Differential Geometry: a concise introduction

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Contents

Supplementary reading

- C Bär: *Elementary differential geometry*; CUP, Cambridge (2010)
- M do Carmo: *Differential geometry of curves and surfaces*; Prentice-Hall, Englewood Cliffs (1976)
- N J Hicks: *Notes on differential geometry*; Van Norstrand, New York (1965)
- M Spivak: *A comprehensive introduction to differential geometry*; Publish or Perish, Berkeley (1979)
- D J Struik: *Lectures on Classical Differential Geometry*; 2^{nd} Ed: Dover, New York (1988)

Manifesto

Our Mission: Study the geometry of curves and surfaces using methods from calculus, i.e., differentiation and (to a lesser degree) integration (thus "*differential* geometry").

In this study we will only be interested in properties that are independent of the position of the curve or surface, that is, that are *invariant under Euclidean motions*.

What does "geometry" mean? The term "geometry" comes from the greek

$$\gamma\epsilon\omega + \mu\epsilon\tau\rho\acute{\iota}\alpha \;\simeq\; \begin{cases} \gamma\eta & = \text{ “earth”}, \\ \mu\epsilon\tau\rho\acute{\omega} & = \text{ “measure”}. \end{cases}$$

"Geometry" originated from the task of measuring the earth (for example, a farmer's fields). The greeks turned this "applied science" into "pure mathematics" by studying geometric objects on an abstract level: Euclid's "Elements" (ca 300BC) is the most famous example of this work. Still today, modern cartography relies on (differential) geometry.

"Differential geometry" developed much later after calculus was developed in the early 1700's: then it was possible to study more complicated geometrical objects, such as, arbitrarily curved "curves" and "surfaces" in 3-space. C F Gauss' book "Disquisitiones generales circa superficies curvas" (1827) was one of the mile-stones in this development.

Much of this text will be about the description of how objects (curves and surfaces) in space are "curved" or, more generally speaking, about their "shape". Clearly, the "shape" of a curve or surface does not depend on its position in space — hence the requirement for *invariance under Euclidean motions*

$$p \mapsto Ap + c, \quad \text{where} \quad A \in SO(3) \text{ and } c \in \mathbb{R}^3.$$

What are "curves" and "surfaces"? A "curve" can be thought of as the shape that a thin bent wire in space (or in a plane) would take; equally, it can be thought of as the trace of a moving particle in space (we shall see in what sense "equally"). Thus a curve is a 1-dimensional object and will, mathematically, be described by an $\mathbb{R}^3$-valued function $t \mapsto \gamma(t)$ of *one* variable (or, equivalently, by *two* equations for $(x, y, z) \in \mathbb{R}^3$).

A "surface" can be thought of as, for example, a soap film or the film of a soap bubble or the surface of a body or the earth. Thus a surface is a 2-dimensional object and will therefore be described by an $\mathbb{R}^3$-valued function $(u, v) \mapsto \sigma(u, v)$ of *two* variables (or, equivalently, by *one* equation for $(x, y, z) \in \mathbb{R}^3$).

This rough idea of a "curve" or a "surface" will be made precise later. A "submanifold" or a "manifold" is a generalization (and abstraction) of a curve or surface; these notions resolve some issues that the notions of "curve" and "surface" as discussed in the first three sections of this text create.

What is all this good for? First of all, differential geometry is a beautiful subject in pure mathematics. But, secondly, there is also a variety of applications: in the natural sciences, most notably, in physics (for example, when considering a moving particle or planet or when studying the shape of thin plates) and also in engineering or architecture, where more complicated shapes need to be modelled (for example, when designing the shape of a car or a building).

<u>*Example*</u>. Consider the ellipse E in the plane $\mathbb{R}^2$:

$$E = \{(x, y) \,|\, (\tfrac{x}{a})^2 + (\tfrac{y}{b})^2 = 1\}.$$

This is given in *implicit form*, i.e., by an equation between the two coordinates (x, y) of $\mathbb{R}^2$.

Can we describe it in *parametric form*, i.e., $E = \{\gamma(t) \,|\, t \in I\}$, where I is some interval and $\gamma : I \to \mathbb{R}^2$ a suitable function?

1. *As a graph:* $E \supset \{(x, \pm b\sqrt{1 - (\tfrac{x}{a})^2}) \,|\, -a < x < a\}$ — but we do not get the whole ellipse in this way. Note that we cannot have $x = \pm a$ without loosing differentiability.
2. *In parametric form:* $E = \{\gamma(t) := (a \cos t, b \sin t) \,|\, t \in \mathbb{R}\}$ — but we cover the ellipse infinitely often. Note that $\gamma' \neq 0$ everywhere.

There are theoretical tools (the *implicit* and *inverse mapping theorems*), which show that one can pass from an implicit description to a parametric description and vice versa — under certain conditions (see Appendix).

1 Curves

1.1 Parametrization & Arc length

Def. $\gamma : \mathbb{R} \overset{\circ}{\supset} I \to \mathbb{R}^3$ *is called* regular *if* $\gamma'(t) \neq 0$ *for all* $t \in I$.
We call the image $\gamma(I) \subset \mathbb{R}^3$ *of a regular map* $\gamma : I \to \mathbb{R}^3$ *from an open interval* I *into* $\mathbb{R}^3$ *a* curve*;*
γ *is called a* parametrization *of the curve or a* parametrized curve.

Examples.

(1) *Straight line.*

$$\mathbb{R} \ni t \mapsto \gamma(t) := a + b\,t \in \mathbb{R}^3,$$

where $a \in \mathbb{R}^3$ and $b \in \mathbb{R}^3 \setminus \{0\}$ are (constant) vectors.

(2) *Circle.*

$$\mathbb{R} \ni t \mapsto \gamma(t) := c + r\,(\mathbf{e}_1 \cos t + \mathbf{e}_2 \sin t) \in \mathbb{R}^3,$$

where $c \in \mathbb{R}^3$ is the centre of the circle, $r > 0$ its radius and $(\mathbf{e}_1, \mathbf{e}_2)$ an orthonormal basis of its plane.

(3) *Circular helix.*

$$\mathbb{R} \ni t \mapsto \gamma(t) := (r \cos t, r \sin t, h\,t) \in \mathbb{R}^3,$$

where $r > 0$ and $h \in \mathbb{R}$;
note that the curve $\gamma(\mathbb{R})$ lies on the cylinder $x^2 + y^2 = r^2$.
If $h \neq 0$ then the helix can also be written as a graph over its axis (the z-axis):

$$x = r \cos \tfrac{z}{h} \quad \text{and} \quad y = r \sin \tfrac{z}{h}, \quad \text{where} \quad z \in \mathbb{R}.$$

(4) *Hyperbola.*

$$H := \{(x, y, z) \,|\, y = 0, (\tfrac{x}{a})^2 - (\tfrac{z}{b})^2 = 1\};$$

parametrizations of the two branches are obtained by solving for x,

$$z \mapsto \gamma(z) := (\pm a \sqrt{1 + (\tfrac{z}{b})^2}, 0, z);$$

however, these contain square roots (usually undesirable!), a better choice is given by a "reparametrization" $z = b\sinh t$, giving

$$t \mapsto \gamma(t) = (\pm a\cosh t, 0, b\sinh t);$$

note that there cannot be a smooth (hence continuous) parametrization of *both* branches at the same time; hence each branch is a curve in our sense, but the hyperbola (consisting of two connected components) is *not*.

Problem 1. Find parametrizations for the conic sections

$$C = \{(x, y, z) \,|\, x^2 + y^2 = z^2,\ \ x\cos\alpha + z\sin\alpha = d\},$$

$\alpha \in [0, \frac{\pi}{2}]$ and $d \neq 0$. [Hint: distinguish $\alpha < \frac{\pi}{4}$, $\alpha = \frac{\pi}{4}$ and $\alpha > \frac{\pi}{4}$.]

Def. *A* reparametrization *of a parametrized curve $I \ni t \mapsto \gamma(t) \in \mathbb{R}^3$ is a new parametrized curve*

$$\tilde\gamma(\tilde t) = \gamma(\varphi(\tilde t)), \quad \textit{where} \quad \varphi : \tilde I \to I \textit{ is onto and } \varphi' \neq 0.$$

Remark. The condition $\varphi'(\tilde t) \neq 0$ for all $\tilde t$ ensures that a reparametrization $\tilde\gamma$ of γ is regular (chain rule), hence a parametrization.

Problem 2. Prove that $t \mapsto \gamma(t)$ is a straight line if $\gamma''(t)$ and $\gamma'(t)$ are linearly dependent for all t.

Motivation. Thinking of a (parametrized) curve $t \mapsto \gamma(t)$ as the path of a particle moving in time, we may think of

- $\gamma'(t)$ as the *velocity vector* at given time t; and of
- $|\gamma'(t)|$ as the *speed* of the particle at given time t.

The distance travelled by the particle between two given times t_0 and t_1 is then

$$\int_{t_0}^{t_1} |\gamma'(t)|\, dt.$$

Def. *The* arc length *of a (parametrized) curve $t \mapsto \gamma(t)$, measured from $\gamma(t_0)$, is*

$$s(t) := \int_{t_0}^{t} |\gamma'(t)|\, dt.$$

Remark. The arc length is indeed the length of the curve between $\gamma(t_0)$ and $\gamma(t)$, as can be proved by polygonal approximation of the curve.

Hence, the arc length does not depend the parametrization.

Problem 3. Use substitution to show that the arc length is invariant under reparametrization of a parametrized curve.

Lemma & Def. *Any curve $t \mapsto \gamma(t)$ can be reparametrized by arc length, i.e., so that it has constant speed 1. This is called an* arc length parametrization *of γ and usually denoted by $s \mapsto \gamma(s)$.*

Proof. Fix t_0 and observe that $s'(t) = |\gamma'(t)| > 0$ for all t. Hence we can invert s to obtain $t = t(s)$ and let $\tilde{\gamma}(s) := \gamma(t(s))$. Then

$$|\tilde{\gamma}'(s)| = |\gamma'(t)|\, t'(s) = \frac{|\gamma'(t)|}{s'(t)} = 1$$

has length 1 (note: $t'(s) = \frac{1}{s'(t)}$ by chain rule). ∎

Remark. An arc length parametrization is unique up to choice of "initial point" and sense of direction (orientation) of the curve.

Examples.

(1) *Circle*. Parametrization: $t \mapsto \gamma(t) = (r\cos t, r\sin t, 0)$;
Arc length: $s(t) = \int_0^t |\gamma'(t)|dt = \int_0^t r\,dt = rt$, thus $t(s) = \frac{s}{r}$;
Arc length (re)parametrization: $s \mapsto \tilde{\gamma}(s) = (r\cos\frac{s}{r}, r\sin\frac{s}{r}, 0)$.

(2) *Ellipse*. Parametrization: $t \mapsto \gamma(t) = (a\cos t, b\sin t, 0)$;
Arc length: $s(t) = \int_0^t |\gamma'(t)|dt = \int_0^t \sqrt{b^2 + (a^2 - b^2)\sin^2 t}\,dt$,
...which is an elliptic integral so that we cannot write down an arc length (re)parametrization in terms of elementary functions.

(3) *Circular helix*. Parametrization: $t \mapsto \gamma(t) = (r\cos t, r\sin t, ht)$;
Arc length: $s(t) = \int_0^t \sqrt{r^2 + h^2}\,dt = \sqrt{r^2 + h^2}\,t$;
Arc length (re)param: $s \mapsto (r\cos\frac{s}{\sqrt{r^2+h^2}}, r\sin\frac{s}{\sqrt{r^2+h^2}}, \frac{hs}{\sqrt{r^2+h^2}})$.

Problem 4. Consider the curve given implicitly by $(\frac{x}{a})^2+(\frac{y}{b})^2+(\frac{z}{c})^2 = 1$ and $a\sqrt{b^2-c^2}\,z = c\sqrt{a^2-b^2}\,x$, where $a > b > c$. Compute its arc length and find an arc length (re)parametrization.

1.2 Ribbons & Frames

If we think of an (arc length) parametrized curve $t \mapsto \gamma(t)$ as the path of a body in space (moving at constant speed) it will be useful not only to describe the direction of movement of the particle but also its orientation in space — like a flying air plane, which does not only have a notion of "forward" but also a notion of "upward".

Def. *Let $t \mapsto \gamma(t)$ be a parametrized curve. A (unit) smooth vector field $t \mapsto N(t)$ so that $N(t) \perp \gamma'(t)$ for all t is called a* (unit) normal field *along γ.*
The pair (γ, N) of a parametrized curve and a unit normal field will be called a ribbon*.*

Rem & Def. *At each point, a regular curve has a unique* normal plane

$$\mathcal{N}(t) := \{p \in \mathbb{R}^3 \,|\, p - \gamma(t) \perp T(t)\}, \quad \text{where} \quad T := \tfrac{\gamma'}{|\gamma'|}$$

denotes the unit tangent vector field *of γ; hence, at each point, a curve has a circle's worth of unit normal vectors.*
A normal plane $\mathcal{N}(t)$ inherits a natural linear structure (with origin $\gamma(t) \simeq 0$) from the 2-dimensional vector subspace $\{T(t)\}^\perp \subset \mathbb{R}^3$ since

$$\mathcal{N}(t) = \gamma(t) + \{T(t)\}^\perp = \gamma(t) + \{N \in \mathbb{R}^3 \,|\, N \perp T(t)\}.$$

Remark. A unit normal field N along γ defines a "horizontal" plane $\mathcal{T}(t)$ at each point $\gamma(t)$ of the curve in a similar way:

$$p \in \mathcal{T}(t) \quad :\Leftrightarrow \quad p - \gamma(t) \perp N(t),$$

i.e., we obtain a second family of planes varying smoothly along the curve. Conversely, any such family of planes defines a unit normal field N uniquely up to sign.

Notation. We will denote the standard basis of $\mathbb{R}^3$ by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$.

Def. *$t \mapsto F(t) \in SO(3)$ is called an* (adapted) frame for a parametrized curve $t \mapsto \gamma(t)$ *if*

$$F(t)\mathbf{e}_1 = T(t) = \tfrac{\gamma'(t)}{|\gamma'(t)|};$$

and an (adapted) frame for a ribbon (γ, N) *if, additionally,*

$$F(t)\mathbf{e}_2 = N(t).$$

Remark. We obtain a second unit normal field alon γ as

$$B := T \times N = F\mathbf{e}_3;$$

thus we will (not entirely appropriately) also write $F = (T, N, B)$.

Then, any normal field $\tilde{N}$ along γ can be written as

$$\tilde{N} = \lambda N + \mu B$$

with suitable functions λ and μ and any two adapted frames F and $\tilde{F}$ are related by a *normal rotation*,

$$\tilde{F} = F \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

Lemma & Def. *Let F be an adapted frame for (γ, N); then there are unique functions κ_n, κ_g and τ so that*

$$F' = F\,\Phi \quad \textit{with} \quad \Phi = |\gamma'| \begin{pmatrix} 0 & -\kappa_n & \kappa_g \\ \kappa_n & 0 & -\tau \\ -\kappa_g & \tau & 0 \end{pmatrix}; \qquad (*)$$

$(*)$ *are called the* structure equations *of the ribbon* (γ, N), *and*

- κ_n *its* normal curvature,
- κ_g *its* geodesic curvature *and*
- τ *its* torsion.

Proof. Since $t \mapsto F(t) \in SO(3)$ we have $F^tF \equiv id$ so that

$$0 = (F^tF)' = (F^tF') + (F^tF')^t = \Phi + \Phi^t,$$

that is, $\Phi(t) \in \mathfrak{o}(3)$ is *skew symmetric* for all t. Hence we can find functions κ_n, κ_g and τ so that Φ is of the above form. ∎

Problem 5. Let (γ, N) be a ribbon and $\tilde{N} := N\cos\varphi + B\sin\varphi$, where φ is smooth and $B = T \times N$. Show that $(\gamma, \tilde{N})$ is a ribbon and compute how the structure equations (i.e., κ_n, κ_g and τ) change.

Remark. κ_n, κ_g and τ are geometric quantities, i.e., are independent of the position of γ in space as well as of the parametrization of γ:

- if $(\tilde{\gamma}, \tilde{N}) = (A\gamma + c, AN)$, where $A \in SO(3)$ and $c \in \mathbb{R}^3$, is a *Euclidean motion* of (γ, N) then $\tilde{F} = AF$ and $\tilde{\Phi} = \Phi$;
- if $(\tilde{\gamma}, \tilde{N})(s) = (\gamma, N)(t(s))$ is an orientation preserving (i.e., $t' > 0$) reparametrization of γ, then $\tilde{\Phi}(s) = t'\Phi(t(s))$ and $|\tilde{\gamma}'| = |t'\gamma'|$, hence $\tilde{\kappa}_n(s) = \kappa_n(t(s))$, etc.

Problem 6. Let (γ, N) be a ribbon and $\tilde{\gamma} = \gamma \circ t$ a reparametrization of γ with $t' > 0$; set $\tilde{N} := N \circ t$. Show that $(\tilde{\gamma}, \tilde{N})$ is a ribbon with $\tilde{\kappa}_n = \kappa_n \circ t$, $\tilde{\kappa}_g = \kappa_g \circ t$ and $\tilde{\tau} = \tau \circ t$.

Remark. If γ is arc length parametrized then the structure equations read

$$F' = F\,\Phi \quad \Leftrightarrow \quad \begin{cases} T' = & & \kappa_n N & -\kappa_g B \\ N' = & -\kappa_n T & & +\ \tau B \\ B' = & +\kappa_g T & -\tau N & \end{cases}$$

Thus κ_n and κ_g measure how fast the tangent line changes, that is, "how strongly the curve is curved", and τ measures how fast N (and hence also B) rotate around the curve.

Problem 7. Let γ parametrize a straight line, $\gamma' \times \gamma'' \equiv 0$, and let F denote any adapted frame for γ. Show that $\kappa_n = \kappa_g = 0$. Find a unit normal field N so that $\tau \equiv 1$.

Examples.

(1) *Circular helix*. $\gamma(t) = (r\cos t, r\sin t, ht)$ with "arc length element" $ds = \sqrt{r^2 + h^2}\,dt$; as a unit normal field, we choose

$$N(t) := -(\cos t, \sin t, 0)$$

note that, for all t,

$$N(t) \perp T(t) = \tfrac{\gamma'(t)}{|\gamma'(t)|} = \tfrac{1}{\sqrt{r^2+h^2}}(-r\sin t, r\cos t, h)$$

and consider the ribbon (γ, N); with the third frame vector field

$$B(t) = (T \times N)(t) = \tfrac{1}{\sqrt{r^2+h^2}}(h\sin t, -h\cos t, r)$$

the structure equations become

$$\begin{array}{rlll} T' = & & \frac{r}{\sqrt{r^2+h^2}}N & \\ N' = & -\frac{r}{\sqrt{r^2+h^2}}T & & +\frac{h}{\sqrt{r^2+h^2}}B \\ B' = & & -\frac{h}{\sqrt{r^2+h^2}}N & \end{array}$$

so that

$$\kappa_n = \tfrac{r}{r^2+h^2}, \quad \kappa_g \equiv 0, \quad \tau = \tfrac{h}{r^2+h^2}.$$

(2) *Spherical curve*. Let $s \mapsto \gamma(s) \in S^2(r)$ i.e., $|\gamma|^2 \equiv r^2$, be arc length parametrized, i.e., $|\gamma'|^2 \equiv 1$. Observe that

$$\gamma' \cdot \gamma = \tfrac{1}{2}(|\gamma|^2)' \equiv 0$$

so that $N := \frac{1}{r}\gamma$ defines a unit normal field and

$$F = (\gamma', \tfrac{1}{r}\gamma, \tfrac{1}{r}\gamma' \times \gamma)$$

an adapted frame for the ribbon $(\gamma, \frac{1}{r}\gamma)$. Then

$$\begin{array}{ccccccc} \kappa_n & = & T' \cdot N & = & \frac{1}{r}\gamma'' \cdot \gamma & \equiv & -\frac{1}{r} \\ \kappa_g & = & B' \cdot T & = & \frac{1}{r}(\gamma' \times \gamma)' \cdot \gamma' & = & \frac{1}{r}\det(\gamma, \gamma', \gamma'') \\ \tau & = & N' \cdot B & = & \frac{1}{r^2}\gamma' \cdot (\gamma' \times \gamma) & \equiv & 0 \end{array}$$

and the structure equations read

$$\begin{array}{rlll} T' = & & -\frac{1}{r}N & -\frac{\det(\gamma,\gamma',\gamma'')}{r}B \\ N' = & \frac{1}{r}T & & \\ B' = & \frac{\det(\gamma,\gamma',\gamma'')}{r}T & & \end{array}$$

<u>*Remark*</u>. Note that, in the first example, $\kappa_g \equiv 0$ whereas, in the second example, $\tau \equiv 0$. These two conditions characterize two prominent classes of ribbons/frames that we will discuss in more detail later.

Def. *A ribbon (γ, N) is called*

- asymptotic ribbon *if* $\kappa_n \equiv 0$,
- geodesic ribbon *if* $\kappa_g \equiv 0$,
- curvature ribbon *if* $\tau \equiv 0$.

And here is one of the key theorems in curve theory:

Lemma (Fundamental theorem for ribbons). *Fix three functions*

$$s \mapsto \kappa_n(s), \kappa_g(s), \tau(s).$$

Then there is an arc length parametrized curve γ and a unit normal field N along γ so that κ_n, κ_g and τ are the normal and geodesic curvatures and the torsion of the ribbon (γ, N), respectively. Moreover, this ribbon is unique up to Euclidean motion

Proof. We seek a ribbon (γ, N) with $|\gamma'|^2 \equiv 1$ so that the adapted frame

$$F = (T, N, B), \quad \text{where} \quad T = \gamma' \text{ and } B = T \times N,$$

satisfies

$$F' = F \cdot \Phi, \quad \Phi = \begin{pmatrix} 0 & -\kappa_n & \kappa_g \\ \kappa_n & 0 & -\tau \\ -\kappa_g & \tau & 0 \end{pmatrix}. \qquad (\star)$$

$(\star)$ is a first order linear homogeneous system of ode's. By the Picard-Lindelöf Thm, this has a unique, globally defined solution $s \mapsto F(s)$ for any given initial value $F(s_0) = F_0$.

Next we need to verify that $F(s) \in SO(3)$ for all s, so that it qualifies as a frame. To this end:

1. note that $(FF^t)' = F(\Phi + \Phi^t)F^t = 0$ so that $s \mapsto F(s) \in O(3)$ as soon as $F_0 \in O(3)$;
2. $F(s) \in O(3) \Rightarrow \det F(s) = \pm 1$ and $s \mapsto \det F(s) \in \{-1, +1\}$ is continuous, hence cannot change sign (Intermediate value theorem), so that $s \mapsto F(s) \in SO(3)$ as soon as $F_0 \in SO(3)$.

Now fix s_0 and $F_0 = \mathrm{id}_{\mathbb{R}^3}$ and take

$$T := F\mathbf{e}_1, \quad N := F\mathbf{e}_2, \quad B := F\mathbf{e}_3 \quad \text{and} \quad \gamma(s) := \int_{s_0}^{s} T(s)ds.$$

Clearly, γ is arc length parametrized, $|\gamma'| = |T| \equiv 1$, and F is an adapted frame for the ribbon (γ, N) so that the structure equations $(\star)$ hold.

To see uniqueness of γ up to Euclidean motion suppose that F and $\tilde{F}$ are two solutions of $(\star)$; then

$$(\tilde{F}F^{-1})' = \tilde{F}'F^{-1} - \tilde{F}F^{-1}F'F^{-1} = \tilde{F}(\Phi - \Phi)F^{-1} = 0,$$

that is $\tilde{F}F^{-1} \equiv \tilde{F}(s_0)F^{-1}(s_0) =: A$ is a constant special orthogonal transformation as soon as $\tilde{F}(s_0), F(s_0) \in SO(3)$. Hence

$$\tilde{T} = AT, \quad \tilde{N} = AN, \quad \tilde{B} = AB, \quad \text{and} \quad \tilde{\gamma} = A\gamma + c,$$

where c is a constant of integration: that is, $(\tilde{\gamma}, \tilde{N}) = (A\gamma + c, AN)$ is obtained by a Euclidean motion from (γ, N). ■

Problem 8. Prove that an arc length parametrized curve $s \mapsto \gamma(s)$ is planar if and only if it has an adapted frame so that $\kappa_g = \tau \equiv 0$.

1.3 Normal connection & Parallel transport

We shall now go on to study certain special frames for space curves: frames that become particularly "simple" or well adapted to study certain problems or notions in curve theory. These can be characterized by the vanishing of one of the curvatures, κ_n or κ_g, or of the torsion, τ.

We start with $\tau \equiv 0$. Consider the problem of lying out a fibre cable without twist (to minimize waste of material or interference between currents): thus we wish the material not to twist around its soul, that is, we have to solve the purely geometric problem of finding a frame without torsion along the soul.

Def. *A normal field $t \mapsto N(t)$ along $t \mapsto \gamma(t)$ is called* parallel *if*

$$\nabla^{\perp} N := (N')^{\perp} = N' - (N' \cdot T)\, T \equiv 0,$$

where $\nabla^\perp$ *denotes the* normal connection *along* γ*.*
An adapted frame $F = (T, N, B)$ *is* parallel *if* N *and* B *are parallel.*

Note. In this definition, we do not assume $|N| \equiv 1$.

Lemma. *The normal connection* $\nabla^\perp$ *is* metric*, i.e.,*

$$(N_1 \cdot N_2)' = \nabla^\perp N_1 \cdot N_2 + N_1 \cdot \nabla^\perp N_2;$$

parallel normal fields have constant length and make constant angles.

