MATH 415, FINAL EXAM

Due at 5:00pm on Monday, Dec. 15

General Rules: This is a take-home examination. There is no fixed time limit other than the submission deadline, and you are not required to complete your work in a single sitting. All numbered problems are worth 10 points each. *Please write your answers in an exam booklet, or on sheets of paper stapled together, including this signed sheet as a cover page.*

Your answers should be obtained and written without collaboration or any direct assistance from other people. You may ask the instructor for clarification of the meaning or intent of the problems; however hints will not be provided. Permitted sources of information include the textbook, your class notes and exams, and material posted on the course website.

Ethics Statement: The strength of the university depends on academic and personal integrity. In this course, you must be honest and truthful. Ethical violations include cheating on exams, plagiarism, reuse of assignments, improper use of the Internet and electronic devices, unauthorized collaboration, alteration of graded assignments, forgery and falsification, lying, facilitating academic dishonesty, and unfair competition.

Please sign and date this statement: I certify that I have followed the rules of this examination, and that all anwsers are entirely my own work.

Signature	Date

If you cannot truthfully sign the statement, it is best to contact me immediately to discuss the issues.
Read the Ethics Board Constitution, Article V, for further information.

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1. Let $E \subseteq \mathbb{R}$. Prove that the following two statements about a number $x \in \mathbb{R}$ are equivalent:
i) $x = \sup\{E \cap (-\infty, x)\}$.
ii) There is a strictly increasing sequence of numbers $(x_n) \subseteq E$ with $\lim_{n \to \infty} x_n = x$.

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2. Consider the sets $A = \{$Bounded sequences of real numbers $(x_n)\}$
and $B = \{$Bounded sequences (x_n) with $\lim_{n \to \infty} x_n = 0\}$.

Show that $A \sim B$ (as defined in chapter 2).

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3. Suppose a sequence $(a_n) \subseteq \mathbb{R}$ has $\|(a_n)\|_1 \leq \frac{3}{2}\|(a_n)\|_\infty$.

Show that the largest term in $(|a_n|)$ is at least twice as big as the second-largest term.

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4. [Related to Exercise 4.31] Given disjoint closed sets $A, B \subseteq M$, show that there exist disjoint open sets U, V with $A \subseteq U$ and $B \subseteq V$.

[Suggestion: For any $x \in A$, the distance $d(x, B) = \inf\{d(x, y) : y \in B\} > 0$.
For identical reasons, $d(y, A) > 0$ when $y \in B$.]

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5. Prove that any subset of $\mathbb{Q}$ with more than one point is disconnected.

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6. Suppose a metric space M has the property that for any $\epsilon > 0$, there is a totally bounded set $E_\epsilon \subseteq M$ with $\sup\limits_{x \in M} d(x, E_\epsilon) < \epsilon$.

Show that M is totally bounded. [Suggestion: Modify a covering of $E_{\frac{\epsilon}{2}}$]

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7. [Exercise 8.80] Show that the definite integral $I(f) = \int_a^b f(t)\, dt$ is continuous from $C([a, b])$ into $\mathbb{R}$. What is $\|I\|$?

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8. Let $p_n(x) = \sum\limits_{k=0}^{n} c_{kn} x^k$ be a polynomial degree n in $C([0, 1])$.

8a) Show that $|c_{0n}| \leq \|p_n\|_\infty$ regardless of n.

8b) Describe a sequence of polynomials (p_n) with $\|p_n\|_\infty \leq 1$ and $|c_{1n}| \to \infty$.
[Suggestion: Express $\sin(2n+1)x$ in terms of powers of $\sin x$. Can you compute $p_n'(0)$?]

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9. [Exercise 11.51] Let X be a compact metric space, and let (f_n) be a sequence in $C(X)$ that is uniformly convergent. Show that (f_n) is both uniformly bounded and equicontinuous.

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10. The proof of Theorem 11.18 contains the (slightly edited) sentence "We need to show that (f_n) has a Cauchy subsequence in the norm of $C(X)$."

In other words, we need to prove an inequality like

$$|f_{n_j}(x) - f_{n_\ell}(x)| < \epsilon \text{ for all } j, \ell > J \text{ and every } x \in X.$$

There are four principal objects here: the sequence (f_n), a subsequence $(f_{n_j})_{j=1}^{\infty}$, an integer $J < \infty$, and a number $\epsilon > 0$. In what order should they be chosen to make a valid proof?

In what order are they chosen in the textbook's argument?

Extra Credit Assignment. [up to 5 points]

Write a complete and correct proof of the Arzelà-Ascoli Theorem.
[Suggestion: Read about Helly's Selection Principle (Theorem 13.13) before you get started.]