1.4.5

Lemma For each $n \in \mathbb{N}$, the polynomial $p_n(x) := 1 - x^n$ can be factored as
$$1 - x^n = (1-x)(1 + x + x^2 + \cdots + x^{n-1}).$$

Proof

For $n=1$, both sides of the identity are just "$1-x$". In general, if we multiply out the right-hand-side and "collect terms" we get the left-hand-side. $\square$

Claim Let $a, b \in \mathbb{R}$ with $a \geq 0$, $b \geq 0$. Then $\forall\, n \in \mathbb{N}$

(a) $a < b \iff a^n < b^n$

and

(b) $a < b \iff a^{1/n} < b^{1/n}$.

Proof

First note that (a) $\iff$ (b) (check this!)*, so it suffices to establish (a).

Second, we may assume that $b > 0$. Thus we can "divide thru by b", and so writing $x := a/b$ we see that (a) is equivalent to the assertion that, for $x \geq 0$,
$$x < 1 \iff x^n < 1.$$

To verify this we observe that
$$x^n < 1 \iff 1 - x^n > 0$$
and since $1 - x^n = (1-x)(1 + x + \dots + x^{n-1})$, and $1 + x + \dots + x^{n-1} > 0$, we have
$$1 - x^n > 0 \iff 1 - x > 0.$$

Finally,
$$1 - x > 0 \iff x < 1.$$

* Use the fact that $(x^{1/n})^n = x$.