Proof. First we prove that $\nabla^\perp$ is metric, i.e., satisfies *Leibniz' rule*:

$$\nabla^\perp N_1 \cdot N_2 + N_1 \cdot \nabla^\perp N_2 = N_1' \cdot N_2 + N_1 \cdot N_2' = (N_1 \cdot N_2)'.$$

Hence, $(N_1 \cdot N_2)' = 0$ if N_1 and N_2 are parallel.

In particular, $(|N|^2)' = 2N \cdot \nabla^\perp N = 0$ for a parallel normal field N, showing that N has constant length; and two parallel normal fields make a constant angle α since

$$\cos\alpha = \tfrac{N_1 \cdot N_2}{|N_1|\,|N_2|} \equiv const$$

if N_1 and N_2 are parallel. ∎

Problem 9. Prove that any two parallel frames of a curve $t \mapsto \gamma(t)$ are related by a constant rotation in the normal plane.

Lemma. *If* F *is an adapted frame along* γ *and* $N = F\mathbf{e}_2$ *is a parallel normal field then* F *is a parallel frame.*

Proof. We need to show that $B = T \times N = F\mathbf{e}_3$ is parallel:

$$\begin{cases} \nabla^\perp B \cdot T = 0 & (\text{def of } \nabla^\perp), \\ \nabla^\perp B \cdot N = (B \cdot N)' - B \cdot \nabla^\perp N = 0 & (N \text{ parallel}), \\ \nabla^\perp B \cdot B = \frac{1}{2}(|B|^2)' = 0 & (|B|^2 \equiv 1); \end{cases}$$

hence $\nabla^\perp B = 0$. ∎

In the course of the proof we have also learned:

Cor. *(γ, N) is a curvature ribbon iff N is a parallel unit normal field.*

Proof. Writing $F = (T, N, B)$ we have

$$0 = \tau = \tfrac{1}{|\gamma'|}(N' \cdot B) = \tfrac{1}{|\gamma'|}(\nabla^\perp N \cdot B) \quad \Leftrightarrow \quad 0 = \nabla^\perp N$$

since $\nabla^\perp N \perp T, N$. ∎

Example. *Circular helix*. $\gamma(t) = (r\cos t, r\sin t, ht)$ and (see above)

$$\begin{array}{rclcl} T(t) & = & \frac{\gamma'(t)}{|\gamma'(t)|} & = & \frac{1}{\sqrt{r^2+h^2}}\,(-r\sin t, r\cos t, h) \\ N(t) & = & \frac{\gamma''(t)}{|\gamma''(t)|} & = & -(\cos t, \sin t, 0) \\ B(t) & = & T(t)\times N(t) & = & \frac{1}{\sqrt{r^2+h^2}}\,(h\sin t, -h\cos t, r) \end{array}$$

yields an adapted frame with

$$\kappa_n = \tfrac{r}{r^2+h^2}, \quad \kappa_g \equiv 0, \quad \tau = \tfrac{h}{r^2+h^2}.$$

We seek a parallel normal field

$$\tilde N(t) = \cos\varphi(t)\, N(t) + \sin\varphi(t)\, B(t);$$

thus compute

$$\begin{array}{rclclcl} \nabla^\perp N & = & N' - (N'\cdot T)\,T & = & |\gamma'|\,\tau B & = & \frac{h}{\sqrt{r^2+h^2}}\,B \\ \nabla^\perp B & = & B' - (B'\cdot T)\,T & = & -|\gamma'|\,\tau N & = & -\frac{h}{\sqrt{r^2+h^2}}\,N \end{array}$$

and

$$0 \overset{!}{=} \nabla^\perp \tilde N = \{\varphi' + \tfrac{h}{\sqrt{r^2+h^2}}\}\,\{-\sin\varphi\, N + \cos\varphi\, B\}.$$

Hence $F = (T, \tilde N, \tilde B)$ with

$$\begin{array}{rcl} \tilde N(t) & := & \cos\frac{ht}{\sqrt{r^2+h^2}}\,N(t) - \sin\frac{ht}{\sqrt{r^2+h^2}}\,B(t) \\ \tilde B(t) & := & \sin\frac{ht}{\sqrt{r^2+h^2}}\,N(t) + \cos\frac{ht}{\sqrt{r^2+h^2}}\,B(t) \end{array}$$

yields a parallel frame for γ.

Note that every other parallel frame is obtained by a constant rotation of the given one: this is reflected by the constant of integration for φ.

The following lemma asserts that we can always find parallel normal fields along a curve:

Lemma. *Let $t \mapsto \gamma(t)$ be regular and $\xi_0 \perp \gamma'(t_0)$. Then there is a unique parallel normal field $t \mapsto \xi(t)$ along γ with $\xi(t_0) = \xi_0$.*

Proof. Wlog., $|\gamma'|^2 \equiv 1$; let $F = (T, N, B)$ be an adapted frame.

With the ansats $\xi = \alpha N + \beta B$ we compute

$$\nabla^\perp \xi = \{\alpha' - \tau\beta\} N + \{\beta' + \tau\alpha\} B;$$

thus $\nabla^\perp \xi = 0$ iff

$$\alpha = r_0 \sin(\varphi_0 + \varphi) \quad \text{and} \quad \beta = r_0 \cos(\varphi_0 + \varphi)$$

with $\varphi(t) = \int_{t_0}^t \tau(t) dt$ and $r_0, \varphi_0 \in \mathbb{R}$.

In order to obtain $\xi(t_0) = \xi_0$ we need to choose $r_0 = |\xi_0|$ and $\varphi_0 \in \mathbb{R}$ so that $\frac{\xi_0}{|\xi_0|} = N \sin\varphi_0 + B \cos\varphi_0$. ■

Cor & Def. *Parallel normal fields along γ yield a linear isometry from the normal plane $\mathcal{N}(t_0)$ at $\gamma(t_0)$ to the normal plane $\mathcal{N}(t)$ at $\gamma(t)$. This isometry is called* parallel transport *along γ.*

Remark. This explains the term "connection" for $\nabla^\perp$: it provides a way to identify normal planes of a curve at different points.

Remark. For "linear" to make sense, recall that a normal plane $\mathcal{N}(t)$ carries a natural linear structure, with $\gamma(t)$ as the origin.

Proof. Fix some $\xi_0 \perp \gamma'(t_0)$; by the preceding lemma there is a unique normal field ξ with $\nabla^\perp \xi = 0$ along γ with $\xi(t_0) = \xi_0$. Thus, there is a unique map $\mathcal{N}(t_0) \to \mathcal{N}(t)$.

As the equation $\nabla^\perp \xi = 0$ is linear in ξ, constant linear combinations of parallel normal fields are parallel ("superposition principle"); hence this map is linear.

As parallel normal fields have constant length and make constant angles, it is an isometry. ■

Problem 10. Show that a curve takes values in a sphere if and only if the curvatures κ_n and κ_g of a parallel frame satisfy the equation of a line in

the plane.
How can the radius of the sphere be read off from this equation?

1.4 Frenet curves

In this section we study a second special type of frames, satisfying a "normal form" of the structure equations: after requiring $\tau \equiv 0$ in the last section we will now require $\kappa_g \equiv 0$, which leads to the "classical curve theory" of the 18th and 19th century described in most text books.

The difference between curvature and geodesic ribbons is illustrated by the example of the motion of an air plane during taxi and during the flight. Apart from forward or backward forces (caused by change of speed), the passenger experiences the forces caused by change of direction as sideways forces during taxi but as up- or downward forces during flight — this is achieved by "twist" (torsion) of the plane during flight.

Note that, if $F = (T, N, B)$ denotes an adapted frame, a constant normal rotation of the frame leads to a similar "rotation" in the "curvature plane":

$$(\tilde{N}, \tilde{B}) = (N, B) \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} \tilde{\kappa}_n \\ \tilde{\kappa}_g \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \kappa_n \\ \kappa_g \end{pmatrix};$$

in particular, for a 90°-rotation of the normal frame,

$$(N, B) \to (-B, N) \quad \Rightarrow \quad (\kappa_n, \kappa_g) \to (\tilde{\kappa}_n, \tilde{\kappa}_g) = (\kappa_g, -\kappa_n).$$

Hence, the geometry of geodesic ribbons ($\kappa_g \equiv 0$) and of asymptotic ribbons ($\kappa_n \equiv 0$) will be very similar (but differ in interpretation).

Def. *A parametrized curve* $t \mapsto \gamma(t)$ *is called a* Frenet curve *if*

$$\forall t : (\gamma' \times \gamma'')(t) \neq 0.$$

Remark. The Frenet-condition is invariant under reparametrization.

Problem 11. Convince yourself that the Frenet-condition is invariant under reparametrization.

Lemma & Def. *If $t \mapsto \gamma(t)$ is a Frenet curve then $T'(t) \neq 0$ for all t.*

$$t \mapsto N(t) := \frac{T'(t)}{|T'(t)|}$$

is called the principal normal field *of γ.*

Proof. By the Frenet condition $(\gamma' \times |\gamma'|T')(t) = (\gamma' \times \gamma'')(t) \neq 0$ for all t; hence $T'(t) \neq 0$ for all t. Further, $0 = (|T|^2)' = 2T \cdot T'$ showing that $T'(t) \perp \gamma'(t)$ for all t so that N defines a unit normal field of γ. ■

Remark. Let $t \mapsto \gamma(t)$ parametrize a Frenet curve. Then (γ, N) is a geodesic ribbon iff $\pm N$ is the principal normal field of γ.

Problem 12. Let $t \mapsto \gamma(t)$ be a Frenet curve. Prove that (γ, N) is a geodesic ribbon if and only if $\pm N$ is the principal normal field of γ.

Rem & Def. *At each point of the curve we can, in addition to $\mathcal{N}(t)$ and $\mathcal{T}(t)$, consider the plane*

$$\mathcal{O}(t) := \gamma(t) + \mathrm{span}\{\gamma'(t), \gamma''(t)\} = \gamma(t) + \{(\gamma' \times \gamma'')(t)\}^{\perp},$$

which intersects both $\mathcal{N}(t)$ and $\mathcal{T}(t)$ orthogonally. This is the osculating plane *of the curve at $\gamma(t)$.*

Remark. Thinking again of a parametrized curve $t \mapsto \gamma(t)$ as the path of a particle moving in time,

- $\gamma'(t)$ is the *velocity vector* at time t; and of
- $\gamma''(t)$ is the *acceleration vector* of the particle at time t.

Thus the osculating plane is the plane of the forces acting on the body: any force causing a change of speed as well as the "*centripetal force*" (perpendicular to the velocity and opposite of the "*centrifugal force*"), which "holds" the body on its path. If the body is moving at constant speed then this centripetal force is given by γ'' as

$$\gamma'' \cdot \gamma' = \tfrac{1}{2}(|\gamma'|^2)' = 0.$$

Geometrically, the tangent line of a curve is the line through two "infinitely close points" $\gamma(s)$ and $\gamma(s+ds)\simeq\gamma(s)+\gamma'(s)ds$; the osculating plane is the plane of three "infinitely close points"

$$\gamma(s) \quad \text{and} \quad \gamma(s\pm ds)\simeq\gamma(s)\pm\gamma'(s)ds+\tfrac{1}{2}\gamma''(s)ds^2$$

(as long as they are not collinear: Frenet condition).

<u>*Examples*</u>.

(1) *Planar curve*. If $t\mapsto\gamma(t)$ parametrizes a planar (Frenet) curve,

$$\gamma\cdot n\equiv c$$

for some $n\in S^2$ and $c\in\mathbb{R}$, then the plane $\pi=\{p\,|\,p\cdot n=c\}$ of the curve is its (fixed) osculating plane as $\gamma',\gamma''\perp n$.

(2) *Circular helix*.

$$\begin{aligned}\gamma(t) &= (r\cos t, r\sin t, ht),\\ \gamma'(t) &= (-r\sin t, r\cos t, h),\\ \gamma''(t) &= (-r\cos t, -r\sin t, 0)\end{aligned}$$

so that $\mathcal{O}(t)=\gamma(t)+\mathrm{span}\{\gamma'(t),\gamma''(t)\}$ contains, apart from the tangent line, the "radial" line perpendicular to the axis and passing through $\gamma(t)$.

Def & Lemma. *Let $t\mapsto\gamma(t)$ be a Frenet curve and $t\mapsto N(t)$ its principal normal field. The adapted frame $F=(T,N,B)$ of the ribbon (γ,N) is called the* principal frame *or* Frenet frame *of γ.*
Its structure equations take the form

$$F'=F\,\Phi \quad \text{with} \quad \Phi=|\gamma'|\begin{pmatrix}0 & -\kappa & 0\\ \kappa & 0 & -\tau\\ 0 & \tau & 0\end{pmatrix} \qquad (*)$$

with $\kappa>0$. These are the Frenet equations *of γ.*
κ and τ are the curvature *and* torsion *of the Frenet curve.*

<u>*Remark*</u>. Thus, for a Frenet frame, $\kappa:=\kappa_n$ and $\kappa_g\equiv 0$.

Proof. Fristly, $\kappa_g \equiv 0$ since $T' \cdot B = T' \cdot (T \times \frac{T'}{|T'|}) = 0$;
and secondly $\kappa = \kappa_n > 0$ as $T' \cdot N = T' \cdot \frac{T'}{|T'|} = |T'| > 0$. ∎

Problem 13. Let $s \mapsto \gamma(s)$ be an arc-length parametrized Frenet curve and define the *Darboux vector field* by $D := \tau T + \kappa B$. Prove that the Frenet equations can be written as

$$T' = D \times T, \quad N' = D \times N, \quad B' = D \times B.$$

Problem 14. Express curvature and torsion of a Frenet curve in terms of κ_n and κ_g of a parallel frame, and vice versa.

Lemma. *Curvature and torsion of a Frenet curve are given by*

$$\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} \quad \text{and} \quad \tau = \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2}.$$

In particular, they can be uniquely defined in terms of the curve (without reference to a choice of normal field or frame).

Remark. Recall that κ and τ are invariant under reparametrization.

Proof. First assume that $s \mapsto \gamma(s)$ is arc-length parametrized; then

$$\kappa = T' \cdot N = T' \cdot \frac{T'}{|T'|} = |T'| = |\gamma''| = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3}$$

since $T = \gamma'$ and $\gamma'' = T' \perp T = \gamma'$, and

$$\tau = (T \times N) \cdot N' = \det(T, N, N') = \frac{\det(T, T', T'')}{|T'|^2} = \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2}$$

since $N = \frac{T'}{|T'|}$ and $N' = \frac{T''}{|T'|} + \dots T'$.

Now, if $s \mapsto \tilde\gamma(s) := \gamma(t(s))$ is a reparametrization of $t \mapsto \gamma(t)$ then

$$\begin{aligned}
\tilde\gamma'(s) &= t'(s)\gamma'(t(s)), \\
\tilde\gamma''(s) &= t'^2(s)\gamma''(t(s)) + \dots \gamma'(t(s)), \\
\tilde\gamma'''(s) &= t'^3(s)\gamma'''(t(s)) + \dots \gamma''(t(s)) + \dots \gamma'(t(s));
\end{aligned}$$

hence

$$\begin{aligned}
|\tilde\gamma' \times \tilde\gamma''|(s) &= |t'(s)|^3 \, |\gamma' \times \gamma''|(t(s)), \\
\det(\tilde\gamma', \tilde\gamma'', \tilde\gamma''')(s) &= t'^6(s) \det(\gamma', \gamma'', \gamma''')(t(s)).
\end{aligned}$$

Thus
$$\tilde{\kappa}(s) = \kappa(t(s)) \quad \text{and} \quad \tilde{\tau}(s) = \tau(t(s)),$$
showing invariance of κ and of τ under reparametrization. ■

Problem 15. Let $t \mapsto \gamma(t)$ be a Frenet curve. Prove by direct computation that
$$\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} \quad \text{and} \quad \tau = \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2}.$$
Conclude that κ and τ are invariant under Euclidean motions of γ.

For Frenet curves, our earlier Fundamental theorem for ribbons specializes to a central theorem of classical curve theory:

Thm (Fundamental Theorem for Frenet curves). *Fix two functions*
$$s \mapsto \kappa(s), \tau(s) \quad \textit{with} \quad \forall s : \kappa(s) > 0.$$
Then there is an arc-length parametrized Frenet curve $s \mapsto \gamma(s)$ with curvature and torsion κ and τ, respectively.
Moreover, this curve is unique up to Euclidean motion.

Proof. By the fundamental theorem for ribbons there is a ribbon (γ, N) with $|\gamma'|^2 \equiv 1$, $\kappa_n = \kappa$, $\kappa_g \equiv 0$ and torsion τ; this ribbon is unique up to Euclidean motion
$$\gamma \to A\gamma + c \quad \text{with} \quad A \in SO(3) \text{ and } c \in \mathbb{R}^3.$$
We need to prove that γ is a Frenet curve and that N is its principal normal field: as $\kappa_g = 0$ and $\kappa \neq 0$,
$$T' = \kappa_n N = \kappa N \neq 0 \quad \text{and} \quad |T'| = \kappa,$$
so that $\gamma' \times \gamma'' = T \times T' = \kappa T \times N \neq 0$ and $s \mapsto \gamma(s)$ is a Frenet curve; moreover, $N = \frac{T'}{\kappa} = \frac{T'}{|T'|}$ is its principal normal field. ■

Remark. There is a similar, simpler statement for (planar) curves in $\mathbb{R}^2$, where only one function $s \mapsto \kappa(s)$ appears.

Problem 16. Formulate a Fundamental theorem for curves in $\mathbb{R}^2$.
Prove it without using the Picard-Lindelöf Theorem.

Example. Let $\kappa > 0$ and $\tau \in \mathbb{R}$ be two numbers. Then there is a unique (up to Euclidean motion) curve $s \mapsto \gamma(s)$ with curvature κ and torsion

τ by the Fundamental Theorem for space curves. On the other hand, we know that the circular helix

$$s \mapsto \gamma(s) = \left(r \cos \tfrac{s}{\sqrt{r^2+h^2}}, r \sin \tfrac{s}{\sqrt{r^2+h^2}}, \tfrac{sh}{\sqrt{r^2+h^2}}\right),$$

where

$$r = \tfrac{\kappa}{\kappa^2+\tau^2} \quad \text{and} \quad h = \tfrac{\tau}{\kappa^2+\tau^2},$$

is a curve with the given curvature and torsion. Thus every curve with constant curvature and torsion is a circular helix:

Thm (Classification of Circular helices). *A Frenet curve is a circular helix if and only if it has constant curvature and torsion.*

2 Surfaces

2.1 Parametrization & Metric

Def. $\sigma : \mathbb{R}^2 \overset{\circ}{\supset} U \to \mathbb{R}^3$ *is called* regular *if* $d_{(u,v)}\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ *injects for every* $(u,v) \in U$*; such regular maps are also called* immersions.
We call the image $\sigma(U) \subset \mathbb{R}^3$ *of a regular map* $\sigma : U \to \mathbb{R}^3$ *from an open and connected subset* $U \subset \mathbb{R}^2$ *into* $\mathbb{R}^3$ *a* surface*;*
σ *is called a* parametrization *of the surface or a* parametrized surface.

Remark. σ is regular iff $\sigma_u(u,v)$ and $\sigma_v(u,v)$ are linearly independent for all $(u,v) \in U$, that is, iff $(\sigma_u \times \sigma_v)(u,v) \neq 0$ for all $(u,v) \in U$.

Examples.

(1) *Plane*. A plane $\{p \in \mathbb{R}^3 \,|\, p \cdot n = d\}$, where $n \in S^2$ is a unit normal and $d \in \mathbb{R}$ its directed distance from the origin, can be parametrized by
$$(u,v) \mapsto \sigma(u,v) := p_0 + u\,e_1 + v\,e_2,$$
where $p_0 \in \pi$ is some point and (e_1, e_2) is a basis of the linear subspace $\{n\}^\perp \subset \mathbb{R}^3$; σ is regular since $e_1 \times e_2 \neq 0$.

(2) *Sphere*. A common parametrization is given by
$$(u,v) \mapsto \sigma(u,v) := (\cos u \cos v, \cos u \sin v, \sin u);$$
but there is a problem: the parametrization ceases to be regular for $\cos u = 0$ and $\sin u = \pm 1$ ("north" and "south poles" of the sphere); this problem is symptomatic and cannot be resolved: *there is no regular parametrization of the whole sphere at once*. This shows a weakness of our definition of a surface and can only be resolved by a "better" definition.

Problem 1. Show that the (twice punctured) ellipsoid
$$E = \{(x,y,z) \,|\, (\tfrac{x}{a})^2 + (\tfrac{y}{b})^2 + (\tfrac{z}{c})^2 = 1, |z| < c\},$$
$a > b > c > 0$, is a surface by finding a regular (prove it) parametrization.

(3) *Hyperboloids*. The equations

$$(\tfrac{x}{a})^2 + (\tfrac{y}{b})^2 - (\tfrac{z}{c})^2 = \pm 1,$$

$a, b, c > 0$, each pose 1 constraint on the 3 coordinates of space, hence we may expect them to describe surfaces;
<u>in the $+1$-case</u>

$$(u, v) \mapsto \sigma(u, v) := (a \cosh u \cos v, b \cosh u \sin v, c \sinh u)$$

gives a (regular) parametrization, hence the equation defines a surface: the 1-*sheeted hyperboloid*;
<u>in the -1-case</u>, the described set is a 2-*sheeted hyperboloid*, which has two connected components, hence does not have a chance of being a surface in our sense — however, each component is, as these can be (regularly) paramtrized by

$$(u, v) \mapsto \sigma^{\pm}(u, v) := (x, y, \pm c\sqrt{1 + (\tfrac{x}{a})^2 + (\tfrac{y}{b})^2});$$

note that

$$(u, v) \mapsto (a \sinh u \cos v, b \sinh u \sin v, \pm c \cosh u)$$

does *not* give regular parametrizations.

<u>*Problem 2*</u>. Consider $T^2 := \{(x, y, z) \,|\, (\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2\}$, where $0 < r < R$. Show that the torus T^2 is a surface.

Def. *A* reparametrization *of a parametrized surface* $\sigma : U \to \mathbb{R}^3$ *is a new parametrized surface*

$$\tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma(\varphi(\tilde{u}, \tilde{v})), \quad \text{where} \quad \varphi : \tilde{U} \to U \ \text{ is a diffeomorphism,}$$

i.e., a smooth bijection with smooth inverse φ^{-1}.

<u>*Remark*</u>. If $\tilde{\sigma}(\tilde{u}, \tilde{v}) = \sigma(\varphi(\tilde{u}, \tilde{v}))$, where $\varphi(\tilde{u}, \tilde{v}) = (u(\tilde{u}, \tilde{v}), v(\tilde{u}, \tilde{v}))$, then

$$\tilde{\sigma}_{\tilde{u}} \times \tilde{\sigma}_{\tilde{v}} = \det \begin{pmatrix} u_{\tilde{u}} & u_{\tilde{v}} \\ v_{\tilde{u}} & v_{\tilde{v}} \end{pmatrix} (\sigma_u \times \sigma_v) \circ \varphi;$$

thus a reparametrization of a parametrized surface is regular.

Rem & Def. *At each point $\sigma(u,v)$, a surface has a* tangent plane

$$\mathcal{T}(u,v) = \sigma(u,v) + d_{(u,v)}\sigma(\mathbb{R}^2) = \sigma(u,v) + \{(\sigma_u \times \sigma_v)(u,v)\}^\perp,$$

which inherits a natural linear structure (with origin $\sigma(u,v) \simeq 0$) from the 2-dimensional vector subspace $\{(\sigma_u \times \sigma_v)(u,v)\}^\perp \subset \mathbb{R}^3$, the tangent space *of the surface at $\sigma(u,v)$.*
As $d_{(u,v)}\sigma$ injects

$$d_{(u,v)}\sigma : \mathbb{R}^2 \to \mathbb{R}^3$$

can be used to identify tangent vectors *with vectors in $\mathbb{R}^2$.*

Next we introduce a way to measure length and angles of tangent vectors:

Lemma & Def. *Let $U \ni (u,v) \mapsto \sigma(u,v) \in \mathbb{R}^3$ be a parametrized surface; then*

$$\mathrm{I} := d\sigma \cdot d\sigma$$

defines a positive definite, symmetric bilinear form for each $(u,v) \in U$. I *is the* induced metric *or* first fundamental form *of σ.*

<u>*Notation*</u>. The first fundamental form is often written as

$$\mathrm{I} = E\,du^2 + 2F\,du\,dv + G\,dv^2 \quad \text{or} \quad \mathrm{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

where $E := |\sigma_u|^2$, $F := \sigma_u \cdot \sigma_v$ and $G := |\sigma_v|^2$.

<u>*Remark*</u>. If $(u,v) \in U$ and $w_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \in \mathbb{R}^2$, $i = 1, 2$, are two vectors then

$$\begin{aligned} &\mathrm{I}|_{(u,v)}(w_1, w_2) \\ &= d_{(u,v)}\sigma(w_1) \cdot d_{(u,v)}\sigma(w_2) \\ &= E(u,v)\, x_1x_2 + F(u,v)\,(x_1y_2 + x_2y_1) + G(u,v)\, y_1y_2 \\ &= \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}^t \begin{pmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}. \end{aligned}$$

You should think of the first fundamental form as the $\mathbb{R}^3$-scalar product restricted, at each point, to the tangent space of the surface; the above form $E\,du^2 + 2F\,du\,dv + G\,dv^2$ is its representation in the coordinates (u,v) of the surface.

Proof. Clearly, the first fundamental form is a symmetric bilinear form on $\mathbb{R}^2$ at each $(u,v) \subset U$. If $(u,v) \in U$ and $w \in \mathbb{R}^2$ then

$$0 = \mathrm{I}_{(u,v)}(w,w) = |d_{(u,v)}\sigma(w)|^2 \quad \Rightarrow \quad 0 = w$$

since $d_{(u,v)}\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ injects; hence $\mathrm{I}_{(u,v)}$ is positive definite. ∎

Examples.

