^h(\mathcal{J}_x) < +\infty$ and $\text{HD}(\mathcal{J}_x) = h$.
- The measures $\mu$, $\mathcal{H}^h$, and $\mathcal{P}^h$ are all mutually equivalent with Radon-Nikodym derivatives separated away from zero and infinity.
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From (a) we get that the Bogenschütz-Ochs conjecture does not hold. Moreover, essential conformal random systems are entirely new objects, drastically different from deterministic self-conformal sets since we get:
Corollary

Suppose that $f : J \to J$ is essential. Then, for $m$-a.e. $x \in X$, the following hold.

1. The fiber $J_x$ is not bi-Lipschitz equivalent to any deterministic nor quasi-deterministic self-conformal set.
2. $J_x$ is not a geometric circle nor even a piecewise smooth curve.
3. If $J_x$ has a non-degenerate connected component (for example if $J_x$ is connected), then $h = \text{HD}(J_x) > 1$.
4. Let $d$ be the dimension of the ambient Riemannian space $Y$. Then $\text{HD}(J_x) < d$. 
Corollary

Suppose that $f : \mathcal{I} \to \mathcal{I}$ is essential. Then, for $m$-a.e. $x \in X$, the following hold.

1. The fiber $\mathcal{I}_x$ is not bi-Lipschitz equivalent to any deterministic nor quasi-deterministic self-conformal set.
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3. If $\mathcal{I}_x$ has a non-degenerate connected component (for example if $\mathcal{I}_x$ is connected), then $h = \text{HD}(\mathcal{I}_x) > 1$.
4. Let $d$ be the dimension of the ambient Riemannian space $Y$. Then $\text{HD}(\mathcal{I}_x) < d$.

Finally we also ...
... get a positive answer to the question by Brück-Bürger concerning polynomial systems:

**Theorem**

If $d \geq 2$ is an integer, $0 < \delta < \delta(d)$, the skew-product map $f_{d,\delta} : \mathcal{J} \to \mathcal{J}$ is given by the formula

$$f_{d,\delta}(\omega, z) = (\sigma(\omega), f_{d,\omega_0}(z)) = (\sigma(\omega), z^d + \omega_0),$$

and if $m_0$ is an arbitrary Borel probability measure on $\overline{B}(0, \delta)$, different from $\delta_0$, the Dirac $\delta$ measure supported at 0. Then we have

$$1 < \text{HD}(\mathcal{J}_\omega) < 2 \quad \text{for m-a.e. } \omega \in \overline{B}(0, \delta)^\mathbb{Z}.$$
Theorem

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