#2.3.7 Let $E \subseteq \mathbb{R}$ be non-empty & bounded. Thus E has a least upper bound, say $L := \text{l.u.b.}(E) \in \mathbb{R}$.

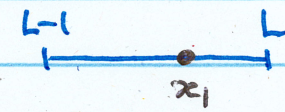
Claim Suppose $L \notin E$. Then there exists a strictly increasing sequence $(x_n)_1^\infty$ in E that converges to L.

Proof We make repeated use of the Approximation Property for LUBs, which in our case says that

$(*)$ $\quad \forall \varepsilon > 0, \ \exists x \in E$ s.t. $L - \varepsilon < x \leq L$.

In fact, since $L \notin E$, we can replace the last ineq$^{\underline{n}}$ with $x < L$.

1/ Let $\varepsilon_1 := 1$. Appeal to $(*)$ to get a pt $x_1 \in E$ with $L - 1 < x_1 < L$.



Let $\varepsilon_2 := \min\{\frac{1}{2}, L - x_1\} > 0$. Appeal to $(*)$ to get $x_2 \in E$ with $L - \varepsilon_2 < x_2 < L$. Note that

$$\varepsilon_2 \leq 1/2 \implies L - 1/2 < x_2 < L$$

and

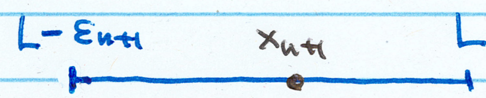
$$\varepsilon_2 \leq L - x_1 \implies x_1 = L - (L - x_1) \leq L - \varepsilon_2 < x_2. \ \therefore x_1 < x_2.$$

Let $\varepsilon_3 := \min\{\frac{1}{3}, L - x_2\}$ & use $(*)$ to get $x_3 \in E$ w/ ....

2/ Assume that we have defined $\varepsilon_1, \ldots, \varepsilon_n$ as above & selected $x_1, \ldots, x_n \in E$ so that $x_1 < x_2 < \cdots < x_n$ & so that

$$\forall \ 1 \leq k \leq n, \quad L - 1/k < x_k < L.$$

Let $\varepsilon_{n+1} := \min\left\{\frac{1}{n+1}, L - x_n\right\} > 0$. According to $(*)$, there is a point x_{n+1} in E with



$$L - \varepsilon_{n+1} < x_{n+1} < L.$$

Then $\varepsilon_{n+1} \leq 1/n+1$ tells us that $L - \frac{1}{n+1} < x_{n+1} < L$, and

$\varepsilon_{n+1} \leq L - x_n \implies x_n = L - (L - x_n) \leq L - \varepsilon_{n+1} < x_{n+1}$, so $x_n < x_{n+1}$.

3/ Therefore by the Principle of Mathematical Induction, for each $n \in \mathbb{N}$ there is a point $x_n \in E$ such that

$$L - 1/n < x_n < L$$

and moreover, $\forall n \in \mathbb{N}$ we have $x_n < x_{n+1}$. Thus we do indeed have a strictly increasing seq $(x_n)_{n=1}^{\infty}$ in E.

Moreover, since $\frac{1}{n} \longrightarrow 0$ as $n \longrightarrow \infty$, $L - 1/n \longrightarrow L$.

Hence the Squeeze Principle permits us to assert that

$$\lim_{n \to \infty} x_n = L.$$

$\boxtimes$