(1) *Cylinder*. Let $(u,v) \mapsto (x(u), y(u), v)$; then

$$E(u,v) = (x'^2 + y'^2)(u), \quad F(u,v) = 0, \quad \text{and} \quad G(u,v) = 1.$$

In particular, if the planar curve $u \mapsto (x(u), y(u), 0)$ is parametrized by arc length then $E = G = 1$ and $F = 0$, that is,

$$\mathrm{I} = du^2 + dv^2$$

and the surface is parametrized *isometrically*.

(2) *Helicoid*. This is a regular surface, traced out by a straight normal line moved along a helix:

$$(u,v) \mapsto \sigma(u,v) = (\sinh u \cos v, \sinh u \sin v, v)$$

so that

$$\sigma_u(u,v) = (\cosh u \cos v, \cosh u \sin v, 0)$$
$$\sigma_v(u,v) = (-\sinh u \sin v, \sinh u \cos v, 1)$$

and

$$\mathrm{I}|_{(u,v)} = \cosh^2 u\, du^2 + (1 + \sinh^2 u)\, dv^2 = \cosh^2 u\,(du^2 + dv^2);$$

that is, the helcoid is parametrized *conformally*.

Problem 3. Compute the induced metric of the catenoid

$$(u,v) \mapsto \sigma(u,v) := (\cosh u\ \cos v, \cosh u\ \sin v, u).$$

Def. *A surface parametrization $(u,v) \mapsto \sigma(u,v)$ is called* conformal *if $E = G$ and $F = 0$. It is called* isometric *if $\mathrm{I} = du^2 + dv^2$.*

Problem 4. Find a conformal parametrization $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ of the unit sphere with its north pole removed [Hint: consider the "stereographic projection" from the north pole, obtained by drawing a line through the north pole and any given point on the sphere to obtain a point in $\mathbb{R}^2$ as the intersection of this line with the equator plane of the sphere.]

Remark. A parametrization is conformal iff it preserves angles, i.e., if the angle of any two tangent vectors of the surface can be measured in $\mathbb{R}^2$.

Problem 5. Convince yourself that a surface is parametrized conformally if and only if the parametrization preserves angles.

Remark. Isometric parametrizations are very special and do *not* normally exist (not even locally) — in contrast to curves which can always be parametrized by arc-length. We shall later see what the obstruction is.

In contrast to this:

Thm. *Any surface can (locally) be conformally (re-)parametrized.*

Proof. ...is beyond this text; a beautiful proof uses technology from Complex Analysis. ∎

2.2 Gauss map & Shape operator

Recall. A surface has a *tangent plane* at each point, in terms of a parametrization $(u, v) \mapsto \sigma(u, v)$ this is given by

$$\mathcal{T}(u, v) = \sigma(u, v) + \{(\sigma_u \times \sigma_v)(u, v)\}^{\perp},$$

where $\{(\sigma_u \times \sigma_v)(u, v)\}^{\perp} \subset \mathbb{R}^3$ is its *tangent space* (where *tangent vectors* "live");

$$n(u, v) := \tfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}(u, v)$$

is a unit normal vector to the tangent plane/space at $\sigma(u, v)$.

Note that a unit normal vector of $\mathcal{T}(u, v)$ is unique up to sign.

Def. *The unit (normal) vector field* $n := \frac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|} = \frac{\sigma_u \times \sigma_v}{\sqrt{EG - F^2}}$ *is called the* Gauss map *of the parametrized surface* $(u, v) \mapsto \sigma(u, v)$.

Problem 6. Suppose a surface is given implicitely by $F(x, y, z) = 0$, i.e., a parametrization $(u, v) \mapsto \sigma(u, v)$ satisfies $F \circ \sigma \equiv 0$. Show that its Gauss map is given by $n = \pm \frac{\text{grad}\, F \circ \sigma}{|\,\text{grad}\, F \circ \sigma|}$.

Example. Consider a *surface of revolution*

$$(u, v) \mapsto \sigma(u, v) = (r(u) \cos v, r(u) \sin v, h(u)).$$

Each meridian curve $v \equiv const$ is the orthogonal intersection of the surface with the plane $x \sin v = y \cos v$; hence n is obtained by rotating the unit tangent vector field of the meridian curve by 90°:

$$(u, v) \mapsto n(u, v) = \frac{1}{\sqrt{r'^2(u) + h'^2(u)}}(-h'(u) \cos v, -h'(u) \sin v, r'(u)).$$

Remark. The Gauss map of a parametrized surface is a geometric object: if we apply a Euclidean motion,

$$\sigma \to \tilde{\sigma} = A\sigma + c \quad \Rightarrow \quad n \to \tilde{n} = An,$$

that is, the Gauss map rotates with the surface.

We may run into problems when the surface is non-orientable, that is, if we cannot choose a unit normal vector field *globally*: a *Möbius strip* provides a simple example.

Problem 7. Let $r > 0$ and define a parametrization of a Möbius strip by

$$\sigma(u, v) := r\,(\cos 2u, \sin 2u, 0) + v\,(\cos u \cos 2u, \cos u \sin 2u, \sin u).$$

Show that $\sigma(u + \pi, 0) = \sigma(u, 0)$ but $n(u + \pi, 0) = -n(u, 0)$.

Agreement. All our surfaces will be orientable.

For Frenet curves, curvature measured how the principal normal field changes. For surfaces, the Gauss map may change differently in different directions — we pick up a "second fundamental form":

Def. *Given a parametrized surface σ with Gauss map n,*

$$\mathrm{I\!I} := -dn \cdot d\sigma$$

is called the second fundamental form *of σ.*

Lemma. II *is a symmetric bilinear form for each (u, v).*

Proof. Clearly, $\mathrm{II}_{(u,v)}$ is a bilinear form on $\mathbb{R}^2$, and

$$\begin{aligned}\mathrm{II}(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)) - \mathrm{II}(\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)) &= -n_u \cdot \sigma_v + n_v \cdot \sigma_u \\ &= n \cdot \sigma_{vu} - n \cdot \sigma_{uv} \\ &= 0\end{aligned}$$

showing that it symmetric. ∎

<u>*Notation*</u>. The second fundamental form is often written as

$$\mathrm{II} = e\, du^2 + 2f\, du\, dv + g\, dv^2 \quad \text{or} \quad \mathrm{II} = \left(\begin{smallmatrix} e & f \\ f & g \end{smallmatrix}\right),$$

where $e := -n_u \cdot \sigma_u$, $f := -n_u \cdot \sigma_v = -n_v \cdot \sigma_u$ and $g := -n_v \cdot \sigma_v$.

<u>*Problem 8*</u>. Investigate how the first and second fundamental forms change under Euclidean motion and under reparametrization.

<u>*Example*</u>. For the helicoid $(u, v) \mapsto \sigma(u, v) = (\sinh u \cos v, \sinh u \sin v, v)$ we have

$$\begin{aligned}\sigma_u(u, v) &= (\cosh u \cos v, \cosh u \sin v, 0) \\ \sigma_v(u, v) &= (-\sinh u \sin v, \sinh u \cos v, 1)\end{aligned}$$

so that its Gauss map

$$(u, v) \mapsto n(u, v) = \tfrac{1}{\cosh u}(\sin v, -\cos v, \sinh u)$$

and its second fundamental form becomes

$$\mathrm{II}|_{(u,v)} = -2du\, dv.$$

Lemma & Def. *Let σ be a parametrized surface with Gauss map n; then*

$$d_{(u,v)}n : \mathbb{R}^2 \to \{n(u, v)\}^{\perp}$$

takes values in the tangent space of the surface at $\sigma(u, v)$. Hence

$$(u, v) \mapsto \mathrm{S}_{(u,v)} := -d_{(u,v)}n \circ (d_{(u,v)}\sigma)^{-1} \in \mathrm{End}(\{n(u, v)\}^{\perp})$$

will be called the shape operator *or* Weingarten tensor *of* σ *at* $\sigma(u,v)$.

Proof. Since $|n|^2 \equiv 1$ we find
$$0 = d(|n|^2) = 2n \cdot dn,$$
that is, $d_{(u,v)}n : \mathbb{R}^2 \to \{n(u,v)\}^\perp \subset \mathbb{R}^3$ takes values in the tangent space of σ at $\sigma(u,v)$; also, as $d_{(u,v)}\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ injects it is an isomorphism
$$d_{(u,v)}\sigma : \mathbb{R}^2 \to \{n(u,v)\}^\perp \subset \mathbb{R}^3$$
onto the tangent space, hence can be inverted. As both maps $(d_{(u,v)}\sigma)^{-1}$ and $d_{(u,v)}n$ are linear, $\mathrm{S}_{(u,v)}$ is as well, showing that S is well defined. ■

<u>*Remark*</u>. Since (σ_u, σ_v) is a basis of $\{n\}^\perp$ at each point, we can determine the shape operator by its values on this basis:
$$\begin{array}{rcccl} \mathrm{S}|_{(u,v)}\sigma_u(u,v) & = & -d_{(u,v)}n \begin{pmatrix} 1 \\ 0 \end{pmatrix} & = & -n_u(u,v), \\ \mathrm{S}|_{(u,v)}\sigma_v(u,v) & = & -d_{(u,v)}n \begin{pmatrix} 0 \\ 1 \end{pmatrix} & = & -n_v(u,v); \end{array}$$
The inverse $(d_{(u,v)}\sigma)^{-1}$ in the definition of $\mathrm{S}|_{(u,v)}$ can also be interpreted as the Moore-Penrose pseudoinverse:
$$\mathrm{S} = -dn \circ ((d\sigma)^t d\sigma)^{-1}(d\sigma)^t.$$
However: *this does not provide a useful matrix representation!*

<u>*Example*</u>. Let $(u,v) \mapsto \sigma(u,v) = (r(u)\cos v, r(u)\sin v, h(u))$ be a surface of revolution with arc-length parametrized meridian, $r'^2 + h'^2 \equiv 1$. Then its Gauss map is
$$n(u,v) = (-h'(u)\cos v, -h'(u)\sin v, r'(u));$$
hence
$$d_{(u,v)}\sigma \simeq \begin{pmatrix} r'(u)\cos v & -r(u)\sin v \\ r'(u)\sin v & r(u)\cos v \\ h'(u) & 0 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^3$$
and
$$d_{(u,v)}n \simeq \begin{pmatrix} -h''(u)\cos v & h'(u)\sin v \\ -h''(u)\sin v & -h'(u)\cos v \\ r''(u) & 0 \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^3$$
both take values in $\{n(u,v)\}^\perp \subset \mathbb{R}^3$: note that $r'r'' + h'h'' = 0$ so that
$$n_u + (r'h'' - r''h')\,\sigma_u = 0; \text{ clearly } n_v + \tfrac{h'}{r}\,\sigma_v = 0.$$

Hence w.r.t. the basis $(\sigma_u(u,v),\sigma_v(u,v))$ of $\{n(u,v)\}^\perp$ we obtain the matrix representation

$$\mathrm{S}|_{(u,v)} \simeq \begin{pmatrix} (r'h''-r''h')(u) & 0 \\ 0 & \frac{h'}{r}(u) \end{pmatrix}.$$

<u>*Matrix representation*</u>. Writing $\mathrm{S} \simeq \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ so that

$$(n_u, n_v) = -(\sigma_u, \sigma_v) \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix},$$

we find

$$\begin{pmatrix} e & f \\ f & g \end{pmatrix} = -(\sigma_u,\sigma_v)^t(n_u,n_v) = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}.$$

Hence

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \frac{1}{EG-F^2} \begin{pmatrix} Ge-Ff & Gf-Fg \\ Ef-Fe & Eg-Ff \end{pmatrix}$$

or, equivalently,

$$\begin{aligned} 0 &= n_u + \tfrac{1}{EG-F^2}\{(Ge-Ff)\,\sigma_u + (Ef-Fe)\,\sigma_v\}, \\ 0 &= n_v + \tfrac{1}{EG-F^2}\{(Gf-Fg)\,\sigma_u + (Eg-Ff)\,\sigma_v\}. \end{aligned}$$

<u>*Remark*</u>. Note that, given the first fundamental form, S can be computed from II and vice versa.

<u>*Problem 9*</u>. Compute Gauss map and shape operator of the helicoid.

Lemma. *S is a symmetric endomorphism of tangent spaces.*

Proof. This follows directly from $\mathrm{S}\sigma_u \cdot \sigma_v = f = \sigma_u \cdot \mathrm{S}\sigma_v$. ∎

<u>*Warning*</u>. Even though S is a symmetric endomorphism, its matrix representation usually is *not* ((σ_u,σ_v) is not orthonormal in general).

Def. *Let S denote the shape operator of $(u,v) \mapsto \sigma(u,v)$. Then:*

- $H := \frac{1}{2}\,\mathrm{tr}\,\mathrm{S} = \frac{Eg-2Ff+eG}{2(EG-F^2)}$ *is the* mean curvature*;*
- $K := \det \mathrm{S} = \frac{eg-f^2}{EG-F^2}$ *is the* Gauss curvature*; and*

- *the eigenvalues* $\kappa_i = H \pm \sqrt{H^2 - K}$ *of* S *are the* principal curvatures *of the surface, and*
  its eigendirections are the curvature directions *of* σ*.*

Remark. Note that $H = \frac{1}{2}(\kappa_1 + \kappa_2)$ — hence "mean curvature".

Remark. The shape operator and curvatures are geometric objects:

- if $\tilde{\sigma} = \sigma \circ \varphi$ is a reparametrization of the surface and $\tilde{n} = n \circ \varphi$ then
  $\tilde{\mathrm{S}}_{(\tilde{u},\tilde{v})} = -(d_{\varphi(\tilde{u},\tilde{v})} n \circ d_{(\tilde{u},\tilde{v})}\varphi) \circ (d_{\varphi(\tilde{u},\tilde{v})}\sigma \circ d_{(\tilde{u},\tilde{v})}\varphi)^{-1} = \mathrm{S}_{\varphi(\tilde{u},\tilde{v})}$
  so that $\tilde{H} = H \circ \varphi$, $\tilde{K} = K \circ \varphi$, etc.
  Note, however, that a reparametrization changes the basis (σ_u, σ_v) of the tangent space and, consequently, the matrix representation of S *does change* under reparametrization.

- if $\sigma \to \tilde{\sigma} = A\sigma + c$, where $A \in SO(3)$ and $c \in \mathbb{R}^3$, is a Euclidean motion of σ then
  $$\tilde{\mathrm{S}} = -(A\,dn) \circ (A\,d\sigma)^{-1} = A \circ \mathrm{S} \circ A^{-1}$$
  so that the curvatures remain invariant but the curvature directions "rotate with the surface".

Problem 10. Compute mean and principal curvatures of the helicoid.

Def. *A point* $\sigma(u,v)$ *of a surface is called*

- umbilic *if* $\kappa_1(u,v) = \kappa_2(u,v)$*, i.e., if* $(H^2 - K)(u,v) = 0$*;*
- flat point *if* $\mathrm{S}|_{(u,v)} = 0$*.*

Example. Suppose $(u,v) \mapsto \sigma(u,v)$ takes values in a fixed plane
$$\pi = \{p \in \mathbb{R}^3 \,|\, (p - p_0) \cdot m = 0\}.$$
Then $(\sigma - p_0) \cdot m \equiv 0$, hence $\sigma_u, \sigma_v \perp m$ so that $n \equiv \pm m$ and $\mathrm{S} \equiv 0$. Thus every point is a flat point.

Problem 11. Prove that all points of a sphere of radius $r > 0$ are umbilics and compute its Gauss curvature.

Lemma. *If* $(u,v) \mapsto \sigma(u,v)$ *has no umbilics then* $F = f = 0$ *if and only if*

$$0 = n_u + \kappa_1 \sigma_u = n_v + \kappa_2 \sigma_v,$$

that is, the parametrization σ *diagonalizes the shape operator.*

Proof. Clearly, if $F = f = 0$ then

$$\mathrm{S} = \left(\begin{smallmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{smallmatrix}\right) \quad \text{with} \quad \kappa_1 = \tfrac{e}{E} \text{ and } \kappa_2 = \tfrac{g}{G}.$$

Conversely, suppose that $\mathrm{S} = \left(\begin{smallmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{smallmatrix}\right)$ with $\kappa_1 \neq \kappa_2$; as S is symmetric

$$0 = (\mathrm{S}\sigma_u) \cdot \sigma_v - \sigma_u \cdot (\mathrm{S}\sigma_v) = (\kappa_1 - \kappa_2)\, F,$$

that is, $\sigma_u \perp \sigma_v$; hence $f = -n_u \cdot \sigma_v = \kappa_1\, F = 0$ as well. ∎

Remark. In this proof we used the *symmetry of* S,

$$(\mathrm{S}\sigma_u) \cdot \sigma_v = \sigma_u \cdot (\mathrm{S}\sigma_v),$$

to show that the curvature directions intersect orthogonally (if $\kappa_1 \neq \kappa_2$).

Def. $(u,v) \mapsto \sigma(u,v)$ *is a* curvature line parametrization *if* $F = f = 0$.

Thm. *Any surface (locally) admits, away from umbilics, a curvature line parametrization.*

Proof. ...is beyond the text — as for the existence of conformal parametrization. ∎

Problem 12. Find a curvature line reparametrization for the helicoid.

2.3 Covariant differentiation & Curvature tensor

Similarly to the normal connection of a curve we define a "connection" for tangential vector fields of a surface:

Def. *Let* $(u,v) \mapsto \xi(u,v)$ *be a* tangential vector field *along a surface parametrization* $(u,v) \mapsto \sigma(u,v)$*, i.e.,* $\xi(u,v) \perp n(u,v)$ *for all* (u,v)*.*

$$\nabla\xi := (d\xi)^T = d\xi - (d\xi \cdot n)\, n$$

is called its covariant derivative*,* ∇ *is the* Levi-Civita connection *of* σ*.*

Lemma. *The Levi-Civita connection satisfies the Leibniz rule,*

$$\nabla(\alpha\xi) = d\alpha\,\xi + \alpha\nabla\xi \ \text{ for any function } \alpha,$$

and is metric*,*

$$d(\xi \cdot \eta) = (\nabla\xi)\cdot\eta + \xi\cdot(\nabla\eta).$$

Proof. The Leibniz rule: if α is some function and $\xi \perp n$ then

$$\nabla(\alpha\xi) = d\alpha\,\xi + \alpha d\xi - ((d\alpha\,\xi + \alpha d\xi)\cdot n)\, n = d\alpha\,\xi + \alpha\nabla\xi.$$

∇ is metric: if $\xi, \eta \perp n$ then

$$(\nabla\xi)\cdot\eta + \xi\cdot(\nabla\eta) = (d\xi)\cdot\eta + \xi\cdot(d\eta) = d(\xi\cdot\eta)$$

since $(\nabla\xi)\cdot\eta = (d\xi - (d\xi\cdot n)\, n)\cdot\eta = (d\xi)\cdot\eta$. ∎

Remark. Since (σ_u, σ_v) is a basis of the tangent space $\{n\}^\perp$ at every point we can write $\nabla\xi$ in terms of these basis fields. In particular, we can write

$$\begin{array}{rclcl}
\nabla_{\frac{\partial}{\partial u}}\sigma_u & = & \sigma_{uu} - e\,n & = & \Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v, \\
\nabla_{\frac{\partial}{\partial v}}\sigma_u & = & \sigma_{uv} - f\,n & = & \Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_v, \\
\nabla_{\frac{\partial}{\partial u}}\sigma_v & = & \sigma_{vu} - f\,n & = & \Gamma^1_{21}\sigma_u + \Gamma^2_{21}\sigma_v, \\
\nabla_{\frac{\partial}{\partial v}}\sigma_v & = & \sigma_{vv} - g\,n & = & \Gamma^1_{22}\sigma_u + \Gamma^2_{22}\sigma_v.
\end{array} \qquad (\star)$$

Then, writing an arbitrary tangential vector field $\xi = \alpha\sigma_u + \beta\sigma_v$ in terms of the basis fields,

$$\begin{array}{rcl}
\nabla_{\frac{\partial}{\partial u}}\xi & = & (\alpha_u + \alpha\Gamma^1_{11} + \beta\Gamma^1_{21})\,\sigma_u + (\beta_u + \alpha\Gamma^2_{11} + \beta\Gamma^2_{21})\,\sigma_v, \\
\nabla_{\frac{\partial}{\partial v}}\xi & = & (\alpha_v + \alpha\Gamma^1_{12} + \beta\Gamma^1_{22})\,\sigma_u + (\beta_v + \alpha\Gamma^2_{12} + \beta\Gamma^2_{22})\,\sigma_v.
\end{array} \qquad (\star\star)$$

Def. Γ_{ij}^k *are called the* Christoffel symbols *of* σ.

<u>*Matrix representation*</u>. The covariant derivatives $\nabla_{\frac{\partial}{\partial u}}$ and $\nabla_{\frac{\partial}{\partial v}}$ are *differential operators* (not endomorphisms!); nevertheless they admit a representation using matrices: $(\star\star)$ reads

$$\nabla_{\frac{\partial}{\partial u}} d\sigma(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}) = d\sigma((\tfrac{\partial}{\partial u} + \Gamma_1)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}) \text{ with } \Gamma_1 := \begin{pmatrix}\Gamma_{11}^1 & \Gamma_{21}^1\\ \Gamma_{11}^2 & \Gamma_{21}^2\end{pmatrix}$$

and

$$\nabla_{\frac{\partial}{\partial v}} d\sigma(\begin{pmatrix}\alpha\\ \beta\end{pmatrix}) = d\sigma((\tfrac{\partial}{\partial v} + \Gamma_2)\begin{pmatrix}\alpha\\ \beta\end{pmatrix}) \text{ with } \Gamma_2 := \begin{pmatrix}\Gamma_{12}^1 & \Gamma_{22}^1\\ \Gamma_{12}^2 & \Gamma_{22}^2\end{pmatrix};$$

and, in particular, with $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}1\\ 0\end{pmatrix}$ and $\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}0\\ 1\end{pmatrix}$ the defining equations $(\star)$ are recovered.

Lemma. $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Proof. This is because $\sigma_{uv} = \sigma_{vu}$ so that $\nabla_{\frac{\partial}{\partial v}} \sigma_u = \nabla_{\frac{\partial}{\partial u}} \sigma_v$. ∎

Lemma (Koszul's formulas).

$$\begin{array}{rlrl}
\frac{1}{2}E_u = E\Gamma_{11}^1 + F\Gamma_{11}^2, & F_u - \frac{1}{2}E_v = F\Gamma_{11}^1 + G\Gamma_{11}^2;\\
\frac{1}{2}E_v = E\Gamma_{12}^1 + F\Gamma_{12}^2, & \frac{1}{2}G_u = F\Gamma_{12}^1 + G\Gamma_{12}^2;\\
F_v - \frac{1}{2}G_u = E\Gamma_{22}^1 + F\Gamma_{22}^2, & \frac{1}{2}G_v = F\Gamma_{22}^1 + G\Gamma_{22}^2.
\end{array}$$

Proof. Multiplying the first equation of $(\star)$ by σ_u and σ_v, respectively, we obtain the first two equations:

$$\begin{array}{rclcl}
E\Gamma_{11}^1 + F\Gamma_{11}^2 & = & \sigma_u \cdot \nabla_{\frac{\partial}{\partial u}} \sigma_u & = & \frac{1}{2}\frac{\partial}{\partial u}(\sigma_u \cdot \sigma_u),\\
F\Gamma_{11}^1 + G\Gamma_{11}^2 & = & \sigma_v \cdot \nabla_{\frac{\partial}{\partial u}} \sigma_u & = & \frac{\partial}{\partial u}(\sigma_u \cdot \sigma_v) - \sigma_u \cdot \nabla_{\frac{\partial}{\partial u}} \sigma_v\\
 & & & = & \frac{\partial}{\partial u}(\sigma_u \cdot \sigma_v) - \sigma_u \cdot \nabla_{\frac{\partial}{\partial v}} \sigma_u\\
 & & & = & \frac{\partial}{\partial u}(\sigma_u \cdot \sigma_v) - \frac{1}{2}\frac{\partial}{\partial v}(\sigma_u \cdot \sigma_u).
\end{array}$$

The other equations are obtained similarly. ∎

Cor. *The covariant derivative ∇ depends on I only.*

Proof. As $EG - F^2 \neq 0$ we can solve Koszul's formulas for the Γ^k_{ij}:

$$\begin{pmatrix} \Gamma^1_{11} \\ \Gamma^2_{11} \end{pmatrix} = \frac{1}{2}\,\frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}\begin{pmatrix} E_u \\ 2F_u - E_v \end{pmatrix};$$

the other Γ^k_{ij}'s can be computed from I in a similar way. ∎

<u>*Example*</u>. If $(u,v) \mapsto \sigma(u,v)$ is an isometric parametrization, that is, $E = G = 1$ and $F = 0$, then all $\Gamma^k_{ij} = 0$ by Koszul's formulas.

<u>*Problem 13*</u>. Compute the Christoffel symbols of a conformally parametrized surface.

Using the Levi-Civita connection we introduce the curvature tensor; as a consequence of the previous corollary, it will only depend on I as well:

Def. *Let $(u,v) \mapsto \xi(u,v)$ be a tangential vector field along a surface parametrization $(u,v) \mapsto \sigma(u,v)$ and define*

$$\mathrm{R}\xi := \nabla_{\frac{\partial}{\partial u}} \nabla_{\frac{\partial}{\partial v}} \xi - \nabla_{\frac{\partial}{\partial v}} \nabla_{\frac{\partial}{\partial u}} \xi;$$

R *is called the* curvature tensor *of σ.*

<u>*Remark*</u>. This is a simplified version of the "real" curvature tensor (which is sufficient in our setup of 2-dimensional surfaces though).

Lemma. *The curvature tensor R depends on I only.*

Proof. This follows directly from the corresponding property of ∇. ∎

Lemma. R *is a skew symmetric* tensor *on tangent spaces, i.e.,*

$$(\mathrm{R}\xi)\cdot\eta + \xi\cdot(\mathrm{R}\eta) = 0 \quad \textit{and} \quad \mathrm{R}(\alpha\xi) = \alpha\,\mathrm{R}\xi$$

for any function $(u,v) \mapsto \alpha(u,v)$.

