UNIVERSITY OF Cincinnati

Department of Mathematical Sciences
PO Box 210025
Cincinnati OH 45221-0025

4^{th} Floor French Hall West
Phone (513) 556-4050
Fax (513) 556-3417

QUALIFYING EXAM PRACTICE PROBLEMS

$\mathbb{R}$ is the field of real numbers and $\mathbb{R}^n$ is n-dimensional Euclidean space

Proofs, or counter examples, are required for all problems.

(1) Let (X, d) be a metric space and S a subset of X. State the logical implications that hold among the following conditions. (No proofs are required here, but where possible, provide 'names' of appropriate theorems.)
 (a) S is bounded
 (b) S is closed
 (c) S is compact
 (d) S is complete
 (e) S is sequentially compact
 (f) S is totally bounded

What changes if $X = \mathbb{R}^n$ and $d(x, y) = ||x - y||$?

(2) Let $x_n := (-1)^n \dfrac{\sqrt{n^2+1}}{n+1}$. Is $(x_n)_1^\infty$ a Cauchy sequence?

(3) Determine $\limsup_{n\to\infty} x_n$ and $\liminf_{n\to\infty} x_n$ if $x_n := (-1)^n + (-1)^n \dfrac{3^n}{4^{n-2}}$.

(4) Prove that a sequence $(a_n)_1^\infty$ of real numbers that has *no* Cauchy subsequences must be unbounded.

(5) Suppose $[0, \infty) \xrightarrow{f} \mathbb{R}$ is continuous and satisfies $\lim_{x\to\infty} f(x) = 0$. Is f uniformly continuous on $[0, \infty)$? Why, or why not?

(6) Suppose $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is uniformly continuous. For each $n \in \mathbb{N}$, define

$$f_n(x) := f\left(x + \frac{1}{n}\right).$$

Prove that $(f_n)_1^\infty$ converges uniformly, and find the limit function.

(7) Determine whether or not the following series converges.

$$1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \frac{1}{11} - \frac{1}{12} + \frac{1}{13} - \cdots$$

(8) Find the interval of convergence for the series $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{n!}(x-2)^n$.

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(9) Suppose that $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is differentiable at the point a. Prove that

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a-h)}{2h}.$$

(10) Suppose that $\mathbb{R} \xrightarrow{f} \mathbb{R}$ is differentiable with $f'(x) \neq 0$ for all $x \in \mathbb{R}$. Prove that f is injective on all of $\mathbb{R}$.

(11) Let $(a_n)_1^\infty$ be an increasing sequence in $(0,1)$ with limit 1. Define $[0,1] \xrightarrow{f} \mathbb{R}$ by

$$f(x) := \begin{cases} 1 & \text{if } x = a_n \text{ for some } n \in \mathbf{N} \\ 0 & \text{otherwise}. \end{cases}$$

Is f Riemann integrable? Why, or why not?

(12) Let $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0). \end{cases}$$

Determine where f is differentiable.

(13) Let $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ be defined by

$$f(x,y) := \begin{cases} \dfrac{xy^2}{x^2+y^4} & \text{when } (x,y) \neq (0,0) \\ 0 & \text{when } (x,y) = (0,0). \end{cases}$$

(a) Let $u := (a,b)$ with $a \neq 0$. Show that the directional derivative $D_u f(0,0)$ exists and find its value.
(b) Show that f is not differentiable at $(0,0)$. (Hint: Is it continuous there?)

(14) Suppose that $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ is a function with the property that for all $x \in \mathbb{R}^2$, $|f(x)| \leq |x|^2$. Prove that f is differentiable at the origin.

(15) (a) Let $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}$ be defined by $f(x,y) := x + y$. Prove that f is differentiable on $\mathbb{R}^2$ and that for all $(a,b), (x,y) \in \mathbb{R}^2$, $Df(a,b)(x,y) = x + y$.
(b) Suppose $\mathbb{R}^2 \xrightarrow{\varphi} \mathbb{R}$ is defined by

$$\varphi(x,y) := \int_0^{x+y} g(t)\, dt \quad \text{where } \mathbb{R} \xrightarrow{g} \mathbb{R} \text{ is continuous}.$$

Prove that φ is differentiable and find the derivative.

(16) Let V be a vector space on which an inner product is defined. Define the norm for $v \in V$ by $||v|| := \sqrt{< v, v >}$. Show that the norm satisfies the *triangle inequality* $||v + w|| \leq ||v|| + ||w||$ for any $v, w \in V$.

(17) Let A be an invertible symmetric operator on a vector space V. Use the inner product definition of a symmetric operator to show that A^{-1} is also a symmetric operator.

(18) Take $A \in \text{Mat}_{m \times m}(K)$.
(a) A square matrix A is *nilpotent* if $A^n = 0$ for some positive integer n. Show that if A is nilpotent then $I - A$ is invertible.

(b) Show that if $A^3 - A + I = 0$ then A is invertible.

(19) (a) Let $T : V \to W$ be a linear mapping between two vector spaces. Show $T(0) = 0$.
(b) Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping. Suppose

$$L([3,1]) = [1,2] \quad \text{and} \quad L([-1,0]) = [1,1].$$

Compute $L([1,0])$ and $L([0,1])$.
(c) Give an example of a linear mapping that is not injective on its image.

(20) Let V be a finite-dimensional vector space over $\mathbb{R}$ or $\mathbb{C}$ with an inner product. Let A be a linear map. Show that

$$\mathrm{Im} A^{\mathsf{T}} = (\ker A)^{\perp},$$

that is, show the image of A^{T} is the orthogonal complement of the kernel of A.

(21) Let J_{rs} be the $n \times n$ matrix whose rs-entry is c and all other entries are 0. Set $E_{rs} := I + J_{rs}$.
(a) Compute $\det E_{rs}$. Note there are two distinct cases.
(b) Let A be an $n \times n$ matrix. What is the effect of multiplying A on the left by E_{rs}? What is the effect of multiplying A on the right by E_{rs}?

(22) Compute the determinant of an arbitrary upper-triangular $n \times n$ matrix A.

(23) Let $A = (a_{ij}) \in \mathrm{Mat}_{n \times n}(K)$ be such that

$$\sum_{j=1}^{n} a_{ij} = c, \quad i = 1, ..., n$$

for some $c \in K$. Show that c is an eigenvalue for A.

(24) Consider $A \in \mathrm{Mat}_{2 \times 2}(\mathbb{R})$. Does A necessarily have a real eigenvalue? If so, prove it. If not, give a counterexample.

(25) Give a 3×3 matrix with real entries whose eigenspace is exactly two-dimensional. Find a basis of generalized eigenvectors for your matrix.