Name: ________________________________ April 25, 2007

Math 416, Final Exam

Due at 5:00pm on Friday, May 4

General Rules: This is a take-home examination. There is no fixed time limit other than the submission deadline, and you are not required to complete your work in a single sitting. All numbered problems are weighted equally. In questions that are subdivided into several parts, each part is worth an approximately equal share. *Please write your answers on sheets of paper stapled together, including this sheet as a cover page.*

Your answers should be obtained and written without collaboration or help from other people. You may ask the instructor to clarify the meaning or intent of the problems; extra hints will not be provided. Permitted sources of information include the textbook, your class notes, and material posted on the course website.

Ethics Statement: The strength of the university depends on academic and personal integrity. In this course, you must be honest and truthful. Ethical violations include cheating on exams, plagiarism, reuse of assignments, improper use of the Internet and electronic devices, unauthorized collaboration, alteration of graded assignments, forgery and falsification, lying, facilitating academic dishonesty, and unfair competition.

Please Read and Sign the following statement: *I understand the rules of this examination and the ethical policies of Johns Hopkins University. The answers I have submitted represent my own work.*

______________________________ __________

Your Signature Date

1. Does there exist a closed set $E \subseteq [0,1]$ with $m(E) = 1$, aside from the obvious example $E = [0,1]$?

2. Let f be any function in $L^p(\mathbb{R})$, $1 \leq p \leq \infty$. Show that $\lim_{n\to\infty} \left\| f\chi_{\{|f|<\frac{1}{n}\}} \right\|_p = 0$.

3a. Suppose $f \in L^1(E) \cap L^\infty(E)$. Show that $f \in L^p(E)$ for every $1 \leq p \leq \infty$, and suggest an upper bound for $\|f\|_p$ in terms of the quantities $\|f\|_1$ and $\|f\|_\infty$.

3b. Given $f \in L^p(E) \cap L^q(E)$, consider the function $F(r) = \int_E |f(x)|^r \, dx$. Prove that $F(r)$ is a continuous function at every point in the interval $p \leq r \leq q$.

4. For each number $\alpha \in (0,1)$, compute the Fourier coefficients of the function $f_\alpha(x) = e^{i\alpha x}$.

Use this information to derive the series identity $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(\alpha - n)^2} = \frac{\pi^2}{\sin^2(\alpha\pi)}$.

5. Suppose f is a function in $BV[-\pi,\pi]$. Prove that $|\hat{f}(n)| \leq \frac{2}{|n|} V_{-\pi}^{\pi} f$ for each $n \neq 0$.

Hint: Integrate by parts.