Proof. To see skew symmetry observe that

$$(|\xi|^2)_{vu} = 2\xi \cdot \nabla_{\frac{\partial}{\partial u}} \nabla_{\frac{\partial}{\partial v}} \xi + 2(\nabla_{\frac{\partial}{\partial u}} \xi)\cdot(\nabla_{\frac{\partial}{\partial v}} \xi);$$

thus
$$0 = (|\xi|^2)_{vu} - (|\xi|^2)_{uv} = 2\xi \cdot \mathrm{R}\xi$$
and
$$0 = (\xi + \eta) \cdot \mathrm{R}(\xi + \eta) - \xi \cdot \mathrm{R}\xi - \eta \cdot \mathrm{R}\eta = \xi \cdot \mathrm{R}\eta + \eta \cdot \mathrm{R}\xi.$$
To see that R is a tensor compute
$$\begin{array}{rcl} \mathrm{R}(\alpha\xi) & = & \alpha_{vu}\xi + \alpha_v \nabla_{\frac{\partial}{\partial u}} \xi + \alpha_u \nabla_{\frac{\partial}{\partial v}} \xi + \alpha \nabla_{\frac{\partial}{\partial u}} \nabla_{\frac{\partial}{\partial v}} \xi \\ & - & \alpha_{uv}\xi - \alpha_u \nabla_{\frac{\partial}{\partial v}} \xi - \alpha_v \nabla_{\frac{\partial}{\partial u}} \xi - \alpha \nabla_{\frac{\partial}{\partial v}} \nabla_{\frac{\partial}{\partial u}} \xi \\ & = & \alpha\, \mathrm{R}\xi \end{array}$$
Hence, clearly, $\mathrm{R}_{(u,v)} \in \mathsf{End}(\{n(u,v\}^\perp)$. ∎

Lemma. $\mathrm{R} = \varrho\,(\sigma_u \wedge \sigma_v)$ *with some function* ϱ *and*
$$(\sigma_u \wedge \sigma_v)\,\xi := (\xi \cdot \sigma_u)\,\sigma_v - (\xi \cdot \sigma_v)\,\sigma_u.$$

Proof. The vector space of skew symmetric endomorphisms on a 2-dimensional vector space is 1-dimensional; as both $(\sigma_u \wedge \sigma_v)_{(u,v)} \neq 0$ and $\mathrm{R}_{(u,v)}$ are skew symmetric endomorphisms of $\{n(u,v)\}^\perp$ they must be linearly dependent. ∎

Matrix representation. By the previous lemma
$$\mathrm{R}\sigma_u = -\varrho\,\{F\sigma_u - E\sigma_v\} \quad \text{and} \quad \mathrm{R}\sigma_v = -\varrho\,\{G\sigma_u - F\sigma_v\}$$
or, in matrix representation with respect to the basis (σ_u, σ_v),
$$\mathrm{R} \simeq \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \varrho \begin{pmatrix} -F & -G \\ E & F \end{pmatrix} = \begin{pmatrix} 0 & -\varrho \\ \varrho & 0 \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}.$$
On the other hand, by the definition of R and the matrix representation for the covariant derivative,
$$\begin{array}{rcl} \mathrm{R}\sigma_u & = & d\sigma(((\frac{\partial}{\partial u} + \Gamma_1)(\frac{\partial}{\partial v} + \Gamma_2) - (\frac{\partial}{\partial v} + \Gamma_2)(\frac{\partial}{\partial u} + \Gamma_1)) \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \\ & = & d\sigma((\Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2]) \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \text{ and} \\ \mathrm{R}\sigma_v & = & d\sigma((\Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2]) \begin{pmatrix} 0 \\ 1 \end{pmatrix}), \end{array}$$
where $[X, Y] := XY - YX$ denotes the *commutator*; hence, in terms of matrices,
$$\mathrm{R} \simeq \Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2]$$
and ϱ can be computed from ∇ and I and, consequently, from I.

Problem 14. Show that $\mathrm{R} = -\frac{1}{r^2}\sigma_u \wedge \sigma_v$ if σ takes values in the sphere $S^2(r)$ of radius $r > 0$, i.e., if $|\sigma|^2 \equiv r^2$. [Hint: compute $\nabla\sigma_u$ directly from the definition, using $n = \pm\frac{1}{r}\sigma$.]

2.4 Gauss-Weingarten & Gauss-Codazzi equations

Using the notions from the previous sections, fundamental forms and covariant derivative, we can now formulate the structure equations for the *frame* $F := (\sigma_u, \sigma_v, n)$ of a parametrized surface:

Gauss-Weingarten equations. *If* $(u,v) \mapsto \sigma(u,v)$ *is a parametrized surface with Gauss map* $(u,v) \mapsto n(u,v)$ *then*

$$\begin{array}{rclclclcl}
\sigma_{uu} &=& \nabla_{\frac{\partial}{\partial u}}\sigma_u + e\,n &=& \Gamma^1_{11}\sigma_u &+& \Gamma^2_{11}\sigma_v &+& e\,n,\\
\sigma_{uv} &=& \nabla_{\frac{\partial}{\partial v}}\sigma_u + f\,n &=& \Gamma^1_{12}\sigma_u &+& \Gamma^2_{12}\sigma_v &+& f\,n,\\
\sigma_{vv} &=& \nabla_{\frac{\partial}{\partial v}}\sigma_v + g\,n &=& \Gamma^1_{22}\sigma_u &+& \Gamma^2_{22}\sigma_v &+& g\,n;
\end{array}$$

and

$$\begin{array}{rclcl}
n_u &=& -\mathrm{S}\sigma_u &=& -\frac{1}{EG-F^2}\{(Ge-Ff)\,\sigma_u + (Ef-Fe)\,\sigma_v\},\\
n_v &=& -\mathrm{S}\sigma_v &=& -\frac{1}{EG-F^2}\{(Gf-Fg)\,\sigma_u + (Eg-Ff)\,\sigma_v\}.
\end{array}$$

Def. *The* covariant derivative of the shape operator S *is defined by*

$$(\nabla\mathrm{S})\,\xi := \nabla(\mathrm{S}\xi) - \mathrm{S}(\nabla\xi),$$

where ξ *is a tangential vector field.*

Lemma. $\nabla\mathrm{S}$ *is a tensor, i.e., for any function* $(u,v) \mapsto \alpha(u,v)$

$$(\nabla\mathrm{S})(\alpha\xi) = \alpha\,(\nabla\mathrm{S})\xi.$$

Proof. This is a straightforward computation:

$$\begin{array}{rcl}
(\nabla_{\frac{\partial}{\partial u}}\mathrm{S})(\alpha\xi) &=& \nabla_{\frac{\partial}{\partial u}}(\alpha\mathrm{S}\xi) - \mathrm{S}\nabla_{\frac{\partial}{\partial u}}(\alpha\xi)\\
&=& \alpha_u\,\mathrm{S}\xi + \alpha\nabla_{\frac{\partial}{\partial u}}\mathrm{S}\xi - \mathrm{S}(\alpha_u\xi + \alpha\nabla_{\frac{\partial}{\partial u}}\xi)\\
&=& \alpha\,(\nabla_{\frac{\partial}{\partial u}}\mathrm{S})\,\xi.
\end{array}$$

A similar computation works for $\nabla_{\frac{\partial}{\partial v}}\mathrm{S}$. ∎

Matrix representation. If $\Sigma = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$ denotes the matrix of S with respect to the basis (σ_u, σ_v) of the tangent spaces,

$$\mathrm{S}\xi = d\sigma(\Sigma \begin{pmatrix} \alpha \\ \beta \end{pmatrix}) \text{ for } \xi = \alpha\sigma_u + \beta\sigma_v = d\sigma(\begin{pmatrix} \alpha \\ \beta \end{pmatrix}),$$

then

$$\begin{array}{rcl} (\nabla_{\frac{\partial}{\partial u}} \mathrm{S})\xi & = & d\sigma((\frac{\partial}{\partial u} + \Gamma_1)(\Sigma \begin{pmatrix} \alpha \\ \beta \end{pmatrix})) - \Sigma\,(\frac{\partial}{\partial u} + \Gamma_1) \begin{pmatrix} \alpha \\ \beta \end{pmatrix}) \\ & = & d\sigma((\Sigma_u + [\Gamma_1, \Sigma]) \begin{pmatrix} \alpha \\ \beta \end{pmatrix})) \end{array}$$

and similarly for $\nabla_{\frac{\partial}{\partial v}} \mathrm{S}$.

Thus $\nabla_{\frac{\partial}{\partial u}} \mathrm{S}|_{(u,v)}, \nabla_{\frac{\partial}{\partial v}} \mathrm{S}|_{(u,v)} \in \mathrm{End}(\{n(u,v)\}^\perp)$ have matrix representations

$$\begin{array}{rcl} \nabla_{\frac{\partial}{\partial u}} \mathrm{S}|_{(u,v)} & \simeq & (\Sigma_u + [\Gamma_1, \Sigma])|_{(u,v)}, \\ \nabla_{\frac{\partial}{\partial u}} \mathrm{S}|_{(u,v)} & \simeq & (\Sigma_v + [\Gamma_2, \Sigma])|_{(u,v)}. \end{array}$$

Gauss-Codazzi equations. *For a parametrized surface* σ

(G) $\mathrm{R}\sigma_v \cdot \sigma_u = K\,(EG - F^2)$ — Gauss equation,

(C) $(\nabla_{\frac{\partial}{\partial u}} \mathrm{S})\,\sigma_v = (\nabla_{\frac{\partial}{\partial v}} \mathrm{S})\,\sigma_u$ — Codazzi equation.

Proof. First we consider the Codazzi equation:

$$(\nabla_{\frac{\partial}{\partial u}} \mathrm{S})\,\sigma_v = -(\nabla_{\frac{\partial}{\partial u}} n_v + \mathrm{S}\nabla_{\frac{\partial}{\partial u}} \sigma_v);$$

then $n_{uv} = n_{vu}$ and $\sigma_{uv} = \sigma_{vu}$ yields

$$\left.\begin{array}{l} \nabla_{\frac{\partial}{\partial u}} n_v = \nabla_{\frac{\partial}{\partial v}} n_u \\ \nabla_{\frac{\partial}{\partial u}} \sigma_v = \nabla_{\frac{\partial}{\partial v}} \sigma_u \end{array}\right\} \;\Rightarrow\; (\nabla_{\frac{\partial}{\partial u}} \mathrm{S})\,\sigma_v = (\nabla_{\frac{\partial}{\partial v}} \mathrm{S})\,\sigma_u.$$

For the Gauss equation we investigate $(\sigma_v)_{uv} = (\sigma_v)_{vu}$:

$$\begin{array}{rcccl} (\sigma_v)_{vu} & = & (\nabla_{\frac{\partial}{\partial v}} \sigma_v + g\,n)_u & = & (\nabla_{\frac{\partial}{\partial u}} \nabla_{\frac{\partial}{\partial v}} \sigma_v + g\,n_u) + (\ldots)\,n, \\ (\sigma_v)_{uv} & = & (\nabla_{\frac{\partial}{\partial u}} \sigma_v + f\,n)_v & = & (\nabla_{\frac{\partial}{\partial v}} \nabla_{\frac{\partial}{\partial u}} \sigma_v + f\,n_v) + (\ldots)\,n; \end{array}$$

then

$$0 = \mathrm{R}\sigma_v \cdot \sigma_u - (eg - f^2) = \mathrm{R}\sigma_v \cdot \sigma_u - K(EG - F^2)$$

by taking the inner product of the difference with σ_u. ∎

Problem 15. Prove: for a curvature line parametrization the Codazzi equation(s) reads

$$0 = \kappa_{1v} + \tfrac{E_v}{2E}(\kappa_1 - \kappa_2) = \kappa_{2u} - \tfrac{G_u}{2G}(\kappa_1 - \kappa_2).$$

Gauss' Theorema Egregium. *K depends only on* I.

Proof. By the Gauss equation

$$K = \tfrac{1}{EG-F^2}\,\mathrm{R}\sigma_v \cdot \sigma_u = \tfrac{\varrho}{EG-F^2}\,(\sigma_u \wedge \sigma_v)\sigma_v \cdot \sigma_u = -\varrho,$$

where ϱ can be computed from the Γ_{ij}^k and I, hence from I. ∎

Remark. Note that we have also shown that $\mathrm{R} = -K\,\sigma_u \wedge \sigma_v$.

Cor. *If a surface admits an isometric (re-) parametrization then, necessarily, $K \equiv 0$.*

Proof. For an isometric parametrization all $\Gamma_{ij}^k = 0$, hence $\mathrm{R} \equiv 0$ and, consequently, $K \equiv 0$. As the Gauss curvature is a geometric invariant of a surface, $\tilde{K} = K \circ \varphi$ for a reparametrization $\tilde{\sigma} = \sigma \circ \varphi$, we have $K \equiv 0$ as soon as a surface admits an isometric (re-) parametrization. ∎

Problem 16. Prove: for a conformally parametrized surface the Gauss equation reads

$$K = -\tfrac{1}{2E}\Delta \ln E.$$

Def. *A surface is called* totally umbilic *if every point is an umbilic.*

Remark. We already know: if $(u,v) \mapsto \sigma(u,v)$ takes values in a sphere or plane then it parametrizes a totally umbilic surface.

Thm. *A totally umbilic surface is (part of) a plane or a sphere.*

Proof. If $(u,v) \mapsto \sigma(u,v)$ is totally umbilic then, for all (u,v),

$$\mathrm{S}_{(u,v)} = \kappa(u,v)\,\mathrm{id}_{\{n(u,v)\}^\perp}$$

and the Codazzi equation reads

$$0 = (\nabla_{\frac{\partial}{\partial u}} \mathrm{S})\sigma_v - (\nabla_{\frac{\partial}{\partial v}} \mathrm{S})\sigma_u = \kappa_u \sigma_v - \kappa_v \sigma_u.$$

Hence $\kappa_u = \kappa_v \equiv 0$ so that $\kappa \equiv const.$

If $\kappa \equiv 0$ then $n \equiv const$ and the surface is part of a plane.

If $\kappa \equiv const \neq 0$ then $c := \sigma + \frac{1}{\kappa} n \equiv const$ and $|\sigma - c|^2 \equiv \frac{1}{\kappa^2}$, showing that σ takes values in a sphere of radius $\frac{1}{|\kappa|}$ centred at c. ■

2.5 Fundamental Theorem

Recall. The Gauss-Weingarten equations read

$$\begin{array}{rclrcl} \sigma_{uu} & = & \Gamma^1_{11}\sigma_u + \Gamma^2_{11}\sigma_v + e\,n; & \sigma_{uv} & = & \Gamma^1_{12}\sigma_u + \Gamma^2_{12}\sigma_v + f\,n, \\ \sigma_{vu} & = & \Gamma^1_{21}\sigma_u + \Gamma^2_{21}\sigma_v + f\,n; & \sigma_{vv} & = & \Gamma^1_{22}\sigma_u + \Gamma^2_{22}\sigma_v + g\,n, \\ -n_u & = & s_{11}\sigma_u + s_{21}\sigma_v; & -n_v & = & s_{12}\sigma_u + s_{22}\sigma_v. \end{array}$$

Lemma & Def. *$F = (\sigma_u, \sigma_v, n)$ is called an* (adapted) frame of the parametrized surface $(u, v) \mapsto \sigma(u, v)$; *the* structure equations *of F read*

$$F_u = F\Phi \quad \text{and} \quad F_v = F\Psi, \qquad (\star)$$

with

$$\Phi = \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{21} & -s_{11} \\ \Gamma^2_{11} & \Gamma^2_{21} & -s_{21} \\ e & f & 0 \end{pmatrix} \ \text{and} \ \Psi = \begin{pmatrix} \Gamma^1_{12} & \Gamma^1_{22} & -s_{12} \\ \Gamma^2_{12} & \Gamma^2_{22} & -s_{22} \\ f & g & 0 \end{pmatrix}.$$

Proof. $(\star)$ are just the Gauss-Weingarten equations. ■

Recall. The Gauss-Codazzi equations read

$$\mathrm{R} = -K\sigma_u \wedge \sigma_v \ \text{ and } \ (\nabla_{\frac{\partial}{\partial u}} \mathrm{S})\sigma_v = (\nabla_{\frac{\partial}{\partial v}} \mathrm{S})\sigma_u.$$

Given I and II (or I and S), all data required to check the Gauss and Codazzi equations can be computed: in matrix representation the Gauss equation reads

$$\mathrm{R} \simeq \begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

where the r_{ij} can be computed from the Christoffel symbols Γ_{ij}^k,

$$\mathrm{R} \simeq \Gamma_{2u} - \Gamma_{1v} + [\Gamma_1, \Gamma_2],$$

hence from I by using the Koszul's fomulas

$$\mathrm{I}\,\Gamma_1 = \begin{pmatrix} \frac{1}{2}E_u & \frac{1}{2}E_v \\ F_u - \frac{1}{2}E_v & \frac{1}{2}G_u \end{pmatrix} \text{ and } \mathrm{I}\,\Gamma_2 = \begin{pmatrix} \frac{1}{2}E_v & F_v - \frac{1}{2}G_u \\ \frac{1}{2}G_u & \frac{1}{2}G_v \end{pmatrix};$$

the Codazzi equation reads

$$(\Sigma_u + [\Gamma_1, \Sigma]) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\Sigma_v + [\Gamma_2, \Sigma]) \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where $\Sigma = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} e & f \\ f & g \end{pmatrix}$ is the matrix representing S. Since $\Gamma_{21}^k = \Gamma_{12}^k$, i.e., $\Gamma_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \Gamma_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, the Codazzi equation simplifies to

$$s_{2u} + \Gamma_1 s_2 = s_{1v} + \Gamma_2 s_1,$$

where $s_1 := \Sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} s_{11} \\ s_{21} \end{pmatrix}$ and $s_2 := \Sigma \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} s_{12} \\ s_{22} \end{pmatrix}$.

Lemma (Gauss-Codazzi as compatibility for Gauss-Weingarten). *Let $(u,v) \mapsto \Phi(u,v), \Psi(u,v)$ be given; assume $\mathrm{I}\Sigma = \mathrm{II}$ and Koszul's formulas. Then there is (locally) a solution $(u,v) \mapsto F(u,v) \in Gl(3)$ of the Gauss-Weingarten equations $(\star)$ if and only if the Gauss-Codazzi equations are satisfied.*

Proof. If the structure equations $(\star)$ hold then necessarily $F_{uv} = F_{vu}$, hence

$$0 = \Phi_v - \Psi_u - [\Phi, \Psi]; \qquad (\star\star)$$

conversely, if Φ and Ψ satisfy $(\star\star)$ then (by the Maurer-Cartan lemma) there is (locally) a solution $(u,v) \mapsto F(u,v) \in Gl(3)$ of the structure equations $(\star)$.

We compute $\Phi_v - \Psi_u - [\Phi, \Psi]$ using $\mathrm{I}\Sigma = \mathrm{II}$, so that we have matrix representations

$$\Phi = \begin{pmatrix} \Gamma_1 & -s_1 \\ (\mathrm{I}s_1)^t & 0 \end{pmatrix} \text{ and } \Psi = \begin{pmatrix} \Gamma_2 & -s_2 \\ (\mathrm{I}s_2)^t & 0 \end{pmatrix},$$

to obtain

$$\begin{aligned} &\Phi_v - \Psi_u - [\Phi, \Psi] \\ &\quad = \begin{pmatrix} (\Gamma_{1v} - \Gamma_{2u} - [\Gamma_1, \Gamma_2]) + (s_1 s_2^t - s_2 s_1^t)\,\mathrm{I} & (s_{2u} + \Gamma_1 s_2) - (s_{1v} + \Gamma_2 s_1) \\ ((\mathrm{I}s_1)_v^t - (\mathrm{I}s_1)^t \Gamma_2) - ((\mathrm{I}s_2)_u^t - (\mathrm{I}s_2)^t \Gamma_1) & (\mathrm{I}s_1)^t s_2 - (\mathrm{I}s_2)^t s_1 \end{pmatrix}. \end{aligned}$$

Now

- $(\mathrm{I}s_1)^t s_2 - (\mathrm{I}s_2)^t s_1 = s_1^t \mathrm{I}\, s_2 - s_2^t \mathrm{I}\, s_1 = 0$ by the symmetry of I;
- $s_1 s_2^t - s_2 s_1^t = \binom{s_{11}}{s_{21}}(s_{12}, s_{22}) - \binom{s_{12}}{s_{22}}(s_{11}, s_{21}) = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}$;
- $(\mathrm{I}s_2)_u^t - (\mathrm{I}s_2)^t \Gamma_1 = (s_{2u} + \Gamma_1 s_2)^t \mathrm{I} + s_2^t (\mathrm{I}_u - (\mathrm{I}\Gamma_1) - (\mathrm{I}\Gamma_1)^t)$ and, by Koszul's formulas, $(\mathrm{I}\Gamma_1) + (\mathrm{I}\Gamma_1)^t = \mathrm{I}_u$ so that
$$(\mathrm{I}s_2)_u^t - (\mathrm{I}s_2)^t \Gamma_1 = (s_{2u} + \Gamma_1 s_2)^t \mathrm{I}$$
and, similarly,
$$(\mathrm{I}s_1)_v^t - (\mathrm{I}s_1)^t \Gamma_2 = (s_{1v} + \Gamma_2 s_1)^t \mathrm{I}.$$

Thus
$$\begin{array}{rcl} 0 & = & \Phi_v - \Psi_u - [\Phi, \Psi] \\ & = & \begin{pmatrix} -\mathrm{R} + (s_1 s_2^t - s_2 s_1^t)\,\mathrm{I} & (s_{2u} + \Gamma_1 s_2) - (s_{1v} + \Gamma_2 s_1) \\ \{(s_{1v} + \Gamma_2 s_1)^t - (s_{2u} + \Gamma_1 s_2)^t\}\,\mathrm{I} & 0 \end{pmatrix} \end{array}$$
if and only if the Gauss-Codazzi equations are satisfied. ∎

The following theorem is usually attributed to O Bonnet:

Fundamental Theorem for Surfaces. *Suppose that*
$$\mathrm{I} = E\,du^2 + 2F\,du\,dv + G\,dv^2 \ \text{ and } \ \mathrm{II} = e\,du^2 + 2f\,du\,dv + g\,dv^2,$$
I *positive definite, satisfy the Gauss-Codazzi equations* (G) *and* (C). *Then there is (locally) a parametrized surface* $(u, v) \mapsto \sigma(u, v)$ *with* I *and* II *as its first and second fundamental forms.*
Moreover, the surface is unique up to Euclidean motion.

<u>*Remark*</u>. Note that, in contrast to the corresponding theorem for curves, we need the *integrability conditions* (C) and (G) to be satisfied as a necessary (and sufficient) conditions for the existence of σ.

Proof. With the matrix $\Sigma = \mathrm{I}^{-1}\mathrm{II}$ of the shape operator and using Koszul's formulas the Gauss-Weingarten equations $(\star)$ can be formulated. By the above lemma, the Gauss-Codazzi equations are then sufficient to ensure local existence of a solution $(u, v) \mapsto F(u, v) \in Gl(3)$.

By the uniqueness statement of the Maurer-Cartan lemma, such a solution F is unique up to post-composition with some constant $A \in Gl(3)$.

Since $\Gamma_{ij}^k = \Gamma_{ji}^k$ and II is symmetric, $\Psi \mathbf{e}_1 = \Phi \mathbf{e}_2$; hence, by the Poincaré lemma, there is a (regular) map $(u,v) \mapsto \sigma(u,v)$ with

$$\sigma_u = F\mathbf{e}_1 \quad \text{and} \quad \sigma_v = F\mathbf{e}_2.$$

Clearly, σ is unique up to translation.

We seek: σ has first fundamental form I and $F\mathbf{e}_3$ is a unit normal field. Now

$$((F^t)^{-1} \left(\begin{smallmatrix} \mathrm{I} & 0 \\ 0 & 1 \end{smallmatrix}\right) F^{-1})_u = (F^t)^{-1} \begin{pmatrix} \mathrm{I}_u - \mathrm{I}\Gamma_1 - \Gamma_1^t \mathrm{I} & 0 \\ 0 & 0 \end{pmatrix} F^{-1} = 0$$

by Koszul's formulas, and similarly for the v-deritative, so that

$$(F^t)^{-1} \left(\begin{smallmatrix} \mathrm{I} & 0 \\ 0 & 1 \end{smallmatrix}\right) F^{-1} \equiv const \quad \Rightarrow \quad F^t F = \left(\begin{smallmatrix} \mathrm{I} & 0 \\ 0 & 1 \end{smallmatrix}\right)$$

as soon as we choose F to satisfy this equality at an initial point (u_0, v_0).

This choice makes F unique up to post-composition with $A \in O(3)$ since

$$F^t F = \tilde{F}^t \tilde{F} = F^t A^t A F \quad \Rightarrow \quad A^t A = \mathrm{id}_{\mathbb{R}^3}.$$

We seek: $n := F\mathbf{e}_3$ is the Gauss map of σ. By the above choice $F\mathbf{e}_3$ is already a unit normal field; hence all we ask is

$$\det F = \det(\sigma_u, \sigma_v, n) = (\sigma_u \times \sigma_v) \cdot n > 0.$$

As $\det F$ does nowhere vanish this can be achieved by possibly post-composing F with a reflection.

This further choice makes F unique up to post-composition with constant rotations $A \in SO(3)$.

