MATH 415, FINAL EXAM

Due at 5:00pm on Monday, Dec. 17

General Rules: This is a take-home examination. There is no fixed time limit other than the submission deadline, and you are not required to complete your work in a single sitting. All numbered problems are worth 10 points each. *Please write your answers in an exam booklet, or on sheets of paper stapled together, including this signed sheet as a cover page.*

Your answers should be obtained and written without collaboration or any direct assistance from other people. You may ask the instructor for clarification of the meaning or intent of the problems; however hints will not be provided. Permitted sources of information include the textbook, your class notes and exams, and material posted on the course website.

Ethics Statement: The strength of the university depends on academic and personal integrity. In this course, you must be honest and truthful. Ethical violations include cheating on exams, plagiarism, reuse of assignments, improper use of the Internet and electronic devices, unauthorized collaboration, alteration of graded assignments, forgery and falsification, lying, facilitating academic dishonesty, and unfair competition.

Please sign and date this statement: I certify that I have followed the rules of this examination, and that all anwsers are entirely my own work.

Signature ____________________ Date ________

If you cannot truthfully sign the statement, it is best to contact me immediately to discuss the issues. Read the Ethics Board Constitution, Article V, for further information.

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1. [Exercise 3.25] Show that the expression $\|f\|_1 := \displaystyle\int_0^1 |f(x)|\,dx$ defines a norm on the vector space $C([0,1])$.

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2. [Exercise 4.19] Given a set $A \subset M$ and two points $x, y \in \bar{A}$, show that for each $\epsilon > 0$ there exist points $\tilde{x}, \tilde{y} \in A$ so that $d(\tilde{x}, \tilde{y}) > d(x, y) - \epsilon$.

Use this fact to prove that $\mathrm{diam}(\bar{A}) = \mathrm{diam}(A)$.

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3. What is the connected component of $\mathbb{Q}$ containing the point $\{\frac{1}{2}\}$?

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4. Suppose a metric space M has the property that for any $\epsilon > 0$, there is a totally bounded set $E_\epsilon \subseteq M$ with $\sup\limits_{x \in M} d(x, E_\epsilon) < \epsilon$.

Show that M is totally bounded. [Suggestion: Modify a covering of $E_{\frac{\epsilon}{2}}$]

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5. [Exercise 8.34] Prove that a subset $A \subseteq M$ is closed in M if and only if $A \cap K$ is compact for every compact set $K \subseteq M$. [Hint: If $x_n \to x$, then the set $\{x\} \cup \{x_n : n \geq 1\}$ is compact.]

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6. Let $K \subseteq M$ be a compact subset of a metric space, and $f : M \to M$ a continuous function. Use the notation $f^n(x) = \underbrace{f(f(\cdots f}_{n \text{ times}}(x)))$ introduced in chapter 7.

Prove that one of the following statements must be true:

1) There exists $x \in K$ so that $f^n(x) \in K$ for each $n \geq 1$.
2) There is $N < \infty$ so that for any choice of $x \in M$, at least one of the points $\{x, f(x), f^2(x), \ldots, f^N(x)\}$ does not belong to K.

[Hint: What do these statements say about the sets $E_n = \{x \in K : f^n(x) \in K\}$?]

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7. [Exercise 10.5] Suppose that $f_n : [a, b] \to \mathbb{R}$ is an increasing function for each n, and that $f(x) = \lim_{n\to\infty} f_n(x)$ exists for each x in $[a, b]$. Must f be an increasing function?

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8. [Exercise 11.14] Let $f \in C([a, b])$ be continuously differentiable, and let $\epsilon > 0$. Show that there is a polynomial p such that $\|f' - p'\|_\infty < \epsilon$ and $\|f - p\|_\infty < \epsilon$.

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9. Describe a sequence of functions $f_n \in C([0, 1])$ that converges pointwise to $f(x) = \begin{cases} 1, & \text{if } x = 0 \\ 0, & \text{if } 0 < x \leq 1 \end{cases}$

and has $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 3$. It is not necessary to give an explicit formula for each f_n, so long as you can verify that they have the desired properties.

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10. The proof of Theorem 11.18 contains the (slightly edited) sentence "We need to show that (f_n) has a Cauchy subsequence in the norm of $C(X)$."

In other words, we need to prove an inequality like

$$|f_{n_j}(x) - f_{n_\ell}(x)| < \epsilon \quad \text{for all } \ j, \ell > J \ \text{ and every } x \in X.$$

There are four principal objects here: the sequence (f_n), a subsequence $(f_{n_j})_{j=1}^\infty$, an integer $J < \infty$, and a number $\epsilon > 0$. In what order should they be chosen to make a valid proof?

In what order are they chosen in the textbook's argument?

Extra Credit Assignment. [up to 5 points]

Write a complete and correct proof of the Arzelà-Ascoli Theorem.

[Suggestion: Read about Helly's Selection Principle (Theorem 13.13) before you get started.]