#1.2.6 Given $a, b \in \mathbb{R}$ with $a \geq 0$ & $b \geq 0$, we define

$$A(a,b) := \frac{a+b}{2} \quad \text{and} \quad G(a,b) := \sqrt{ab} = (ab)^{1/2}.$$

<u>Claim</u> Let $a, b \in \mathbb{R}$. Suppose that $0 \leq a \leq b$. Then

$$(*) \qquad a \leq G(a,b) \leq A(a,b) \leq b.$$

Moreover, equality holds in any one of the three ineqs in $(*)$ if and only if $a = b$.

Before proving this claim, we record the following.

<u>Lemma</u> Let $x, y \in \mathbb{R}$. Suppose that $x \geq 0$ and $y \geq 0$. Then

$$x \leq y \iff x^2 \leq y^2.$$

<u>Proof</u>

The inequality $x \leq y$ is equivalent to $0 \leq y - x$. The inequality $x^2 \leq y^2$ is equivalent to $0 \leq y^2 - x^2$. Since $y^2 - x^2 = (y-x)(y+x)$, and $y + x > 0$, we see that $y^2 - x^2 \geq 0$ iff $y - x \geq 0$. $\square$

<u>Proof of Claim</u>

We begin by recording some consequences of $0 \leq a \leq b$. By adding a or b to both sides of the inequality $a \leq b$ we see that

$$(\#) \qquad 2a \leq a + b \leq 2b.$$

(Also, here equality holds in either of the "$\leq$" iff $a = b$.)

Similarly, by multiplying both sides of $a \leq b$ by a or by b we get

$$(\dagger) \qquad a^2 \leq ab \leq b^2.$$

(Again, equality holds in either of the "$\leq$" iff $a = b$.)

Now we are ready to verify the first & last "$\leq$" in $(*)$.

We use the abbreviations $A = A(a,b)$ & $G = G(a,b)$.

To see that $a \le G$

If $a = 0$, then $G = 0$ too and so $a \le G$. Assume $a > 0$. Then $b > 0$. From (†) we have $a^2 \le ab$. The Lemma now tells us that $a \le \sqrt{ab} = G$.

i.e., $a^2 \le G^2$

To see that $A \le b$

From (#) we have $a + b \le 2b$, so $A = \frac{a+b}{2} \le b$.

Next we demonstrate that $G \le A$. If $a = 0$, then $G = 0$ and $A = b/2 \ge 0$, so $G \le A$. Assume $a > 0$. Then $b > 0$. Also, $G > 0$ and $A > 0$. Thus by the Lemma, $G \le A$ iff $G^2 \le A^2$. Now

$$G^2 = ab \quad \text{and} \quad A^2 = \left(\frac{a+b}{2}\right)^2 = \frac{a^2 + 2ab + b^2}{4},$$

so

$$G^2 \le A^2 \iff 4ab \le a^2 + 2ab + b^2.$$

Evidently,

$$0 \le (a-b)^2 = a^2 - 2ab + b^2 \implies 4ab \le a^2 + 2ab + b^2.$$

Therefore $G^2 \le A^2$, and so $G \le A$.

It remains to verify the statements about equality. If $a = b$, then from (*) we see that $a = G = A = b$. If $a = G$ or $b = A$, then the corresponding statements about equality in (#) or in (†) give us that $a = b$. Finally, suppose that $G = A$. Then (from our calculations above)

$$4ab = (2G)^2 = (2A)^2 = (a+b)^2 = a^2 + 2ab + b^2$$

$$\implies (a-b)^2 = 0.$$

Therefore $a = b$.