Finally, we seek: II is the second fundamental form of σ. This follows now directly from the construction of Φ and Ψ since

$$n_u = F\Phi\mathbf{e}_3 = (\sigma_u, \sigma_v, n) \begin{pmatrix} -s_1 \\ 0 \end{pmatrix}, \quad n_v = F\Psi\mathbf{e}_3 = (\sigma_u, \sigma_v, n) \begin{pmatrix} -s_2 \\ 0 \end{pmatrix}.$$

Finally, after the above choices, σ is unique up to Euclidean motion, $\sigma \to A\sigma + c$ with $A \in SO(3)$ and $c \in \mathbb{R}^3$. ∎

3 Curves on surfaces

3.1 Natural ribbon & Special lines on surfaces

Let $U \ni (u,v) \mapsto \sigma(u,v) \in \mathbb{R}^3$ be a parametrized surface. Then

$$t \mapsto \gamma(t) = \sigma(u(t), v(t)), \quad \text{where} \quad \forall t : (u'^2 + v'^2)(t) \neq 0,$$

defines a curve on the surface $\sigma(U) \subset \mathbb{R}^3$: since I is positive definite

$$0 = |\gamma'|^2 = |\sigma_u u' + \sigma_v v'|^2 = E\,u'^2 + 2F\,u'v' + G\,v'^2$$

implies $u' = v' = 0$; hence $u'^2 + v'^2 \neq 0$ ensures that γ is regular.

Example & Def. *For a parametrized surface $(u,v) \mapsto \sigma(u,v)$, the curves*

$$t \mapsto \sigma(u+t, v) \quad \text{and} \quad t \mapsto \sigma(u, v+t)$$

are called the parameter lines *of σ.*

Now consider a curve $\gamma(t) = \sigma(u(t), v(t))$ on a parametrized surface σ; the Gauss map of $(u,v) \mapsto \sigma(u,v)$ defines a canonical unit normal field

$$t \mapsto N(t) = n(u(t), v(t)) = \tfrac{\sigma_u \times \sigma_v}{|\sigma_u \times \sigma_v|}(u(t), v(t))$$

along γ — hence we obtain a natural ribbon (γ, N):

Def. *If $t \mapsto \gamma(t) = \sigma(u(t), v(t))$ parametrizes a curve on a surface then*

$$t \mapsto N(t) := n(u(t), v(t))$$

defines the natural ribbon (γ, N) *of γ. The curve γ is called*

- *an* asymptotic line *if (γ, N) is an asymptotic ribbon, i.e., $\kappa_n \equiv 0$;*
- *a* pre-geodesic *if (γ, N) is a geodesic ribbon, i.e., $\kappa_g \equiv 0$;*
- *a* curvature line *if (γ, N) is a curvature ribbon, i.e., $\tau \equiv 0$.*

Problem 1. Suppose a surface is given implicitly by $F(x,y,z) = 0$ so that $\operatorname{grad} F(x,y,z) \neq 0$ whenever $F(x,y,z) = 0$. Show that the natural ribbon of a curve $t \mapsto \gamma(t)$ on this surface, i.e., $F \circ \gamma \equiv 0$, is given by

$$N = +\tfrac{\operatorname{grad} F \circ \gamma}{|\operatorname{grad} F \circ \gamma|} \quad \text{or} \quad N = -\tfrac{\operatorname{grad} F \circ \gamma}{|\operatorname{grad} F \circ \gamma|}.$$

Use this to prove that $t \mapsto \gamma_\pm(t) = (1, t, \pm t)$ are asymptotic as well as pre-geodesic lines, but not curvature lines on the 1-sheeted hyperboloid given by $x^2 + y^2 - z^2 = 1$.

Problem 2. Prove Joachimsthal's Theorem: Suppose that two surfaces intersect along a curve and that the curve is a curvature line for one of the two surfaces; then it is a curvature line for the other as well if and only if the two surfaces intersect at a constant angle.

Rodrigues' equation. *$t \mapsto \gamma(t) = \sigma(u(t), v(t))$ is a curvature line iff*

$$0 = (dn + \kappa d\sigma)\binom{u'}{v'},$$

where $\kappa(u, v)$ is a principal curvature of σ at $(u, v) = (u(t), v(t))$.

Proof. $t \mapsto \gamma(t)$ is a curvature line iff the torsion τ of the natural ribbon (γ, N), where $t \mapsto N(t) = n(u(t), v(t))$, vanishes, that is, iff

$$\begin{aligned} 0 &= \nabla^\perp N \\ &= N' + \kappa_n \gamma' \\ &= (n_u u' + n_v v') + \kappa_n(\sigma_u u' + \sigma_v v') \\ &= (dn + \kappa_n d\sigma)\binom{u'}{v'}; \end{aligned}$$

on the other hand, $dn = -\mathrm{S} \circ d\sigma$, so that Rodrigues' equation holds iff κ_n is a principal curvature and $d\sigma\binom{u'}{v'}$ the corresponding curvature direction. ∎

Remark. Thus $(u, v) \mapsto \sigma(u, v)$ is a curvature line parametrization, that is, $F = f = 0$, if and only if the parameter lines are curvature lines.

Example. We determined the Gauss map of a surface of revolution earlier:

$$(u, v) \mapsto \sigma(u, v) = (r(u) \cos v, r(u) \sin v, h(u)),$$

hence

$$(u, v) \mapsto n(u, v) = \frac{(-h'(u) \cos v, -h'(u) \sin v, r'(u))}{\sqrt{(r'^2 + h'^2)(u)}}.$$

Thus

$$n_v + \frac{h'}{r\sqrt{r'^2 + h'^2}}\, \sigma_v = 0$$

so that the *parallels* $t \mapsto \rho(t) = \sigma(u, t)$ of the surface of revolution are curvature lines. As the *meridians* $t \mapsto \mu(t) = \sigma(t, v)$ intersect the parallels orthogonally, $\sigma_u \cdot \sigma_v \equiv 0$, they must be curvature lines as well.

Lemma. *The normal curvature of a curve* $t \mapsto \gamma(t) = \sigma(u(t), v(t))$ *on a surface is given by*

$$\kappa_n = \frac{\mathrm{II}((\begin{smallmatrix} u' \\ v' \end{smallmatrix}),(\begin{smallmatrix} u' \\ v' \end{smallmatrix}))}{\mathrm{I}((\begin{smallmatrix} u' \\ v' \end{smallmatrix}),(\begin{smallmatrix} u' \\ v' \end{smallmatrix}))}.$$

Proof. The normal curvature of the natural ribbon (γ, N) is given by

$$\kappa_n = \tfrac{1}{|\gamma'|}\, T' \cdot N = \tfrac{1}{|\gamma'|^2}\, \gamma'' \cdot N$$

with

$$|\gamma'|^2 = |\sigma_u u' + \sigma_v v'|^2 = E\, u'^2 + 2F\, u'v' + G\, v'^2 = \mathrm{I}((\begin{smallmatrix} u' \\ v' \end{smallmatrix}),(\begin{smallmatrix} u' \\ v' \end{smallmatrix}))$$

and

$$\begin{array}{rcl} \gamma'' \cdot N &=& (\sigma_u u' + \sigma_v v')' \cdot n \\ &=& (\sigma_{uu} u'^2 + 2\sigma_{uv} u'v' + \sigma_{vv} v'^2) \cdot n + \underbrace{(\sigma_u u'' + \sigma_v v'') \cdot n}_{=0} \\ &=& e\, u'^2 + 2f\, u'v' + g\, u'^2 \\ &=& \mathrm{II}((\begin{smallmatrix} u' \\ v' \end{smallmatrix}),(\begin{smallmatrix} u' \\ v' \end{smallmatrix})), \end{array}$$

which proves the claim. ∎

<u>*Remark*</u>. The normal curvature κ_n of a curve on a surface only depends on the tangent direction of the curve (and *not* on u'' or v''). Thus we also speak of the "normal curvature κ_n of a tangent direction".

Euler's Thm. *The normal curvatures* κ_n *at a point* $\sigma(u, v)$ *satisfy*

$$\kappa_n(\vartheta) = \kappa_1 \cos^2 \vartheta + \kappa_2 \sin^2 \vartheta,$$

where κ_i *are the principal curvatures and* ϑ *is the angle between the tangent direction of* $\kappa_n(\vartheta)$ *and the curvature direction of* κ_1*.*

<u>*Problem 3*</u>. Prove Euler's Theorem. [Hint: show that you can choose a basis $(\mathbf{e}_1, \mathbf{e}_2)$ of $\mathbb{R}^2$ so that $d_{(u,v)}\sigma(\mathbf{e}_i)$ are curvature directions of σ at $\sigma(u, v)$ and so that it is orthonormal w.r.t. $\mathrm{I}_{(u,v)}$, hence orthogonal w.r.t. $\mathrm{II}_{(u,v)}$; then consider $\mathbf{e}_\vartheta = \mathbf{e}_1 \cos\vartheta + \mathbf{e}_2 \sin\vartheta$.]

Cor. *The principal curvatures can be characterized as the extremal values of the normal curvatures at a point of a surface.*

As another application of the above lemma we obtain a characterization of asymptotic lines:

Cor. $t \mapsto \gamma(t) = \sigma(u(t), v(t))$ *is an asymptotic line iff*

$$\mathrm{II}(\left(\begin{smallmatrix} u' \\ v' \end{smallmatrix}\right), \left(\begin{smallmatrix} u' \\ v' \end{smallmatrix}\right)) \equiv 0 \quad \Leftrightarrow \quad e\,u'^2 + 2f\,u'v' + g\,v'^2 \equiv 0.$$

Example. *Circular helices as asymptotic lines on the helicoid.* For the helicoid $(u, v) \mapsto \sigma(u, v) = (\sinh u \cos v, \sinh u \sin v, v)$ we computed earlier

$$\mathrm{II}|_{(u,v)} = -2du\,dv;$$

hence, for $r := \sinh u$, the circular helices

$$t \mapsto \gamma(t) = (r \cos t, r \sin t, t) = \sigma(u, t)$$

are asymptotic lines since $g \equiv 0$.

Problem 4. Fix a point $\sigma(u, v)$ on a parametrized surface. Prove that no asymptotic line can pass through $\sigma(u, v)$ if $K(u, v) > 0$; if $K(u, v) < 0$, then an asymptotic line can pass through $\gamma(u, v)$ in two different (independent) directions. What can be said in case $K(u, v) = 0$?

3.2 Geodesics

Geometrically, geodesics can be thought of as the shortest possible curve on a surface between two points (at least locally); equivalently, they can be characterized as the "straight lines" in the surface, i.e., those which are not curved: $\kappa_g \equiv 0$. This is what we call "pre-geodesics".

From a physics point of view, one may think of a geodesic as the path of a particle on a surface which has no forces acting on it (besides the one keeping it on the surface), i.e., it has "no acceleration inside the surface" and it is only accelerated normal to the surface. Thus, additionally to not being curved (inside the surface), a "geodesic" does not change speed:

Def. *Let (γ, N) be the natural ribbon along a curve on a surface and let $t \mapsto \xi(t) \perp N(t)$ be a vector field along γ tangential to the surface.*

$$\tfrac{D}{dt}\xi := \xi' - (\xi' \cdot N)\, N$$

is called the covariant derivative *of ξ along the curve and γ is called a* geodesic *if*

$$\tfrac{D}{dt}\gamma' \equiv 0.$$

<u>*Example*</u>. *Circular helices as geodesics of a cylinder.* Let

$$(u,v) \mapsto \sigma(u,v) = (r\cos v, r\sin v, u)$$

parametrize a cylinder of radius $r > 0$; its Gauss map is

$$(u,v) \mapsto n(u,v) = -(\cos v, \sin v, 0).$$

Thus, for a circular helix

$$t \mapsto \gamma(t) = (r\cos t, r\sin t, ht) = \sigma(ht, t)$$

we find

$$\tfrac{D}{dt}\gamma' = \gamma'' - (\gamma'' \cdot N)\, N = r\,N - r\,N = 0$$

so that γ is a geodesic of the cylinder.

<u>*Remark*</u>. Writing $\gamma(t) = \sigma(u(t), v(t))$ and*) $\xi = \alpha\sigma_u + \beta\sigma_v$ we find

$$\begin{array}{rcl} \tfrac{D}{dt}\xi &=& \alpha'\sigma_u + \alpha(u'\nabla_{\frac{\partial}{\partial u}}\sigma_u + v'\nabla_{\frac{\partial}{\partial v}}\sigma_u) \\ &+& \beta'\sigma_v + \beta(u'\nabla_{\frac{\partial}{\partial u}}\sigma_v + v'\nabla_{\frac{\partial}{\partial v}}\sigma_v) \\ &=& \{\alpha' + \alpha(u'\Gamma^1_{11} + v'\Gamma^1_{12}) + \beta(u'\Gamma^1_{21} + v'\Gamma^1_{22})\}\,\sigma_u \\ &+& \{\beta' + \alpha(u'\Gamma^2_{11} + v'\Gamma^2_{12}) + \beta(u'\Gamma^2_{21} + v'\Gamma^2_{22})\}\,\sigma_v; \end{array}$$

in particular, $\frac{D}{dt}\xi$ can be computed from I alone.

With $\xi = \gamma' = u'\sigma_u + v'\sigma_v$, we see that γ is a geodesic iff

$$\begin{array}{rcl} 0 &=& u'' + \Gamma^1_{11}u'^2 + 2\Gamma^1_{12}u'v' + \Gamma^1_{22}v'^2, \\ 0 &=& v'' + \Gamma^2_{11}u'^2 + 2\Gamma^2_{12}u'v' + \Gamma^2_{22}v'^2. \end{array} \qquad (\star)$$

---

*) More precisely, $t \mapsto \xi(t) = \alpha(t)\sigma_u(u(t), v(t)) + \beta(t)\sigma_v(u(t), v(t))$.

Thm. *Given a point $p_0 = \sigma(u_0, v_0)$ on a surface and a tangent direction $t_0 \perp n(u_0, v_0)$ at p_0, there is a unique geodesic γ with*

$$\gamma(0) = p_0 \quad \textit{and} \quad \gamma'(0) = t_0.$$

Proof. Write $\gamma(t) = \sigma(u(t), v(t))$ and $t_0 = \alpha_0 \sigma_u(u_0, v_0) + \beta_0 \sigma_v(u_0, v_0)$.

With $w = (u, u', v, v')$ the equations $(\star)$ for γ to be a geodesic form a system of ODEs of the form $w' = f(w)$, where f is differentiable:

$$\begin{aligned}
w_1' &= w_2, \\
w_2' &= -\Gamma^1_{11}(w_1, w_3)\, w_2^2 - 2\Gamma^1_{12}(w_1, w_3)\, w_2 w_4 - \Gamma^1_{22}(w_1, w_3)\, w_4^2, \\
w_3' &= w_4, \\
w_4' &= -\Gamma^2_{11}(w_1, w_3)\, w_2^2 - 2\Gamma^2_{12}(w_1, w_3)\, w_2 w_4 - \Gamma^2_{22}(w_1, w_3)\, w_4^2.
\end{aligned}$$

Hence, the sought geodesic is obtained from a solution of the initial value problem

$$w' = f(w), \;\; w(0) = (u_0, \alpha_0, v_0, \beta_0),$$

and the claim follows from the Picard-Lindelöf Thm (1st special case). ■

<u>*Problem 5*</u>. Find the geodesics γ of a plane $\pi \subset \mathbb{R}^3$ with $\gamma(0) = p_0 \in \pi$.

Thm. *Geodesics are the constant speed pre-geodesics.*

Proof. Let (γ, N) be the natural ribbon of $t \mapsto \gamma(t) = \sigma(u(t), v(t))$.

As $\gamma'' = \frac{D}{dt}\gamma' + (\gamma'' \cdot N)\, N$

$$\tfrac{1}{2}(|\gamma'|^2)' = \gamma' \cdot \gamma'' = \gamma' \cdot \tfrac{D}{dt}\gamma'$$

and, by the structure equations for ribbons,

$$\kappa_g = -\frac{T' \cdot (T \times N)}{|\gamma'|} = \frac{\det(N, T, T')}{|\gamma'|} = \frac{\det(N, \gamma', \gamma'')}{|\gamma'|^3} = \frac{\det(N, \gamma', \frac{D}{dt}\gamma')}{|\gamma'|^3}.$$

Thus, if γ is a geodesic, $\frac{D}{dt}\gamma' = 0$, then $|\gamma'| \equiv const$ and $\kappa_g = 0$.

Conversely, suppose that $|\gamma'|^2 \equiv const$ and $\kappa_g \equiv 0$. Then

$$\frac{D}{dt}\gamma' \perp \begin{cases} N & \text{by definition of } \frac{D}{dt}, \\ \gamma' & \text{since } |\gamma'|^2 \equiv const, \\ N \times \gamma' & \text{since } \kappa_g \equiv 0; \end{cases}$$

hence $\frac{D}{dt}\gamma' \equiv 0$ and γ is a geodesic. ∎

Remark. As $\frac{D}{dt}\gamma'$ can be computed from I alone, so can geodesics and geodesic curvature. In particular,

$$\kappa_g = \frac{\sqrt{EG-F^2}}{\sqrt{Eu'^2+2Fu'v'+Gv'^2}^3}\ \det\begin{pmatrix} u' & u''+\Gamma^1_{11}u'^2+2\Gamma^1_{12}u'v'+\Gamma^1_{22}v'^2 \\ v' & v''+\Gamma^2_{11}u'^2+2\Gamma^2_{12}u'v'+\Gamma^2_{22}v'^2 \end{pmatrix}.$$

Thm (Clairaut's theorem). *For a geodesic on a surface of revolution the product*

$$r\sin\theta \equiv const,$$

where $r = r(s)$ is the distance from the axis and $\theta = \theta(s)$ is the angle that the geodesic makes with the meridians.

Proof. Let $s \mapsto \gamma(s) = (r(s)\cos\varphi(s), r(s)\sin\varphi(s), h(s))$ be a geodesic on a surface of revolution, wlog., arc length parametrized.

We denote

$$\alpha_t(s) := \begin{pmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \gamma(s);$$

note that $s \mapsto \alpha_t(s)$ is an arc-length parametrized geodesic for each t. Further let

$$X(s) := \frac{\partial}{\partial t}\Big|_{t=0}\alpha_t(s) = \mathbf{e}_3 \times \gamma(s),$$

where $\mathbf{e}_3 = (0,0,1)$ spans the axis of the surface of revolution. Then

$$r\sin\theta = r\cos(\tfrac{\pi}{2}-\theta) = \gamma'\cdot X = (\tfrac{\partial}{\partial s}\alpha_t \cdot \tfrac{\partial}{\partial t}\alpha_t)|_{t=0},$$

and

$$\frac{\partial}{\partial s}(\tfrac{\partial}{\partial s}\alpha_t \cdot \tfrac{\partial}{\partial t}\alpha_t) = \underbrace{\tfrac{\partial}{\partial s}\tfrac{\partial}{\partial s}\alpha_t}_{\parallel n} \cdot \underbrace{\tfrac{\partial}{\partial t}\alpha_t}_{\perp n} + \tfrac{1}{2}\tfrac{\partial}{\partial t}\underbrace{(\tfrac{\partial}{\partial s}\alpha_t \cdot \tfrac{\partial}{\partial s}\alpha_t)}_{\equiv 1} = 0$$

hence $r\sin\theta \equiv const$. ∎

Remark. Note that Clairaut's theorem provides a *necessary* condition for a geodesic, not a *sufficient* condition: there are curves satisfying Clairaut's relation $r\sin\theta \equiv const$ that are not geodesics.

Example. Let $p_0 \in S^2$ and $t_0 \perp p_0$ be a unit tangent vector at p_0, then

$$s \mapsto \gamma(s) = p_0\cos s + t_0 \sin s$$

is the geodesic with $\gamma(0) = p_0$ and $\gamma'(0) = t_0$ since $\gamma'' = -\gamma$ so that

$$\tfrac{D}{dt}\gamma' = -\gamma + \gamma = 0 \quad \text{and} \quad \gamma(0) = p_0, \quad \gamma'(0) = t_0.$$

Now consider S^2 as a surface of revolution with the z-axis $\mathbb{R}\mathbf{e}_3$ as its axis of rotation; then

$$r = |\mathbf{e}_3 \times \gamma| \quad \text{and} \quad \sin\theta = \tfrac{\gamma'\cdot(\mathbf{e}_3\times\gamma)}{|\gamma'|\,|\mathbf{e}_3\times\gamma|} = \tfrac{\gamma'\cdot(\mathbf{e}_3\times\gamma)}{|\mathbf{e}_3\times\gamma|}$$

so that $r\sin\theta = \gamma' \cdot (\mathbf{e}_3 \times \gamma) = \det(\mathbf{e}_3, \gamma, \gamma') = \det(\mathbf{e}_3, p_0, t_0) \equiv const.$

Thus, when $\mathbf{e}_3$ is not in the plane of the geodesic, the geodesic becomes horizontal where the distance from the axis is the smallest and it is steepest where it crosses the equator.

Note however that the curves $z = const \neq 0$ are *not* (pre-) geodesics on S^2, even though $r\sin\theta = \sqrt{1-z^2}\cdot 1 \equiv const.$

Problem 6. Let $(u, v) \mapsto \sigma(u, v) = (r(u)\cos v, r(u)\sin v, h(u))$ be a surface of revolution. Prove that:

(a) if a parallel $t \mapsto \sigma(u, t)$ is a geodesic then we must have $r'(u) = 0$;

(b) if $r'^2 + h'^2 \equiv 1$ then the meridians $t \mapsto \sigma(t, v)$ are geodesics.

3.3 Geodesic polar coordinates & Minding's theorem

Let $\Sigma \subset \mathbb{R}^3$ be a surface, $p_0 \in \Sigma$ and let $\mathcal{T}(p_0)$ denote the tangent plane of Σ at p_0. Now choose an orthonormal basis $(\mathbf{e}_1, \mathbf{e}_2)$ for $\mathcal{T}(p_0)$, that is, orthonormal vectors so that

$$\mathcal{T}(p_0) = p_0 + \text{span}\{\mathbf{e}_1, \mathbf{e}_2\} = \{p_0 + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2 \,|\, \lambda, \mu \in \mathbb{R}\},$$

and let γ_ϑ denote the unique geodesic in Σ with

$$\gamma_\vartheta(0) = p_0 \quad \text{and} \quad \gamma'_\vartheta(0) = \mathbf{e}_1\cos\vartheta + \mathbf{e}_2\sin\vartheta.$$

There is $\varepsilon > 0$ so that all γ_ϑ are defined for $|t| < \varepsilon$ — hence we can use the γ_ϑ to (locally) parametrize Σ:

Def. *We say that* $(r,\vartheta)\mapsto\sigma(r,\vartheta):=\gamma_\vartheta(r)$ *is a parametrization by* geodesic polar coordinates *around* p_0.

Remark. σ is *not* regular for $r=0$ since $\sigma(0,\vartheta)=p_0$ for all ϑ; however, it is regular for $(r,\vartheta)\in(0,\varepsilon)\times\mathbb{R}$ for some $\varepsilon>0$.

Problem 7. Parametrize $S^2(R)=\{(x,y,z)\,|\,x^2+y^2+z^2=R^2\}$ by geodesic polar coordinates around $p_0=(0,0,R)$. Compute the metric.

Lemma. *In geodesic polar coordinates* (r,ϑ),

$$\mathrm{I}=dr^2+G\,d\vartheta^2 \quad \textit{with} \quad \sqrt{G}|_{r=0}=0 \quad \textit{and} \quad \tfrac{\partial\sqrt{G}}{\partial r}|_{r=0}=1.$$

Proof. First observe that $\sigma(0,\vartheta)=p_0$ for all ϑ, hence $\sigma_\vartheta|_{r=0}=0$.

$E=1$. γ_ϑ is arc-length parametrized, hence $E=|\sigma_r|^2=|\gamma_\vartheta'|^2=1$.

$F=0$. $\sigma_\vartheta|_{r=0}=0$, hence $F|_{r=0}=\sigma_r\cdot\sigma_\vartheta|_{r=0}=0$; moreover

$$F_r=\sigma_{rr}\cdot\sigma_\vartheta+\sigma_r\cdot\sigma_{r\vartheta}=\sigma_{rr}\cdot\sigma_\vartheta+\tfrac{1}{2}E_\vartheta=0$$

since $\sigma_{rr}=\frac{D}{dt}\gamma_\vartheta'+\ldots N$. So, $r\mapsto F(r,\vartheta)\equiv 0$ for all ϑ.

$\sqrt{G}|_{r=0}=0$. $\sigma_\vartheta|_{r=0}=0$, hence $G|_{r=0}=|\sigma_\vartheta|^2\Big|_{r=0}=0$.

$\frac{\partial\sqrt{G}}{\partial r}|_{r=0}=1$. We take it for granted that $\frac{\partial\sqrt{G}}{\partial r}|_{r=0}$ exists and $\neq 0$. Then

$$(\sqrt{G})_r\Big|_{r=0}=\frac{G_r}{2\sqrt{G}}\Big|_{r=0}=\frac{G_{rr}}{2(\sqrt{G})_r}\Big|_{r=0}$$

by de l'Hospital's rule and, as $\sigma_\vartheta|_{r=0}=0$ and $|\frac{d}{d\vartheta}\gamma_\vartheta'(0)|^2=1$,

$$\tfrac{1}{2}G_{rr}\Big|_{r=0}=(\sigma_{rr\vartheta}\cdot\sigma_\vartheta+\sigma_{r\vartheta}\cdot\sigma_{r\vartheta})\Big|_{r=0}=1.$$

Hence $\frac{\partial\sqrt{G}}{\partial r}\Big|_{r=0}=\sqrt{\frac{G_{rr}}{2}}\Big|_{r=0}=1.$ ∎

Problem 8. Prove: $K=-\frac{(\sqrt{G})_{rr}}{\sqrt{G}}$ in geodesic polar coordinates (r,ϑ).

Cor. *Geodesics are (locally) the shortest curves between two points.*

Proof. Let $p_0, p \in \Sigma$ be two points so that p is in a geodesic polar coordinates neighbourhood of p_0, i.e., (r, ϑ) are geodesic polar coordinates around p_0 and $p = \gamma_\Theta(R) = \sigma(R, \Theta)$ for some R and Θ.

