MATH 443, FINAL EXAM

Due at 2:00pm on Thursday, Dec. 21

General Rules: This is a take-home examination. There is no fixed time limit other than the submission deadline, and you are not required to complete your work in a single sitting. All numbered problems are weighted equally. *Please write your answers in an exam booklet, or on sheets of paper stapled together.*

Your answers should be obtained and written without collaboration or any direct assistance from other people. You may ask the instructor for clarification of the meaning or intent of the problems; however hints will not be provided. Permitted sources of information include the textbook, your class notes, and material posted on the course website.

Ethics Statement: The strength of the university depends on academic and personal integrity. In this course, you must be honest and truthful. Ethical violations include cheating on exams, plagiarism, reuse of assignments, improper use of the Internet and electronic devices, unauthorized collaboration, alteration of graded assignments, forgery and falsification, lying, facilitating academic dishonesty, and unfair competition.

1. Let f be an integrable function on the interval $[-\pi, \pi]$, and define $g(x) = f(x+\pi)$ using the usual wrap-around conventions that $x + 2\pi$ is equivalent to x.

If $g(x) = -f(x)$, what information does that give us about the Fourier coefficients $\hat{f}(n)$?

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2. On the interval $[-\pi, \pi]$, consider the function $f(x) = \begin{cases} 1, & \text{if } 0 < x < \pi \\ -1, & \text{if } -\pi < x < 0. \\ 0, & \text{if } x = 0, \pm\pi \end{cases}$

The *Gibbs Phenomenon* is the well-known property that each of the Fourier partial sums $S_N f(x)$ overshoots the discontinuities of the original function f, with $\max |S_N f(x)| \approx 1.18$.

Show that the Cesàro means of f do not experience the Gibbs phenomenon, by proving that $|\sigma_N f(x)| < 1$ for all N and all x.

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3. Suppose $f \in \mathcal{S}(\mathbb{R})$, and define $g_1(x) = \int_{-\infty}^{x} f(y)\,dy$ and $g_2(x) = -\int_{x}^{\infty} f(y)\,dy$.

a.) If we assume that $\int_{-\infty}^{\infty} f(y)\,dy = 0$, show that $g_1(x) = g_2(x)$, and that this function is in $\mathcal{S}(\mathbb{R})$.

b.) With the same assumption as before, express the functions $\widehat{g_1}(\xi)$ and $\widehat{g_2}(\xi)$ (which should be identical to each other) in terms of $\hat{f}(\xi)$.

c.) Prove the identity $\int_{0}^{\infty} f(y)\,dy - \int_{-\infty}^{0} f(y)\,dy = \frac{-1}{\pi i}\int_{-\infty}^{\infty} \frac{\hat{f}(\xi)}{\xi}\,d\xi,$ under the assumption that $\int_{\mathbb{R}} f(y)\,dy = 0$.

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4. What is the value of $\sum_{n=-\infty}^{\infty} \frac{\sin n}{n}$ and of $\sum_{n=1}^{\infty} \frac{\sin n}{n}$?

(This is a special case of a recent homework problem. Most of you were not very careful on that assigment, so this is your chance to get it right.)

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5. Suppose that $u(x,t)$ is a solution to the fourth-order differential equation

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = -\Delta(\Delta u)(x,t), & x \in \mathbb{R}^d,\ t > 0, \\ u(x,0) = f(x) \end{cases}$$

a.) Express $\hat{u}(\xi, t)$ in terms of $\hat{f}(\xi)$.

b.) Show that there exists a function $K(x)$ so that $u(x,t) = t^{-d/4}\int_{\mathbb{R}^d} f(y) K(\frac{x-y}{\sqrt[4]{t}})\,dy$.