Let $t \mapsto \gamma(t) = \sigma(r(t), \vartheta(t))$ be a curve with $\gamma(0) = p_0$ and $\gamma(1) = p$, hence $r(0) = 0$ and $r(1) = R$; then its length

$$\int_0^1 |\gamma'|\, dt = \int_0^1 \sqrt{r'^2 + G(r,\vartheta)\,\vartheta'^2}\, dt \geq \int_0^1 r'\, dt = R$$

with equality iff $\vartheta' \equiv 0$ and $r' > 0$, that is, iff $\gamma = \gamma_\Theta \circ r$. ■

Cor (Minding's theorem). *Any two surfaces with the same constant Gauss curvature are locally isometric, i.e., there are local parametrizations σ^1 and σ^2 so that their first fundamental forms $\mathrm{I}^1 = \mathrm{I}^2$.*

Remark. Gauss Theorema egregium says: two isometric surfaces do necessarily have the same Gauss curvature; Minding's theorem says: for surfaces of *constant Gauss curvatures* this is also sufficient.

Proof. In geodesic polar coordinates (r, ϑ),

$$\mathrm{I} = dr^2 + G\, d\vartheta^2 \quad \text{with} \quad \sqrt{G}|_{r=0} = 0 \quad \text{and} \quad (\sqrt{G})_r|_{r=0} = 1$$

and from the Gauss equation and Koszul's equations

$$K = -\frac{(\sqrt{G})_{rr}}{\sqrt{G}}.$$

Thus, if $K \equiv const$ then G satisfies, for fixed ϑ, the initial value problem

$$0 = (\sqrt{G})_{rr} + K\sqrt{G}, \quad 0 = \sqrt{G}|_{r=0} \quad \text{and} \quad 1 = (\sqrt{G})_r|_{r=0},$$

which has a unique solution

$$\sqrt{G(r,\vartheta)} = \begin{cases} \frac{1}{\sqrt{K}} \sin(\sqrt{K}r) & \text{if } K > 0, \\ r & \text{if } K = 0, \\ \frac{1}{\sqrt{-K}} \sinh(\sqrt{-K}r) & \text{if } K < 0. \end{cases}$$

Hence the metric I is uniquely determined by K and parametrization by geodesic polar coordinates shows that any two surfaces with the same constant Gauss curvature are isometric. ■

4 Manifolds

We have already seen some problems with our definitions of curves and surfaces: for example,

(1) a hyperbola does not qualify as a curve (according to our definition) as it consists of two components, hence cannot be parametrized by a single regular (hence continuous) map defined on an open interval;

(2) the sphere S^2 does not qualify as a surface since there cannot be a (regular) parametrization of all of S^2 defined on an open connected subset $U \subset \mathbb{R}^2$ (by the "hairy ball theorem").

The notion of a k-dimensional submanifold of $\mathbb{R}^n$ ("curve" if $k = 1$ and "surface" if $k = 2$) resolves this problem — at the cost of introducing another restriction, which can in turn be resolved by the notion of an "immersed abstract manifold". At the same time, the following discussions will shed light on the notion of "local" (as opposed by "global"), used previously in this text in an informal way.

Different characterizations for submanifolds will also provide criteria which allow to pass from an implicit representation of a curve or surface to a parametric description, and vice versa — at least theoretically.

1 Submanifolds of $\mathbb{R}^n$

There are several equivalent definitions/characterizations of submanifolds in Euclidean space:

Def 1. *"A submanifold can locally be flattened";*
*$M \subset \mathbb{R}^n$ is called a k-*dimensional submanifold *of $\mathbb{R}^n$ if:*
for every $p \in M$ there is a diffeomorphism $\varphi : U \to \tilde{U}$ between open neighbourhoods $U, \tilde{U} \subset \mathbb{R}^n$ of p and 0, respectively, so that

$$\varphi(M \cap U) = \tilde{U} \cap (\mathbb{R}^k \times \{0\}), \quad \textit{where} \quad \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}.$$

Def 2. *"A submanifold is locally a level set (defined by equations)";*
*$M \subset \mathbb{R}^n$ is called a k-*dimensional submanifold *of $\mathbb{R}^n$ if:*

for every $p \in M$ there is a submersion $F : U \to \mathbb{R}^{n-k}$ from an open neighbourhood $U \subset \mathbb{R}^n$ of p to $\mathbb{R}^{n-k}$ so that

$$M \cap U = F^{-1}(\{0\}).$$

<u>*Remark*</u>. In Def 2, it is sufficient to require d_pF to surject for all $p \in M$: if d_pF surjects then, by the Inertia principle ($q \mapsto d_qF$ is continuous), there is a neighbourhood $\tilde{U} \subset U$ of p so that d_qF surjects for all $q \in \tilde{U}$.

Def 3. *"A submanifold can locally be parametrized";*
*$M \subset \mathbb{R}^n$ is called a k-*dimensional submanifold *of $\mathbb{R}^n$ if:*
for every $p \in M$ there is an immersion $f : V \to U$ from an open neighbourhood $V \subset \mathbb{R}^k$ of 0 *to an open neighbourhood $U \subset \mathbb{R}^n$ of p so that*

$$M \cap U = f(V) \text{ and } f : V \to M \cap U$$

is a homeomorphism (using the induced topology on $U \cap M$).

<u>*Remark*</u>. f being an immersion excludes "kinks"; injectivity of f excludes self-intersections and continuity of the inverse excludes "T-junctions".

Proof. (Equivalence of the three definitions). Throughout the proof we write

$$\pi_1 : \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k, \qquad (x, y) \mapsto x;$$
$$\pi_2 : \mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}, \quad (x, y) \mapsto y.$$

<u>$1 \Rightarrow 2 + 3$</u>. First note that Def 1 easily implies the other two:

(2) for a submersion whose level set is $M \cap U$ let

$$F := \pi_2 \circ \varphi : U \to \mathbb{R}^{n-k};$$

(3) for a local parametrization let $V = \pi_1(\tilde{U}) \subset \mathbb{R}^k$ and

$$f := \varphi^{-1}|_V : V \to U \subset \mathbb{R}^n.$$

Conversely:

<u>$2 \Rightarrow 1$</u>. let $F : U \to \mathbb{R}^{n-k}$ be a submersion so that $U \cap M = F^{-1}(\{0\})$ and choose an orthonormal basis $(t_1, \dots, t_k)$ of $\ker d_pF \subset \mathbb{R}^n$. Define

$$\varphi : U \to \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n, \quad q \mapsto \varphi(q) := (q \cdot t_1, \dots, q \cdot t_k, F(q)).$$

Now $d_pF(v) = 0 \Rightarrow v = \sum_{j=1}^{k}(v \cdot t_j)\, t_j$, hence $d_p\varphi(v) = 0 \Rightarrow v = 0$. Consequently $d_p\varphi : \mathbb{R}^n \to \mathbb{R}^n$ is invertible and, by the Inverse mapping theorem: after possibly making U smaller,

(i) $\varphi : U \to \mathbb{R}^n$ injects (so that $\varphi : U \to \varphi(U)$ is invertible);

(ii) $\tilde{U} := \varphi(U) \subset \mathbb{R}^n$ is open;

(iii) $\varphi^{-1} : \tilde{U} \to U$ is continuously differentiable.

In other words: $\varphi : U \to \tilde{U}$ is a diffeomorphism.

Finally,

$$q \in M \cap U \quad \Leftrightarrow \quad F(q) = 0 \quad \Leftrightarrow \quad \varphi(q) \in \tilde{U} \cap (\mathbb{R}^k \times \{0\}).$$

<u>$3 \Rightarrow 1$</u>. let $f : \mathbb{R}^k \supset V \to U \subset \mathbb{R}^n$ be a local parametrization and choose a basis $(n_1, \ldots, n_{n-k})$ of $(d_0 f(\mathbb{R}^k))^\perp \subset \mathbb{R}^n$. Define

$$V \times \mathbb{R}^{n-k} \ni (x, y) \mapsto g(x, y) := f(x) + \sum_{j=1}^{n-k} y_k n_k \in \mathbb{R}^n.$$

Now $d_0 g \simeq (\frac{\partial f}{\partial x_1}(0), \ldots, \frac{\partial f}{\partial x_k}(0), n_1, \ldots, n_{n-k}) : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ is invertible; hence, by the Inverse mapping theorem, g has a local smooth inverse

$$\varphi := (g|_{\tilde{U}})^{-1} : g(\tilde{U}) \to \tilde{U}.$$

Wlog $g(\tilde{U}) \subset U$; as $f(\pi_1(\tilde{U})) \subset M$ is open we may assume $U = g(\tilde{U})$ by possibly making U smaller. Hence

$$\begin{aligned} q \in U \cap M \quad &\Leftrightarrow \quad \exists x \in \pi_1(\tilde{U}) \subset V : q = f(x) = g(x, 0) \\ &\Leftrightarrow \quad \varphi(q) = (x, 0) \in \tilde{U} \cap (\mathbb{R}^k \times \{0\}). \end{aligned}$$

■

<u>*Problem 1*</u>. Use the Implicit mapping theorem to show directly that an implicitely defined submanifold (Def 2) has local parametrizations (Def 3). [Hint: write $\mathbb{R}^n = \ker d_pF \times (\ker d_pF)^\perp$.]

<u>*Examples*</u>.

(1) *Plane*. $\pi = \{p \in \mathbb{R}^3 \,|\, p \cdot n = d\}$ is a 2-dimensional submanifold of $\mathbb{R}^3$: with

$$\mathbb{R}^3 \ni p \mapsto F(p) := p \cdot n - d \in \mathbb{R}$$

$\pi = F^{-1}(\{0\})$ and $v \mapsto d_pF(v) = v \cdot n$ surjects as $d_pF(n) \neq 0$.

(2) *Sphere.* $S^2 = \{p \in \mathbb{R}^3 \,|\, |p|^2 = 1\}$ is a 2-dimensional submanifold of $\mathbb{R}^3$: taking

$$\mathbb{R}^3 \setminus \{0\} \ni p \mapsto F(p) := |p|^2 - 1 \in \mathbb{R}$$

$S^2 = F^{-1}(\{0\})$ and F is a submersion, i.e., $v \mapsto d_pF(v) = 2v \cdot p$ surjects for all $p \in \mathbb{R}^3 \setminus \{0\}$, since $d_pF(p) = 2|p|^2 \neq 0$.

(3) *Hyperboloids.* $H_\pm = \{(x, y, z) \in \mathbb{R}^3 \,|\, (\frac{x}{a})^2 + (\frac{y}{b})^2 - (\frac{z}{c})^2 = \pm 1\}$ are 2-dimensional submanifolds of $\mathbb{R}^3$: here we take

$$F_\pm(x, y, z) := (\tfrac{x}{a})^2 + (\tfrac{y}{b})^2 - (\tfrac{z}{c})^2 \mp 1;$$

hence $H_\pm = F_\pm^{-1}(\{0\})$ and $\operatorname{grad} F_\pm(x, y, z) = 2(\frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2}) \neq 0$ as soon as $(x, y, z) \neq (0, 0, 0)$. Hence, in order to obtain submersions, we set

$$U := \mathbb{R}^3 \setminus \{0\}, \quad F_\pm : U \to \mathbb{R}.$$

<u>*Problem 2*</u>. Let $F_1, F_2 : \mathbb{R}^3 \ni U \to \mathbb{R}$ with $(\operatorname{grad} F_1 \times \operatorname{grad} F_2)(p) \neq 0$ for all $p \in U$. Prove that the equations $F_1(p) = F_2(p) = 0$ define a 1-dimensional submanifold of $\mathbb{R}^3$. Hence show that the conic sections

$$C_\alpha = \{(x, y, z) \,|\, x^2 + y^2 = z^2, \;\; x \cos\alpha + z \sin\alpha = d\},$$

where $\alpha \in \mathbb{R}$ and $d \neq 0$, are 1-dimensional submanifolds of $\mathbb{R}^3$.

<u>*Example*</u>. *Gerono's lemniscate* is a curve in the plane $\mathbb{R}^2$ defined by the equation

$$x^4 - x^2 + y^2 = 0;$$

a regular (check) parametrization is given by

$$t \mapsto (x(t), y(t)) = \sin t \,(1, \cos t).$$

This curve has a self intersection at $(x, y) = (0, 0) \Leftrightarrow t = k\pi$, $k \in \mathbb{Z}$, hence is not a 1-dimensional submanifold.

<u>*Problem 3*</u>. Prove that Gerono's lemniscate is not a submanifold.

<u>*Remark*</u>. Note that the definition of a k-dimensional submanifold excludes self-intersections — which may be undesirable in some contexts.

<u>*Problem 4*</u>. Prove that $SO(3) \subset M(3 \times 3, \mathbb{R}) \cong \mathbb{R}^9$ is a 3-dimensional submanifold. [Hint: $Sym(3) = \{A \in M(3 \times 3, \mathbb{R}) \,|\, A^t = A\} \cong \mathbb{R}^6$.]

Def. *The* tangent space *of a k-dimensional submanifold $M \subset \mathbb{R}^n$ at $p \in M$ is the k-dimensional subspace*

$$T_pM := d_xf(\mathbb{R}^k) \subset \mathbb{R}^n,$$

where $f : \mathbb{R}^k \supset V \to \mathbb{R}^n$ is a parametrisation of M around $p = f(x)$.

Remark. T_pM is independent of the choice of local parametrisation: if $\tilde{f} : \tilde{V} \to \mathbb{R}^n$ is another local parametrisation around $p = \tilde{f}(\tilde{x})$ then $\tilde{f} = f \circ \mu$ with a diffeomorphism $\mu : \tilde{V} \to V$ so that $x = \mu(\tilde{x})$; hence

$$d_{\tilde{x}}\tilde{f}(\mathbb{R}^k) = d_xf(d_{\tilde{x}}\mu(\mathbb{R}^k)) = d_xf(\mathbb{R}^k).$$

Remark. If $M = F^{-1}(\{0\})$ is defined as a level set of a submersion $F : U \to \mathbb{R}^{n-k}$ then

$$T_pM = \ker d_pF.$$

Namely, $F \circ f \equiv 0$ for a local parametrisation f around $p = f(x)$ so that

$$d_pF \circ d_xf \equiv 0 \quad \Rightarrow \quad T_pM \subset \ker d_pF;$$

then $\dim T_pM = \dim\ker d_pF$ implies $T_pM = \ker d_pF$.

Example. The tangent space of $O(3) = \{A \in Gl(3) \,|\, F(A) = 0\}$ with

$$F : Gl(3) \to Sym(3), \;\; A \mapsto F(A) = A^tA - \mathrm{id}_{\mathbb{R}^3}$$

at $A \in O(3)$ is the 3-dimensional subspace

$$T_AO(3) = \ker d_AF = \{X \in \mathfrak{gl}(3) \,|\, A^{-1}X \in \mathfrak{o}(3)\}.$$

Note that $T_ASO(3) = T_AO(3)$ for $A \in SO(3)$.

2 Functions on submanifolds

Now that we have described submanifolds $M \subset \mathbb{R}^n$ and, in particular, curves and surfaces in $\mathbb{R}^3$ as subsets of the Euclidean ambient space it becomes necessary to discuss analysis issues: previously, functions, vector fields, etc, were defined on a parameter domain, that is, an open subset of $\mathbb{R}$ or $\mathbb{R}^2$ and it was clear what, for example, differentiability meant; now the situation has changed and the domain of a function on a submanifold is no longer an open set in $\mathbb{R}^n$, making it necessary to revisit basic notions of analysis. The key idea is to define the derivative so that the chain rule holds:

Def. *A function $\varphi : M \to \mathbb{R}$ on a submanifold $M \subset \mathbb{R}^n$ is said to be* differentiable *at $p \in M$ with* derivative

$$d_p\varphi := d_0(\varphi \circ f) \circ (d_0 f)^{-1} : d_0 f(\mathbb{R}^k) = T_pM \to \mathbb{R}$$

if $\varphi \circ f : \mathbb{R}^k \overset{\circ}{\supset} V \to \mathbb{R}$ is differentiable at 0 for some local parametrization $f : V \to M$ of M around p with $p = f(0)$.

<u>*Remark*</u>. This definition makes sense as differentiability and derivative of φ do not depend on the choice of parametrization: if $\tilde{f} = f \circ \psi$ is a local reparametrization around a point $p \in M$ then ψ is a diffeomorphism, hence $\varphi \circ \tilde{f} = \varphi \circ f \circ \psi$ is differentiable as soon as $\varphi \circ f$ is.

<u>*Remark*</u>. This definition has an obvious generalization to $\mathbb{R}^m$-valued maps, hence to maps between submanifolds of Euclidean spaces.

<u>*Remark*</u>. If $\Phi : \mathbb{R}^n \to \mathbb{R}$ is differentiable and $M \subset \mathbb{R}^n$ is a submanifold then $\varphi := \Phi|_M : M \to \mathbb{R}$ is differentiable with

$$d_p\varphi = d_p\Phi|_{T_pM} : T_pM \to \mathbb{R}.$$

Namely: φ is clearly differentiable as $\Phi \circ f$ is for any parametrization f; moreover, if $v = d_0 f(w)$ then

$$d_p\varphi(v) = d_0(\varphi \circ f)(w) = d_0(\Phi \circ f)(w) = d_p\Phi(v).$$

Def. *Let $\varphi : M \to \mathbb{R}$ be differentiable; the* gradient *of φ at $p \in M$ is the unique vector* $\operatorname{grad}\varphi(p) \in T_pM$ *with*

$$\forall v \in T_pM : d_p\varphi(v) = v \cdot \operatorname{grad}\varphi(p).$$

<u>*Example*</u>. Consider a parametrized surface $(u, v) \mapsto \sigma(u, v) \in \mathbb{R}^3$ with first fundamental form $\mathrm{I} = E\,du^2 + 2F\,dudv + G\,dv^2$; let

$$\sigma_u^* := \tfrac{1}{EG-F^2}(G\sigma_u - F\sigma_v) \quad \text{and} \quad \sigma_v^* := \tfrac{1}{EG-F^2}(-F\sigma_u + E\sigma_v)$$

and note that $\sigma_u^* \cdot \sigma_u = \sigma_v^* \cdot \sigma_v = 1$ and $\sigma_u^* \cdot \sigma_v = \sigma_v^* \cdot \sigma_u = 0$. Hence, if $\varphi : M = \sigma(U) \to \mathbb{R}$ is differentiable and $\psi := \varphi \circ \sigma$, then

$$(\operatorname{grad}\varphi) \circ \sigma = \psi_u\sigma_u^* + \psi_v\sigma_v^* = \tfrac{G\psi_u - F\psi_v}{EG-F^2}\sigma_u + \tfrac{E\psi_v - F\psi_u}{EG-F^2}\sigma_v.$$

Problem 5. Prove the *Lagrange multiplier theorem*: if $M = F^{-1}(\{0\})$ is a surface in $\mathbb{R}^3$ and $\Phi : \mathbb{R}^3 \to \mathbb{R}$ is differentiable then $p \in M$ is a critical point (hence candidate for an extremum) of $\varphi := \Phi|_M : M \to \mathbb{R}$ iff there is $\lambda \in \mathbb{R}$ so that (λ, p) is a critical point of

$$\mathbb{R} \times \mathbb{R}^3 \ni (\lambda, p) \mapsto \Phi(p) - \lambda F(p) \in \mathbb{R}.$$

After fixing the idea of how to differentiate functions on a submanifold, hence also vector fields, it is fairly straightforward to carry over most notions developed for curves and/or surfaces to submanifolds, such as the first and second fundamental forms, the shape operator, covariant derivative (the curvature tensor will become more complicated when the manifold has higher dimension though). For example:

Def. *Let ξ be a tangential vector field, i.e., $\xi : M \to \mathbb{R}^n$ differentiable so that $\xi(p) \in T_pM$ for all $p \in M$, and let $v \in T_pM$; then*

$$\nabla_v \xi|_p := (d_0(\xi \circ f)(w))^T,$$

where f is a local parametrization of M around p with $f(0) = p$ and $d_0 f(w) = v$; as usual, $(..)^T$ denotes the tangential part, i.e., the orthogonal projection $\mathbb{R}^n \to T_pM$ onto the tangent space.
∇ is called the Levi-Civita connection of M.

This also yields a notion of second derivative for functions:

Def & Lemma. *The* Hessian

$$T_pM \times T_pM \ni (v, w) \mapsto (\mathsf{hess}\,\varphi)|_p(v, w) := w \cdot \nabla_v(\mathsf{grad}\,\varphi)|_p$$

of a smooth function $\varphi : M \to \mathbb{R}$ at $p \in M$ is a symmetric tensor.

Proof. Clearly, $\mathsf{hess}\,\varphi$ is a tensor.

To see symmetry let f be a local parametrization around p and compute

$$\begin{aligned} \tfrac{\partial}{\partial x_i} \tfrac{\partial}{\partial x_j}(\varphi \circ f) &= \tfrac{\partial}{\partial x_i}((\xi_j \cdot \mathsf{grad}\,\varphi) \circ f) \\ &= \{(\nabla_{\xi_i} \xi_j) \cdot \mathsf{grad}\,\varphi + \xi_j \cdot (\nabla_{\xi_i} \mathsf{grad}\,\varphi)\} \circ f \\ &= \{d\varphi(\nabla_{\xi_i} \xi_j) + \mathsf{hess}\,\varphi(\xi_i, \xi_j)\} \circ f, \end{aligned}$$

where we let $\xi_i \circ f = \frac{\partial}{\partial x_i} f$. Clearly the left hand side is symmetric in i and j and*), as in the case of surfaces,

$$(\nabla_{\xi_i}\xi_j - \nabla_{\xi_j}\xi_i) \circ f = (\tfrac{\partial}{\partial x_i}\tfrac{\partial}{\partial x_j} f - \tfrac{\partial}{\partial x_j}\tfrac{\partial}{\partial x_i} f)^T = 0.$$

Hence $\mathrm{hess}\,\varphi(\xi_i, \xi_j) = \mathrm{hess}\,\varphi(\xi_j, \xi_i)$ for $i, j \in \{1, \ldots, k\}$ showing that $\mathrm{hess}\,\varphi|_p$ is a symmetric bilinear form on every tangent space T_pM. ■

Remark. From the above computation we see that the Hessian is the covariant derivative of $d\varphi$:

$$\mathrm{hess}\,\varphi(\xi, \eta) = (\nabla_\xi d\varphi)(\eta) = d(d\varphi(\eta))(\xi) - d\varphi(\nabla_\xi \eta).$$

Note that the Hessian depends on the covariant derivative, hence on the induced metric: *not* just on the differentiable structure on M.

However, if $p = f(x)$ is a critical point of φ, i.e., $\mathrm{grad}\,\varphi(p) = 0$, then

$$\mathrm{hess}\,\varphi|_p(\xi_i(p), \xi_j(p)) = \tfrac{\partial}{\partial x_i}\tfrac{\partial}{\partial x_j}(\varphi \circ f)(x)$$

can be computed without reference to the covariant derivative or induced metric — hence providing a simple(r) criterion to detect local extrema.

Poincaré lemma. *A tangential vector field ξ has a local potential φ, i.e., locally $\xi = \mathrm{grad}\,\varphi$, if and only if $(v, w) \mapsto w \cdot \nabla_v \xi$ is symmetric.*

Proof. We have seen above that symmetry of $(v, w) \mapsto w \cdot \nabla_v \xi$ is a necessary condition for ξ to be a (local) gradient vector field, $\xi = \mathrm{grad}\,\varphi$.

To see that it is also sufficient we use a local parametrization f and employ the above notation $\xi_i \circ f = \frac{\partial f}{\partial x_i}$: thus given ξ we are seeking a function $\psi = \varphi \circ f$ with

$$\tfrac{\partial \psi}{\partial x_i} = (\xi_i \cdot \xi) \circ f.$$

Now, as above,

$$\tfrac{\partial}{\partial x_i}((\xi_j \cdot \xi) \circ f) = \{(\nabla_{\xi_i}\xi_j) \cdot \xi + \xi_j \cdot (\nabla_{\xi_i}\xi)\} \circ f,$$

which is symmetric in i and j as soon as $\xi_j \cdot (\nabla_{\xi_i}\xi)$ is — hence the Poincaré lemma in $\mathbb{R}^k$ yields the result. ■

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*) This is, ∇ is a "torsion free connection".

Def. *The* Laplacian *of φ on M is defined by*

$$\Delta\varphi := \operatorname{tr}\operatorname{hess}\varphi,$$

where trace is taken with respect to the first fundamental form, that is, with an orthonormal basis $(\mathbf{e}_1, \ldots, \mathbf{e}_k)$ of T_pM

$$\operatorname{tr}\operatorname{hess}\varphi|_p = \sum_{j=1}^{k} \operatorname{hess}\varphi|_p(\mathbf{e}_j, \mathbf{e}_j).$$

A function $\varphi : M \to \mathbb{R}$ is called harmonic *if $\Delta\varphi \equiv 0$.*

Remark. First note that the trace of a bilinear form β can be computed as

$$\operatorname{tr}\beta = \beta(w_1, w_1^*) + \beta(w_2, w_2^*)$$

with "dual bases" (w_1, w_2) and (w_1^*, w_2^*), i.e., $w_i^* \cdot w_j = \delta_{ij}$: writing both bases in terms of an orthonormal basis (which is "self-dual"),

$$(w_1, w_2) = (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \text{ and } (w_1^*, w_2^*) = (\mathbf{e}_1, \mathbf{e}_2)\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix}^{-1},$$

the above formula for the trace of β is readily verified.

Now consider a parametrized surface $(u, v) \mapsto \sigma(u, v)$; to compute the Laplacian of a function $\varphi : M = \sigma(U) \to \mathbb{R}$ write $W := \sqrt{EG - F^2}$ and

$$(\operatorname{grad}\varphi) \circ \sigma = \psi_u \sigma_u^* + \psi_v \sigma_v^* = \tfrac{G\psi_u - F\psi_v}{W}\tfrac{\sigma_u}{W} + \tfrac{E\psi_v - F\psi_u}{W}\tfrac{\sigma_v}{W},$$

employing our previous notations. From Koszul's formulas we find

$$\begin{array}{rcccl} \sigma_u^* \cdot \nabla_{\sigma_u}\frac{\sigma_u}{W} + \sigma_v^* \cdot \nabla_{\sigma_v}\frac{\sigma_u}{W} & = & \frac{FE_v - EG_u}{2W^3} + \frac{EG_u - FE_v}{2W^3} & = & 0, \\ \sigma_u^* \cdot \nabla_{\sigma_u}\frac{\sigma_v}{W} + \sigma_v^* \cdot \nabla_{\sigma_v}\frac{\sigma_v}{W} & = & \frac{GE_v - FG_u}{2W^3} + \frac{FG_u - GE_v}{2W^3} & = & 0; \end{array}$$

hence

$$(\Delta\varphi) \circ \sigma = \tfrac{1}{W}\{(\tfrac{G\psi_u - F\psi_v}{W})_u + (\tfrac{E\psi_v - F\psi_u}{W})_v\}.$$

This is often called the *Laplace-Beltrami operator*.

Problem 6. Let $(r, \vartheta) \mapsto \sigma(r, \vartheta)$ be a parametrization by geodesic polar coordinates and suppose that the induced metric is rotationally symmetric, $G_\vartheta \equiv 0$, and has constant Gauss curvature K. Determine all rotationally symmetric (i.e., $\psi_\vartheta = (\varphi \circ \sigma)_\vartheta \equiv 0$) harmonic functions φ.

Example. Let $M^2 \subset \mathbb{R}^3$ be a 2-dimensional submanifold in $\mathbb{R}^3$ and denote the inclusion by $\iota : M^2 \to \mathbb{R}^3$, $p \mapsto \iota(p) = p$. Then

$$\iota \circ \sigma = \sigma \text{ and } d_p\iota = d_x(\iota \circ \sigma) \circ (d_x\sigma)^{-1} = \mathrm{id}_{T_pM}$$

for a (local) parametrization $\sigma : \mathbb{R}^2 \supset V \to M^2 \subset \mathbb{R}^3$ of M^2. With our previous notation for the covariant derivative (along σ)

$$\nabla_{\frac{\partial}{\partial x_i}} \sigma_{x_j} = \nabla_{\frac{\partial}{\partial x_i}} (\xi_j \circ \sigma) = (\nabla_{\xi_i} \xi_j) \circ \sigma, \text{ where } \xi_i \circ \sigma = \tfrac{\partial}{\partial x_i} \sigma,$$

so that, with the Gauss map n of σ,

$$(\mathsf{hess}\, \iota(\xi_i, \xi_j)) \circ \sigma = \tfrac{\partial}{\partial x_i} \tfrac{\partial}{\partial x_j} \sigma - (\nabla_{\xi_i} \xi_j) \circ \sigma = \mathrm{II}(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_i})\, n.$$

Hence

$$(\Delta \iota) \circ \sigma = (\mathsf{tr}\, \mathsf{hess}\, \iota) \circ \sigma = (\mathsf{tr}\, \mathrm{I}^{-1} \mathrm{II})\, n = (\mathsf{tr}\, \mathrm{S})\, n = 2H\, n,$$

that is, the mean curvature H of any parametrization of M^2 vanishes if and only if the inclusion $\iota : M^2 \to \mathbb{R}^3$ is harmonic.

3 Vector fields & Flows

Knowing the velocity vector field of a flow on a surface or in space it is possible to predict the paths of particles dropped into the flow: an obvious example is a paper boat dropped into a brook, where the stream can be obstructed by stones or little islands. This circle of ideas shall be made precise here.

Def. *Let ξ be a (tangential) vector field on a (sub-)manifold $M \subset \mathbb{R}^n$. A curve $\gamma : I \to M$ on an open interval $I \subset \mathbb{R}$ is an* integral curve *of ξ if*

$$\gamma' = \xi \circ \gamma;$$

it is maximal *if γ cannot be extended as an integral curve of ξ.*

Remark. We do not require regularity: for example, if $\xi \equiv 0$ then every integral curve of ξ is constant.

Lemma. *Through any point $p \in M$ passes a unique maximal integral curve of a given (smooth) vector field ξ on M.*

Proof. We only need to prove local existence and uniqueness.

Thus let $f : V \to M$ be a local parametrization of M around $p = f(x_0)$; write

$$\xi \circ f = df \circ y, \text{ i.e., } \forall x \in V : \xi(f(x)) = d_x f(y(x)),$$

with a vector field y on $V \subset \mathbb{R}^k$; the ansatz $\gamma = f \circ x$ then yields

$$\gamma' = \xi \circ \gamma \quad \Leftrightarrow \quad x' = y \circ x$$

since $d_x f : \mathbb{R}^k = T_x V \to T_{f(x)} M$ is an isomorphism for every $x \in V$. Now the claim follows from the Picard-Lindelöf theorem: the initial value problem

$$x' = y \circ x, \;\; x(0) = x_0$$

has a solution $x : J \to V$ on some open interval J with $0 \in J$, which is unique up to extension. ∎

<u>*Remark*</u>. Note the difference with the differential equation for geodesics: here the derivative of the curve is given, whereas it was part of the unknown data in the case of geodesics.

These constructed maximal integral curves of a vector field ξ can be assembled into a single map:

Thm & Def. *Given a tangent vector field ξ on a (sub-)manifold M, its* maximal flow *is the unique smooth map*

$$\Phi : W \to M, \;\; (t,p) \mapsto \Phi_t(p)$$

on an open neighbourhood W of $\{0\} \times M \subset \mathbb{R} \times M$ so that

(i) $\Phi_0 = \mathrm{id}_M$*;*

(ii) $I_p := \{t \,|\, (t,p) \in W\}$ *is an open interval about* 0 *for each* $p \in M$*, and*

(iii) $I_p \ni t \mapsto \Phi_t(p)$ *is the maximal integral curve of* ξ *through* p*.*

Proof. Denoting by γ_p the unique maximal integral curve of ξ,

$$\gamma_p' = \xi \circ \gamma_p, \text{ with } \gamma_p(0) = p,$$

the maximal flow Φ of ξ must be uniquely defined by

$$\bigcup_{p \in M} I_p \times \{p\} = W \ni (t,p) \mapsto \Phi_t(p) = \gamma_p(t) \in M.$$

$W \subset \mathbb{R} \times M$ is open and $\Phi : W \to M$ smooth by the smooth dependence of solutions of the ODE defining γ_p on the initial condition. ■

Warning. There may not be any $\varepsilon > 0$ so that $W \supset (-\varepsilon, \varepsilon) \times M$: for example, with $M = \{(u, v) \in \mathbb{R}^2 \,|\, |u| < 1\}$ and $\xi \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we obtain for fixed $(u, v) \in M$

$$\gamma_{(u,v)} : I_{(u,v)} = (-1 - u, 1 - u) \to M, \;\; t \mapsto \gamma_p(t) = u + t.$$

Rem & Def. If M is compact or, more generally, ξ is compactly supported then

$$W = \mathbb{R} \times M \;\text{ and }\; \forall s, t \in \mathbb{R} : \Phi_{s+t} = \Phi_s \circ \Phi_t;$$

moreover, $\Phi_t : M \to M$ is a diffeomorphism for any fixed $t \in \mathbb{R}$: that is, $\Phi : \mathbb{R} \times M \to M$ is a 1-*parameter group of diffeomorphisms*.

This is what is often also called a "*flow*".

Example. Let $M = S^2 \subset \mathbb{R}^3$ and $\xi(p) := \mathbf{e}_3 \times p$. Then, for $p = (x, y, z)$,

$$\Phi_t(x, y, z) = (x \cos t - y \sin t, x \sin t + y \cos t, z).$$

Note the similarity with $\alpha_t(s)$ from the proof of Clairaut's thm (which is not a coincidence).

Rem & Def. In general, the maximal flow $\Phi : W \to M$ of a vector field ξ is a "*local flow*", that is, there is a neighbourhood $(-\varepsilon, \varepsilon) \times U \subset W$ of $(0, p)$ for any $p \in M$ so that

(a) $\Phi_t|_U : U \to \Phi_t(U)$ is a diffeomorphism for each $t \in (-\varepsilon, \varepsilon)$;

(b) $\Phi_{s+t}(q) = (\Phi_s \circ \Phi_t)(q)$ whenever $q \in U$ and $|s|, |t|, |s + t| < \varepsilon$.

Proof. Fix $p \in M$; as W is open there is a neighbourhood $U \subset M$ of p and $\varepsilon > 0$ so that $W \supset (-\varepsilon, \varepsilon) \times U$. Since $\Phi_0|_U = \mathrm{id}_U$ is a diffeomorphism the $\Phi_t|_U$ are by inertia, after possibly making the neighbourhood smaller.

To prove (b) consider the curve $\gamma(s) := \gamma_q(s + t) = \Phi_{s+t}(q)$: since

$$\gamma' = \xi \circ \gamma \;\text{ and }\; \gamma(0) = \gamma_q(t) = \Phi_t(q),$$

γ is the integral curve of ξ with $\gamma(0) = \Phi_t(q)$, hence $\gamma(s) = \Phi_s(\Phi_t(q))$ and the claim follows. ■

Notation. If ξ is a vector field and φ a function on M we define a new function

$$\xi\varphi : M \to \mathbb{R},\ \ p \mapsto (\xi\varphi)(p) := d_p\varphi(\xi(p)).$$

Thus we think of the vector field ξ as a differential operator, which yields a directional derivative of φ at every point. In particular, $\varphi \mapsto \xi\varphi$ is linear and the Leibniz rule holds,

$$\xi(\varphi\psi) = (\xi\varphi)\psi + \varphi(\xi\psi).$$

Lemma. *If* $(\xi\varphi)(p) = 0$ *for every function* φ *on* M *then* $\xi(p) = 0$.

Proof. We use a local parametrization f around $p = f(x)$ and use the "*coordinate functions*"

$$\varphi_i := (f^{-1})_i = \pi_i \circ (f^{-1}),\ \ i = 1, \dots, k,$$

as test functions: for $y := (d_x f)^{-1}(\xi(p))$ we learn

$$(\xi\varphi_i)(p) = d_x(\varphi_i \circ f)(y) = d_x\pi_i(y) = y_i$$

so that $(\xi\varphi_i)(p) = 0$ implies $y = 0$, hence $\xi(p) = 0$. ∎

Lemma & Def. *Let* ξ *and* η *be two tangent vector fields on* M*. There is a unique vector field* $[\xi, \eta]$ *on* M *so that, for every smooth function* φ *on* M,

$$[\xi, \eta]\varphi = \xi(\eta\varphi) - \eta(\xi\varphi).$$

$[\xi, \eta]$ *is called the* Lie bracket *of* ξ *and* η.

Proof. We use a local parametrization f and set $\xi_i \circ f = \frac{\partial}{\partial x_i} f$. Note that $(\xi_i\psi) \circ f = \frac{\partial}{\partial x_i}(\psi \circ f)$ for any function ψ so that

$$\begin{aligned}\{\xi_i(\xi_j\varphi) - \xi_j(\xi_i\varphi)\} \circ f &= \tfrac{\partial}{\partial x_i}((\xi_j\varphi) \circ f) - \tfrac{\partial}{\partial x_j}((\xi_i\varphi) \circ f) \\ &= \tfrac{\partial}{\partial x_i}\tfrac{\partial}{\partial x_j}(\varphi \circ f) - \tfrac{\partial}{\partial x_j}\tfrac{\partial}{\partial x_i}(\varphi \circ f) \\ &= 0.\end{aligned}$$

Now write

$$\xi = \sum_{i=1}^k \alpha_i\xi_i \ \text{ and } \ \eta = \sum_{i=1}^k \beta_i\xi_i$$

and compute (using also the Leibniz rule)

$$\xi(\eta\varphi) - \eta(\xi\varphi) = \sum_{i=1}^k \sum_{j=1}^k \{\alpha_j(\xi_j\beta_i) - \beta_j(\xi_j\alpha_i)\}\,(\xi_i\varphi).$$

Hence $[\xi,\eta]=\sum_{i,j=1}^{k}\{\alpha_j(\xi_j\beta_i)-\beta_j(\xi_j\alpha_i)\}\,\xi_i$ by the previous lemma, as the above equation holds for any function φ. ∎

Remark. Clearly the Lie bracket is skew symmetric, $[\xi,\eta]+[\eta,\xi]=0$; it also satisfies the (straightforward to verify) *Jacobi identity*,

$$[\xi,[\eta,\zeta]]+[\zeta,[\xi,\eta]]+[\eta,[\zeta,\xi]]=0.$$

Thus, with the Lie bracket as a multiplication, the vector space of smooth vector fields on a (sub-)manifold becomes a *Lie algebra*.

Remark. In the case of a submanifold $M\subset\mathbb{R}^n$ the Lie bracket of two vector fields is related to the Levi-Civita connection:

$$[\xi,\eta]=\nabla_\xi\eta-\nabla_\eta\xi.$$

This characterizes the Levi-Civita connection as "*torsion free*".

The flow $t\mapsto\Phi_t$ of a vector field ξ allows to identify tangent spaces along the integral curves of ξ: if $y\in T_qM$ is a tangent vector at $q=\Phi_t(p)$ then $d_q\Phi_{-t}(y)\in T_pM$ is a tangent vector at p as Φ_{-t} maps a neighbourhood of q to a neighbourhood of p. This observation allows to compute the derivative of a vector field η in the flow direction:

Def. *Let ξ and η be tangent vector fields on a manifold M and let Φ denote the maximal flow of ξ. The* Lie derivative *of η at $p\in M$ in direction $\xi(p)$ is defined by*

$$(\mathcal{L}_\xi\eta)(p):=\tfrac{d}{dt}\Big|_{t=0}d_{\Phi_t(p)}\Phi_{-t}(\eta(\Phi_t(p))).$$

Lemma. $\mathcal{L}_\xi\eta=[\xi,\eta]$.

Proof. Fix $p\in M$ and set $y_t:=d_{\Phi_t(p)}\Phi_{-t}(\eta(\Phi_t(p)))$.

First observe: given $q\in M$ and any function φ on M we have

$$\tfrac{d}{dt}(\varphi\circ\Phi_t)(q)=d_{\Phi_t(q)}\varphi(\xi\circ\Phi_t(q))=(\xi\varphi)(\Phi_t(q))$$

for $t\in I_q$; thus Taylor expansion of the function $t\mapsto(\varphi\circ\Phi_t)(q)$ yields

$$(\varphi\circ\Phi_t)(q)=\varphi(q)+t(\xi\varphi)(q)+r_q(t),\ \text{ where }\ r_q(t)=o(t)$$

is differentiable as a function $(t,q)\mapsto r_q(t)$ since $(t,q)\mapsto(\varphi\circ\Phi_t)(q)$ is.

Secondly: fix a test function φ and observe that

$$\begin{aligned} d_p\varphi(y_t) &= d_p\varphi(d_{\Phi_t(p)}\Phi_{-t}(\eta(\Phi_t(p)))) \\ &= d_{\Phi_t(p)}(\varphi\circ\Phi_{-t})(\eta(\Phi_t(p))) \\ &= \eta(\varphi\circ\Phi_{-t})(\Phi_t(p)) \\ &= \eta(\varphi - t(\xi\varphi) + r(-t))(\Phi_t(p)) \end{aligned}$$

Now:

- $\frac{d}{dt}\Big|_{t=0}(\eta r(-t))\circ\Phi_t = \mathrm{lim}_{t\to 0}(\eta\frac{r(-t)}{t})\circ\Phi_t = 0$ since $r(t) = o(t)$;
- $\frac{d}{dt}\Big|_{t=0} t\eta(\xi\varphi)\circ\Phi_t = \mathrm{lim}_{t\to 0}\,\eta(\xi\varphi)\circ\Phi_t = \eta(\xi\varphi)$ since $\Phi_0 = \mathrm{id}_M$;
- $\frac{d}{dt}\Big|_{t=0}(\eta\varphi)\circ\Phi_t = \xi(\eta\varphi)$ by the first observation.

Consequently, $(\mathcal{L}_\xi\eta)\varphi(p) = d_p\varphi(\frac{dy_t}{dt}\Big|_{t=0}) = [\xi,\eta]\varphi(p)$ and the claim follows since $(\mathcal{L}_\xi\eta - [\xi,\eta])\varphi(p) = 0$ for every test function φ on M at every point $p\in M$. ■

Def. *Two vector fields ξ,η on M are said to* commute *if $[\xi,\eta] = 0$; two (local) flows Φ,Ψ* commute *if (wherever all terms of the equation are defined) $\Phi_t\circ\Psi_s = \Psi_s\circ\Phi_t$.*

Thm. *Two vector fields commute iff their maximal flows do.*

Proof. Let Φ and Ψ denote the maximal flows of ξ and η, respectively.

Using that $\Phi_{\tau+t} = \Phi_\tau\circ\Phi_t = \Phi_t\circ\Phi_\tau$ and writing $q = \Phi_\tau(p)$ we obtain

$$\begin{aligned} d_{\Phi_{\tau+t}(p)}\Phi_{-(\tau+t)}(\eta(\Phi_{\tau+t}(p))) &= d_{\Phi_t(q)}(\Phi_{-\tau}\circ\Phi_{-t})(\eta(\Phi_t(q))) \\ &= d_q\Phi_{-\tau}(d_{\Phi_t(q)}\Phi_{-t}(\eta(\Phi_t(q)))) \end{aligned}$$

hence

$$\frac{d}{dt}\Big|_{t=\tau} d_{\Phi_t(p)}\Phi_{-t}(\eta(\Phi_t(p))) = d_{\Phi_\tau(p)}\Phi_{-\tau}(\mathcal{L}_\xi\eta(\Phi_\tau(p))). \qquad (\star)$$

Now write $\gamma(s,t) := (\Phi_{-t}\circ\Psi_s\circ\Phi_t)(p)$ and observe that

$$\begin{aligned} \frac{\partial}{\partial s}\gamma(s,t) &= d_{\Phi_t(\gamma(s,t))}\Phi_{-t}(\frac{d}{ds}(\Psi_s\circ\Phi_t)(p)) \\ &= d_{\Phi_t(\gamma(s,t))}\Phi_{-t}(\eta(\Phi_t(\gamma(s,t)))). \end{aligned} \qquad (\star\star)$$

Thus, if Φ and Ψ commute, $\Phi_{-t} \circ \Psi_s \circ \Phi_t = \Psi_s$, then $\gamma(s,t) = \Psi_s(p)$ depends on s only and, in particular, $\gamma(0,t) = p$ for all t. Consequently,

$$d_{\Phi_t(p)}\Phi_{-t}(\eta(\Phi_t(p))) = \tfrac{\partial}{\partial s}\gamma(0,t) = \tfrac{\partial}{\partial s}\gamma(0,0) = \eta(p)$$

by $(\star\star)$, hence

$$\mathcal{L}_\xi \eta(p) = \tfrac{d}{dt}\Big|_{t=0} d_{\Phi_t(p)}\Phi_{-t}(\eta(\Phi_t(p))) = \tfrac{\partial}{\partial t}\tfrac{\partial}{\partial s}\gamma(0,t) = 0.$$

Conversely, if ξ and η commute, $\mathcal{L}_\xi \eta \equiv 0$, then $(\star)$ yields

$$t \mapsto d_{\Phi_t(q)}\Phi_{-t}(\eta(\Phi_t(q))) \equiv const = \eta(q)$$

for any q. In particular, for $q = \gamma(s,t)$ we learn from $(\star\star)$ that

$$\tfrac{\partial}{\partial s}\gamma(s,t) = \eta(\gamma(s,t));\ \text{ with }\ \gamma(0,t) = p$$

this shows that $s \mapsto \gamma(s,t)$ is the integral curve of η through $p = \gamma(0,t)$ for each fixed t, that is, $\gamma(s,t) = \Psi_s(p)$ for every fixed t. ∎

Thm & Def. *Let ξ_i, $i = 1, \dots, k$, be pairwise commuting vector fields on a k-dimensional manifold M that are linearly independent at every point of M. Then there is a local parametrization f around each point $p \in M$ so that*

$$\xi_i \circ f = \tfrac{\partial}{\partial x_i} f \ \text{ for } \ i = 1, \dots, k.$$

$(\xi_1, \dots, \xi_k)$ *is called the* Gaussian basis field *of f.*

<u>*Remark*</u>. We already saw that the vector fields of a Gaussian basis field commute: if $\xi_i \circ f = \frac{\partial}{\partial x_i} f$ then $[\xi_i, \xi_j] = 0$.

Proof. Let Φ^k denote the maximal flows of ξ_k. Fix $p \in M$ and define

$$f(x_1, \dots, x_k) := (\Phi^1_{x_1} \circ \dots \circ \Phi^k_{x_k})(p)$$

on a suitable neighbourhood of $0 \in \mathbb{R}^k$ (so that the expression is defined).

Using that the flows Φ^i commute and Φ^k is the flow of ξ_k, we compute

$$\begin{aligned}
\tfrac{\partial}{\partial x_k} f(x_1, \dots, x_k) &= \tfrac{\partial}{\partial x_k}(\Phi^k_{x_k} \circ \Phi^1_{x_1} \circ \ldots \circ \Phi^{k-1}_{x_{k-1}})(p) \\
&= (\xi_k \circ \Phi^k_{x_k} \circ \Phi^1_{x_1} \circ \ldots \circ \Phi^{k-1}_{x_{k-1}})(p) \\
&= (\xi_k \circ f)(x_1, \dots, x_k)
\end{aligned}$$

and similarly for $i = 1, \ldots, k-1$. In particular, $d_0 f : \mathbb{R}^k \to T_pM$ is an isomorphism so that f is a local diffeomorphism by the Inverse mapping theorem, hence qualifies as a local parametrization. ■

4 Surfaces revisited

We shall now return to surfaces in $\mathbb{R}^3$ and re-investigate their geometry in the light of we have learned in this chapter.

Thus let $M^2 \subset \mathbb{R}^3$ be a 2-dimensional submanifold. We assume that M^2 is orientable, that is, there exists a smooth map

$$\nu : M^2 \to S^2 \subset \mathbb{R}^3 \text{ such that } \forall p \in M : \nu(p) \perp T_pM;$$

ν is called a *Gauss map* of the *surface* $M^2 \subset \mathbb{R}^3$. Note that such an orientable surface $M^2 \subset \mathbb{R}^3$ admits two Gauss maps — a choice of Gauss map equips M^2 with an *orientation*, that is, turns $M^2 \subset \mathbb{R}^3$ into an *oriented submanifold*.

Def. *The* shape operator S *of an oriented surface* (M^2, ν) *(with respect to* ν*) is given by*

$$\mathrm{S} := -d\nu.$$

Remark. If $\sigma : \mathbb{R}^2 \supset V \to M$ is a local parametrization around $p = \sigma(x)$ and $n := \nu \circ \sigma$ then

$$\mathrm{S}_p = -d_p\nu = -d_x(\nu \circ \sigma) \circ (d_x\sigma)^{-1} = -d_x n \circ (d_x\sigma)^{-1}$$

yields our earlier definition of the shape operator. Thus the principal, mean and Gauss curvatures of (M^2, ν) are the eigenvalues, trace and determinant of S, respectively.

Def. *Two vector fields* ξ *and* η *on* M^2 *are said to form a* local basis field (ξ, η) *around* $p \in M^2$ *if*

$$\forall q \in U : \mathrm{span}\{\xi(q), \eta(q)\} = T_qM^2,$$

where $U \subset M^2$ *is an open neighbourhood of* p*.* (ξ, η) *is called*

- orthonormal *if* $|\xi|^2 = |\eta|^2 = 1$ *and* $\xi \cdot \eta = 0$ *on* U*;*

- principal *if* $S\xi = k_1\xi$ *and* $S\eta = k_2\eta$ *in* U.

Remark. A manifold M^2 does not necessarily carry a non-vanishing vector field: for example, every vector field ξ on S^2 must have at least one zero by the "hairy ball theorem".

Lemma. *If* $[\xi, \eta] = a\xi - b\eta$ *for an orthonormal local basis field then*

$$\nabla_\xi \xi = -a\eta, \ \ \nabla_\eta \xi = b\eta, \ \ \nabla_\xi \eta = a\xi, \ \ \nabla_\eta \eta = -b\xi.$$

Proof. Using that ∇ is torsion free

$$a\xi - b\eta = [\xi, \eta] = \nabla_\xi \eta - \nabla_\eta \xi,$$

where $\nabla_\xi \eta \parallel \xi$ and $\nabla_\eta \xi \parallel \eta$ since $0 = \xi(|\eta|^2) = 2\eta \cdot \nabla_\xi \eta$ and similarly for $\nabla_\eta \xi$. Hence

$$\nabla_\xi \eta = a\xi \ \text{ and } \ \nabla_\eta \xi = b\eta.$$

Further $\nabla_\xi \xi \parallel \eta$ and $\nabla_\eta \eta \parallel \xi$ by the same argument so that

$$0 = \xi(\xi, \eta) = (\nabla_\xi \xi) \cdot \eta + \xi \cdot (\nabla_\xi \eta) = (\nabla_\xi \xi) \cdot \eta + a,$$

hence $\nabla_\xi \xi = -a\eta$ and, similarly, $\nabla_\eta \eta = -b\xi$. ∎

Lemma. *If* (ξ, η) *is a principal orthonormal local basis field then*

$$[\xi, \eta] = -\tfrac{1}{k_1 - k_2}((\eta k_1)\xi + (\xi k_2)\eta).$$

Proof. Since (ξ, η) is a local basis field $[\xi, \eta] = a\xi - b\eta$ with suitable functions a and b; then, using the previous lemma and by the Codazzi equation,

$$\begin{aligned} 0 &= (\nabla_\xi S)\eta - (\nabla_\eta S)\xi \\ &= \{-(\eta k_1) - a(k_1 - k_2)\}\, \xi + \{(\xi k_2) - b(k_1 - k_2)\}\, \eta \end{aligned}$$

hence $a = -\frac{\eta k_1}{k_1 - k_2}$ and $b = \frac{\xi k_2}{k_1 - k_2}$. ∎

Cor. *Any surface of constant mean curvature $M^2 \subset \mathbb{R}^3$ admits (away from umbilics) local conformal curvature line parametrizations.*

Proof. Let (ξ, η) denote a local principal orthonormal basis field (this is where we need to stay away from umbilics),

$$\mathrm{S}\xi = k_1\xi, \ \ \mathrm{S}\eta = k_2\eta \ \text{ with } \ k_1 = H + e^{-2\varphi}, \ \ k_2 = H - e^{-2\varphi},$$

where $H \equiv const$ and φ is some function. Then

$$[e^{\varphi}\xi, e^{\varphi}\eta] = e^{2\varphi}\{[\xi, \eta] + (\xi\varphi)\eta - (\eta\varphi)\xi\} = 0$$

by the previous lemma. Hence around every point $p \in M^2$ there is a local parametrization $(u, v) \mapsto \sigma(u, v)$ so that

$$e^u\xi = \sigma_u \ \text{ and } \ e^u\eta = \sigma_v;$$

in particular, $\mathrm{I} = e^{2\varphi}(du^2 + dv^2)$, that is, σ is a conformal curvature line parametrization. ■

Cor. *A surface of constant negative Gauss curvature admits local curvature line parametrizations $(u, v) \mapsto \sigma(u, v)$ so that, with a suitable function ω, the induced metric becomes*

$$\mathrm{I} = \cos^2\omega\, du^2 + \sin^2\omega\, dv^2.$$

Proof. Writing $k_1 = c\tan\omega$ and $k_2 = -c\cot\omega$, where $K = -c^2$ and ω is a function with values in $(0, \frac{\pi}{2})$, the above lemma yields

$$[\cos\omega\,\xi, \sin\omega\,\eta] = \cos\omega\sin\omega\,[\xi, \eta] + \cos^2\omega\,(\xi\omega)\eta + \sin^2\omega\,(\eta\omega)\xi = 0.$$

Hence, as above, every point $p \in M^2$ has a neighbourhood, where M^2 has a local parametrization $(u, v) \mapsto \sigma(u, v)$ so that

$$\cos\omega\,\xi = \sigma_u \ \text{ and } \ \sin\omega\,\eta = \sigma_v$$

and the induced metric has the claimed form. ■

Cor & Def. *Any surface of constant negative Gauss curvature carries locally asymptotic* Chebyshev nets, *that is, parametrizations so that the parameter lines are asymptotic and arc length parametrized.*

Proof. Using the same notations as in the previous corollary set

$$\xi_\pm := \cos\omega\,\xi \pm \sin\omega\,\eta$$

and observe that
$$|\xi_\pm|^2 = 1 \text{ and } \mathrm{S}\xi_\pm \perp \xi_\pm.$$
As constant linear combinations of commuting vector fields the $\xi_\pm$ commute,
$$[\xi_+, \xi_-] = 0,$$
hence give rise to a local parametrization $(u, v) \mapsto \sigma(u, v)$ (note that $\xi_\pm$ are linearly independent, hence (ξ_+, ξ_-) is a local basis field) with
$$\sigma_u = \xi_+ \text{ and } \sigma_v = \xi_-,$$
so that that parameter lines are asymptotic and parametrized by arc length. ∎

Remark. Using similar ideas one can prove that *every* surface $M^2 \subset \mathbb{R}^3$ can locally be parametrized by curvature lines. This is *not* true for

- surfaces $M^2 \subset \mathbb{R}^n$ for $n \geq 4$;
- *hypersurfaces* $M^k \subset \mathbb{R}^{k+1}$.

Appendix - Tools from Analysis

Obviously, differentiability is a key issue in differential geometry. Perhaps less obviously, the Inverse and Implicit mapping theorems and solutions of (ordinary and partial) differential equations as well as their uniqueness are also key issues.

Thus, after recalling the notion of differentiability and differentiation rules (and, in the process, fixing notations), we briefly discuss the Inverse and Implicit mapping theorems and collect some theorems about differential equations that are used in the text.

Recall. A map $f : \mathbb{R}^m \overset{\circ}{\supset} U \to \mathbb{R}^n$ is *differentiable* at $p \in U$ if there is a linear map $d_pf \in \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^n)$ from $\mathbb{R}^m$ to $\mathbb{R}^n$ so that

$$\lim_{h\to 0} \frac{f(p+h)-f(p)-d_pf(h)}{|h|} = 0.$$

If f is differentiable at $p \in U \overset{\circ}{\subset} \mathbb{R}^m$ then its derivative is given by the Jacobi matrix

$$d_pf \simeq \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(p) & \dots & \frac{\partial f_1}{\partial x_m}(p) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(p) & \dots & \frac{\partial f_n}{\partial x_m}(p) \end{pmatrix}.$$

Note, however, that existence of the Jacobi matrix does not prove differentiability (see problem below) — but if the Jacobi matrix depends continuously on p then f is *continuously differentiable* and, in particular, differentiable at every point $p \in U$.

We will often write partial derivatives using subscripts: $f_u := \frac{\partial f}{\partial u}$, etc.

Problem 1. Compute the Jacobi matrix of

$$f : \mathbb{R}^2 \to \mathbb{R}, \ (u, v) \mapsto f(u, v) := \begin{cases} \frac{uv^2\sqrt{u^2+v^2}}{u^2+v^4} & \text{for } (u, v) \neq (0, 0), \\ 0 & \text{for } (u, v) = (0, 0) \end{cases}$$

at $(u, v) = (0, 0)$ and prove that f is not differentiable at $(u, v) = (0, 0)$.

Agreement. For the purposes of this text we will assume that every function is as often differentiable as we like, i.e., every function is C^∞. Such functions are often called "*smooth*".

Two differentiation rules are paramount in differential geometry:

1. *Product rule*: if $\odot : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^k$ denotes some "product", i.e., a bilinear map, and $U \ni p \mapsto f_i(p) \in \mathbb{R}^{n_i}$, $i = 1, 2$, are smooth then their product $p \mapsto (f_1 \odot f_2)(p)$ is smooth with derivative
$$d_p(f_1 \odot f_2)(h) = (d_p f_1(h)) \odot f_2(p) + f_1(p) \odot (d_p f_2(h));$$
2. *Chain rule*: if $g : U \to V$ and $f : V \to \mathbb{R}^n$ are smooth then their composition $f \circ g : U \to \mathbb{R}^n$ is smooth with derivative
$$d_p(f \circ g)(h) = d_{g(p)} f(d_p g(h)) = (d_{g(p)} f \circ d_p g)(h),$$
i.e., the Jacobi matrices get multiplied (observe the order!).

Problem 2. Let $\beta : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ be tri-linear; assuming that β is differentiable show that
$$d_{(a_1,a_2,a_3)}\beta(x_1, x_2, x_3) = \beta(x_1, a_2, a_3) + \beta(a_1, x_2, a_3) + \beta(a_1, a_2, x_3).$$
Conclude that $\det : Gl(3) \to \mathbb{R}$ is differentiable with
$$\frac{d_A \det(X)}{\det A} = \mathrm{tr}(A^{-1}X).$$

1 Analysis: Inverse & Implicit mapping theorems

Two theorems are key in understanding the notion of a submanifold and the equivalence of the different characterizations: more specifically, these theorems are the key to relating the implicit and explicit (parametric) representations of curves or surfaces.

Inverse Mapping Theorem. *Suppose that* $f : \mathbb{R}^n \overset{\circ}{\supset} U \to \mathbb{R}^n$ *is continuously differentiable and that* $d_p f : \mathbb{R}^n \to \mathbb{R}^n$ *is invertible at some* $p \in U$*. Then there is an open neighbourhood* $B \subset U$ *of* p *so that:*

(i) $f|_B : B \to \mathbb{R}^n$ *injects (so that* $f : B \to f(B)$ *is invertible);*

(ii) $f(B) \subset \mathbb{R}^n$ *is open;*

(iii) $f^{-1} : f(B) \to B$ *is continuously differentiable with, for* $q \in f(B)$,
$$d_q f^{-1} = (d_{f^{-1}(q)} f)^{-1}.$$

<u>*For short*</u>. A smooth map $f : \mathbb{R}^n \supset U \to \mathbb{R}^n$ has, locally, a smooth inverse where its derivative is invertible (and the derivative of the inverse is the inverse of the derivative, as obtained from the chain rule).

Implicit Mapping Theorem. *Let* $F : \mathbb{R}^m \times \mathbb{R}^k \overset{\circ}{\supset} U \times V \to \mathbb{R}^k$ *be continuously differentiable and suppose that the* $\mathbb{R}^k$*-part*
$$d_{(p,q)} F|_{\{0\}\times\mathbb{R}^k} : \{0\} \times \mathbb{R}^k \to \mathbb{R}^k$$
of $d_{(p,q)}F$ *is invertible for some* $(p,q) \in U \times V$. *Then there are*

- *open neighbourhoods* $A \overset{\circ}{\subset} U$ *of* p *and* $B \overset{\circ}{\subset} V$ *of* q, *and*
- *a (unique) continuously differentiable function* $g : A \to B$

so that $g(p) = q$ *and* $F(x, g(x)) \equiv F(p,q)$; *moreover,*
$$\forall (x,y) \in A \times B : F(x,y) = F(p,q) \Rightarrow y = g(x).$$

<u>*For short*</u>. The equation $F(x,y) \equiv F(p,q)$ can locally, around p, be solved for y if the equation $d_{(p,q)}F(v,w) = 0$ can be solved for w.

<u>*Problem 3*</u>. Use the Implicit mapping theorem to show that, for any point
$$(x,y,z) \in E = \{(x,y,z) \in \mathbb{R}^3 \mid (\tfrac{x}{a})^2 + (\tfrac{y}{b})^2 + (\tfrac{z}{c})^2 = 1\}$$
on the ellipsoid $E \subset \mathbb{R}^3$, there is a neighbourhood U of (x,y,z) so that the intersection $E \cap U$ can be parametrized as a graph of a (real valued) function over one of the coordinate planes.

<u>*Remark*</u>. The Implicit and Inverse mapping theorems are equivalent.

To prove the Inverse mapping theorem from the Implicit mapping theorem: let f satisfy the assumptions of the Inverse mapping theorem, i.e., let $f \in C^1(U, \mathbb{R}^n)$ so that $d_p f$ is invertible for some $p \in U$, and consider
$$F : U \times \mathbb{R}^n \to \mathbb{R}^n, \quad (x,y) \mapsto F(x,y) := f(x) - y.$$

Now let $q := f(p)$ and observe that

$$d_{(p,q)}F = (d_pf, -id) \quad \text{so that} \quad d_{(p,q)}F|_{\mathbb{R}^n\times\{0\}} = d_pf$$

is invertible. Hence, by the Implicit mapping theorem, there is some continuously differentiable map

$$g : \mathbb{R}^n \overset{\circ}{\supset} B \to A \overset{\circ}{\subset} U$$

with $p \in A$ and $q \in B$ so that

$$0 = F(p,q) = F(g(y), y) = f(g(y)) - y$$

for all $y \in B$; moreover, if $f(x) = y$ for $(x,y) \in A \times B$ then $x = g(y)$: hence, $f : A \to B$ injects and g is the inverse of $f|_A$.

Problem 4. Prove the Implicit from the Inverse mapping theorem.

Def. *Let $f : \mathbb{R}^m \overset{\circ}{\supset} U \to \tilde{U} \overset{\circ}{\subset} \mathbb{R}^n$ be smooth. Then f is called:*

- *an* immersion *if d_pf injects for all $p \in U$ (in partiular, $m \leq n$);*
- *a* submersion *if d_pf surjects for all $p \in U$ (in particular, $m \geq n$);*
- *a* diffeomorphism *if it has a smooth inverse (hence, $m = n$).*

2 ODEs: the Picard-Lindelöf theorem

Recall that an *ordinary differential equation* (of order n) is an equation

$$x^{(n)}(t) = f(t, x(t), x'(t), \dots, x^{(n-1)}(t)) \tag{†}$$

for an unknown function $x = x(t)$ which depends on a (real) variable t; however, x may be $\mathbb{R}^m$-valued.

Any such ODE can be re-written as a (system of) ODE(s) of order $n = 1$ by introducing the derivatives as new functions: with $y_k := y^{(k-1)}$ the equation (†) is equivalent to the system

$$\begin{array}{rcl} x_1'(t) & = & x_2(t) \\ & \vdots & \\ x_{n-1}'(t) & = & x_n(t) \\ x_n'(t) & = & f(t, x_1(t), \dots, x_n(t)). \end{array}$$

Hence we never need to think about higher order ODEs.

Picard-Lindelöf theorem. *Let* $\mathbb{R}\times\mathbb{R}^n \overset{\circ}{\supset} I\times U \ni (t,x) \mapsto f(t,x) \in \mathbb{R}^n$ *be continuous and Lipschitz continuous in* y *and* $(t_0, x_0) \in I \times U$*; then there is* $\varepsilon > 0$ *so that the* initial value problem

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0 \tag{$\star$}$$

*has a unique**) *solution on* $(t_0 - \varepsilon, t_0 + \varepsilon)$*.*

Proof. Can be found in any good analysis text book. ■

<u>*Special cases*</u>. Two special cases of the Picard-Lindelöf theorem are of particular importance to us:

(1) if $x \mapsto f(t,x) = f(x)$ is differentiable then $(\star)$ has a unique local solution (prove it for $n = 1$!);

(2) if $y \mapsto f(t,x) = A(t)\,x$ is linear then $(\star)$ has a unique global(!) solution $x : I \to \mathbb{R}^n$.

<u>*Problem 5*</u>. Let $t \mapsto \kappa(t)$ be some function. Find the solution of

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -\kappa \\ \kappa & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{with} \quad \begin{pmatrix} x \\ y \end{pmatrix}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{$*$}$$

[Hint: write (x, y) in polar coordinates.]

<u>3 PDEs: the Poincaré and Maurer-Cartan lemmas</u>

Partial differential equations come in many different flavours. For us the following two (systems of) partial differential equations will be of particular importance.

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*) Of course, choosing another (smaller) ε gives a restriction of the previous solution, hence "another" solution. Compare this with the Peano theorem which only requires continuity but does not assert uniqueness (expl.: $x'^2 = |x|$, $x(0) = 0$).

Poincaré lemma. *Given $\varphi = \varphi(u,v)$ and $\psi = \psi(u,v)$ the partial differential equation*

$$d\sigma = \varphi\, du + \psi\, dv \quad \Leftrightarrow \quad \begin{cases} \sigma_u = \varphi \\ \sigma_v = \psi \end{cases}$$

has a local (on simply connected domains) solution σ iff $\varphi_v = \psi_u$. Moreover, the solution is unique up to an additive constant.

Proof. Can be found in any good analysis text book. ■

The following theorem is less commonly found in analysis textbooks:

Maurer-Cartan lemma. *Given $\Phi = \Phi(u,v), \Psi = \Psi(u,v) \in \mathfrak{gl}(n)$ the partial differential equation*

$$dF = F \cdot (\Phi\, du + \Psi\, dv) \quad \Leftrightarrow \quad \begin{cases} F_u = F \cdot \Phi \\ F_v = F \cdot \Psi \end{cases} \tag{$\star$}$$

can locally (on small open sets) be solved to get $F = F(u,v) \in Gl(n)$ iff

$$\Phi_v - \Psi_u = [\Phi, \Psi] := \Phi\Psi - \Psi\Phi. \tag{$\star\star$}$$

The solution is unique up to left multiplication by a constant matrix.

Proof. First we show that the *Maurer-Cartan equation* ($\star\star$) is necessary: if F is a solution of ($\star$) then

$$\begin{aligned} 0 &= (F_u)_v - (F_v)_u \\ &= F_v\Phi + F\Phi_v - F_u\Psi - F\Psi_u \\ &= F\,(\Psi\Phi + \Phi_v - \Phi\Psi - \Psi_u). \end{aligned}$$

To show that ($\star\star$) is also a sufficient condition suppose that Φ and Ψ are defined on $(-\varepsilon,\varepsilon)^2$ and satisfy ($\star\star$). We first use the Picard-Lindelöf theorem twice to obtain F:

(1) fix $v = 0$ and consider the initial value problem

$$F_u(u,0) = F(u,0)\Phi(u,0), \quad F(0,0) = \mathrm{id}_{\mathbb{R}^n},$$

which is a linear system of ordinary differential equations, hence has a unique solution $u \mapsto F(u,0)$ by the Picard-Lindelöf theorem;

(2) now fix u and consider the initial value problem

$$F_v(u,v) = F(u,v)\Psi(u,v), \quad F(u,0) \text{ as obtained in (1)},$$

which is again has a unique solution $v \mapsto F(u,v)$ by the Picard-Lindelöf theorem.

Now we got $F(u,v)$ at every $(u,v) \in (-\varepsilon,\varepsilon)^2$. Taking differentiability of F for granted we have now to verify that F satisfies $(\star)$. By construction in (2), $F_v = F\Psi$ so that we only need to verify $F_u = F\Phi$. Thus compute

$$\begin{aligned} (F_u - F\Phi)_v &= F_{vu} - F_v\Phi - F\Phi_v \\ &= (F\Psi)_u - F\Psi\Phi - F\Phi_v \\ &= F_u\Psi + F(\Psi_u - \Phi_v - \Psi\Phi) \\ &= (F_u - F\Phi)\Psi \end{aligned}$$

by $(\star\star)$; which, as a linear system of ODEs (u is fixed), has the unique solution $F_u - F\Phi \equiv 0$ since $(F_u - F\Phi)(u,0) = 0$ by construction in (1).

Next we wish to show that $F(u,v) \in Gl(n)$ for all $(u,v) \in (-\varepsilon,\varepsilon)^2$. Suppose $F(u_0,v_0)$ was not invertible at some point (u_0,v_0). Then $F(u_0,v_0)$ would not surject and there we could find a non-zero vector $v \in \mathbb{R}^n$ so that $v^tF(u_0,v_0) = 0$. On the other hand, v^tF satisfies

$$(v^tF)_u = (v^tF)\Phi \quad \text{and} \quad (v^tF)_v = (v^tF)\Psi,$$

which is a linear system of partial differential equations, thus has a unique solution by a similar argument as above. As $v^tF \equiv 0$ is a solution with the given initial value $v^tF(u_0,v_0) = 0$, this would be it and we also would have $v^t = v^tF(0,0) = 0$, contradicting the initial assumption.

Finally we wish to examine uniqueness: suppose that $\tilde{F}$ is another solution of $(\star)$. Using that

$$0 = (\mathrm{id}_{\mathbb{R}^3})_u = (FF^{-1})_u = F_uF^{-1} + F(F^{-1})_u$$

so that

$$(F^{-1})_u = -F^{-1}F_uF^{-1},$$

we compute

$$(\tilde{F}F^{-1})_u = (\tilde{F}_u)F^{-1} - \tilde{F}(F^{-1}F_uF^{-1}) = \tilde{F}(\Phi - \Phi)F^{-1} = 0,$$

and similarly for $(\tilde{F}F^{-1})_v$, showing that $\tilde{F} = AF$ with constant A. ∎

Problem 6. Let $(u,v) \mapsto \Phi(u,v), \Psi(u,v) \in \mathfrak{gl}(2)$ be trace free. Prove that a solution $(u,v) \mapsto F(u,v) \in Gl(2)$ of $F_u = F\Phi$ and $F_v = F\Psi$ has constant determinant. [Hint: verify that $(\det F)_u = \det F \ \mathrm{tr}(F^{-1}F_u)$.]

Appendix - Vector algebra

The following notations, definitions and formulas are used throughout the text without further comment or explanation. It should make a good exercise to prove any unfamiliar identities.

1 Products of vectors in $\mathbb{R}^3$

$\underline{\mathbb{R}^3 \times \mathbb{R}^3 \ni (a,b) \mapsto a \cdot b \in \mathbb{R}}$, the *Euclidean inner product* on $\mathbb{R}^3$:

the inner product is *symmetric*, $a \cdot b = b \cdot a$; two vectors a and b are said to be *orthogonal* or *perpendicular* if $a \cdot b = 0$; more generally, the *angle* $\angle(a,b)$ between two vectors is given by

$$a \cdot b = |a|\,|b| \cos \angle(a,b).$$

$\underline{\mathbb{R}^3 \times \mathbb{R}^3 \ni (a,b) \mapsto a \times b \in \mathbb{R}^3}$, the *cross product* on $\mathbb{R}^3$:

the cross product is *skew-symmetric*, $a \times b = -b \times a$; two vectors a and b are linearly dependent if and only if $a \times b = 0$.

Also,

$$|a \times b|^2 = |a|^2|b|^2 - (a \cdot b)^2 = |a|^2|b|^2 \sin^2 \angle(a,b),$$

giving the area of the parallelogram spanned by a and b;

$$a \cdot (b \times c) = \det(a,b,c)$$

showing that $b, c \perp b \times c$ and $(c \times a) \cdot b + a \cdot (c \times b) = 0$.

$\underline{\mathbb{R}^3 \times \mathbb{R}^3 \ni (a,b) \mapsto a \wedge b \in \mathfrak{o}(3)}$, the *wedge product*, defined by

$$(a \wedge b)\, x := (x \cdot a)\, b - (x \cdot b)\, a = (a \times b) \times x;$$

the product is *skew-symmetric*, $a \wedge b = -b \wedge a$, and $a \wedge b$ is a *skew-symmetric endomorphism*,

$$((a \wedge b)\, x) \cdot y + x \cdot ((a \wedge b)\, y) = 0.$$

Remark. The inner and wedge products generalize to $\mathbb{R}^n$ in an entirely straightforward way, the cross product does not.

2 Transformations of $\mathbb{R}^3$

Endomorphisms are linear maps from a vector space to itself:

$$\mathrm{End}(\mathbb{R}^3) := \{X : \mathbb{R}^3 \to \mathbb{R}^3 \,|\, X \text{ linear}\} \cong M(3 \times 3, \mathbb{R}),$$

where a basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of $\mathbb{R}^3$ is used to identify an endomorphism with a matrix:

$$X\mathbf{e}_j = \sum_{i=1}^{3} \mathbf{e}_i x_{ij}.$$

Linear transformations of $\mathbb{R}^3$ are invertible endomorphisms; important transformation groups are:

$$\begin{array}{rcl} Gl(3) & := & \{A \in \mathrm{End}(\mathbb{R}^3) \,|\, \det A \neq 0\}, \\ Sl(3) & := & \{A \in Gl(3) \,|\, \det A = 1\}, \\ O(3) & := & \{A \in \mathrm{End}(\mathbb{R}^3) \,|\, \forall v, w \in \mathbb{R}^3 : (Av) \cdot (Aw) = v \cdot w\} \\ & \cong & \{A \in M(3 \times 3, \mathbb{R}) \,|\, A^t A = \mathrm{id}_{\mathbb{R}^3}\}, \\ SO(3) & := & \{A \in O(3) \,|\, \det A = 1\}, \end{array}$$

the *general* and *special linear groups* and the *orthogonal* and *special orthogonal groups*, respectively. Note that

$$A \in O(3) \Rightarrow \det A = \pm 1.$$

When $t \mapsto A(t) \in G$ is differentiable with $A(0) = \mathrm{id}_{\mathbb{R}^3}$ then $A'(0) \in \mathfrak{g}$, where $\mathfrak{g}$ corresponding to G from the above list is[†)]:

$$\begin{array}{rcl} \mathfrak{gl}(3) & := & \mathrm{End}(\mathbb{R}^3), \\ \mathfrak{sl}(3) & := & \{X \in \mathrm{End}(\mathbb{R}^3) \,|\, \mathrm{tr}\, X = 0\}, \\ \mathfrak{o}(3) & := & \{X \in \mathrm{End}(\mathbb{R}^3) \,|\, \forall v, w \in \mathbb{R}^3 : (Xv) \cdot w + v \cdot (Xw) = 0\} \\ & \cong & \{X \in M(3 \times 3, \mathbb{R}) \,|\, X^t + X = 0\}; \end{array}$$

note that, if $t \mapsto A(t) \in O(3)$ then, in fact, $t \mapsto A(t) \in SO(3)$ since the determinant $t \mapsto \det A(t) \in \{-1, +1\}$ is continuous and $A(0) \in SO(3)$.

[†)] The first of these is fairly obvious, the others are not — for the second see Problem 2 (Tools from Analysis), the third is discussed in Chapter 1.

The *Euclidean motions* also form a transformation group:

$$\mathbb{R}^3 \ni x \mapsto Ax + c \in \mathbb{R}^3, \quad \text{where} \quad A \in SO(3) \text{ and } c \in \mathbb{R}^3;$$

thus a Euclidean motion is the composition of a *rotation* and a *translation*. Note, however, that Euclidean motions are generally not linear since $0 \not\mapsto 0$ (if $c \neq 0$).

If $A \in SO(3)$ then $a_i = A\mathbf{e}_i$ form a *positively oriented orthonormal basis* of $\mathbb{R}^3$:

$$a_i \cdot a_j = \delta_{ij} \quad \text{and} \quad \det(a_1, a_2, a_3) > 0$$

(δ_{ij} denotes the Kronecker symbol). In particular,

$$a_1 \times a_2 = a_3, \;\; a_2 \times a_3 = a_1, \;\; a_3 \times a_1 = a_2$$

and

$$(Av) \cdot (Aw) = v \cdot w, \quad (Av) \times (Aw) = A(v \times w).$$

Remark. Every fact in this section — apart from those that involve the cross product — generalizes to higher dimensions in an entirely straightforward way, i.e., without any non-obvious changes.

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