Name: ______________________________ May 11, 2006

Section Number/Time ______________________

TA ______________________

MATH 302, FINAL EXAM

Instructor: Michael Goldberg

Directions: This is a pencil-and-paper exam. You are asked to put away all books, notes, calculators, cell phones, and other computing and/or telecommunications equipment. The last page of this booklet is blank and is intended for use as scrap paper. Additional sheets of paper are available upon request.

Grading: Each numbered question is worth 10 points, for a total of 100 points.

1. Find the solution of the equation $y'(x) + \dfrac{3x^2+2}{2y(x)} = 0$ which goes through the point $(1, -1)$.

Find the values of $y(-2)$ and $y(2)$, if they exist.

Name ____________________

Section/TA ____________________

2a. Find all solutions to the differential equation $y''(t) - 6y'(t) + 10y(t) = 0$.

2b. Find all solutions to the differential equation $y''(t) - 6y'(t) + 10y(t) = 6e^{2t}$ which have $y(0) = 1$.

Name ______________________________

Section/TA __________________________

3a. Consider the second-order linear differential equation $(t-1)y''(t) - ty'(t) + y = 0$, which is solved by both $y_1(t) = t$ and $y_2(t) = e^t$.

What is the Wronskian $W[y_1, y_2](t)$?
Is this a fundamental set of solutions at time $t = 0$?

3b. Find the solution to this equation with initial conditions $y(0) = 1$ and $y'(0) = 1 - e$.

What is unusual about this solution at $t = 1$, and why isn't it a violation of the Existence and Uniqueness Theorem?

Name ____________________________

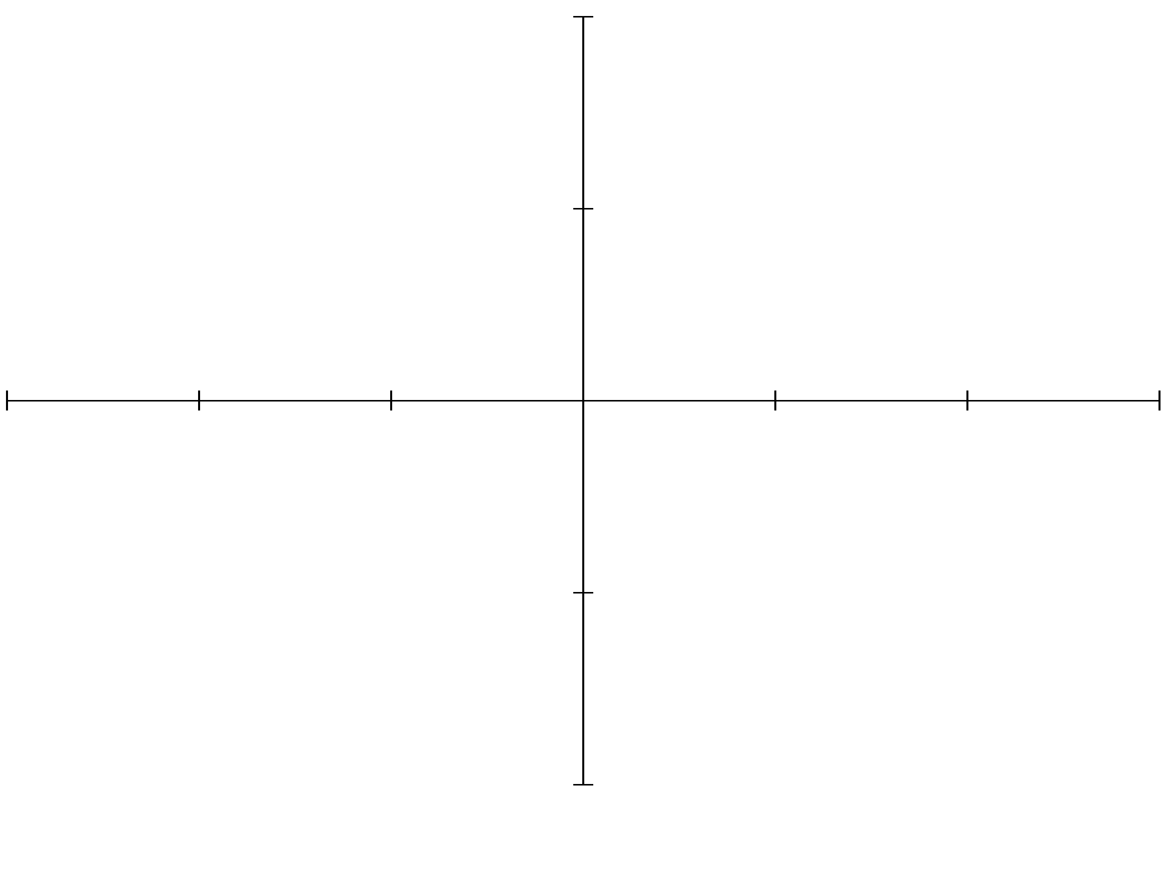
Section/TA ____________________________

4a. Consider the linear system of equations $\begin{cases} x_1'(t) = 3x_1(t) \\ x_2'(t) = -2x_1(t) + x_2(t) \end{cases}$

What kind of critical point is there at the origin?

What are the eigenvalues and eigenvectors?

4b. Sketch the phase portrait for this system on the axes provided.



Name ______________________________

Section/TA ___________________________

Problems 5 – 6 will concern the autonomous system of equations $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} y \\ x - x^2 - xy \end{bmatrix}$.

5a. Find the critical points of this system.
For each critical point, write down the matrix of the associated linear system.

5b. Classify each of the critical points above [as a node, saddle point, spiral point, or center] and indicate whether it is stable.

Name ______________________

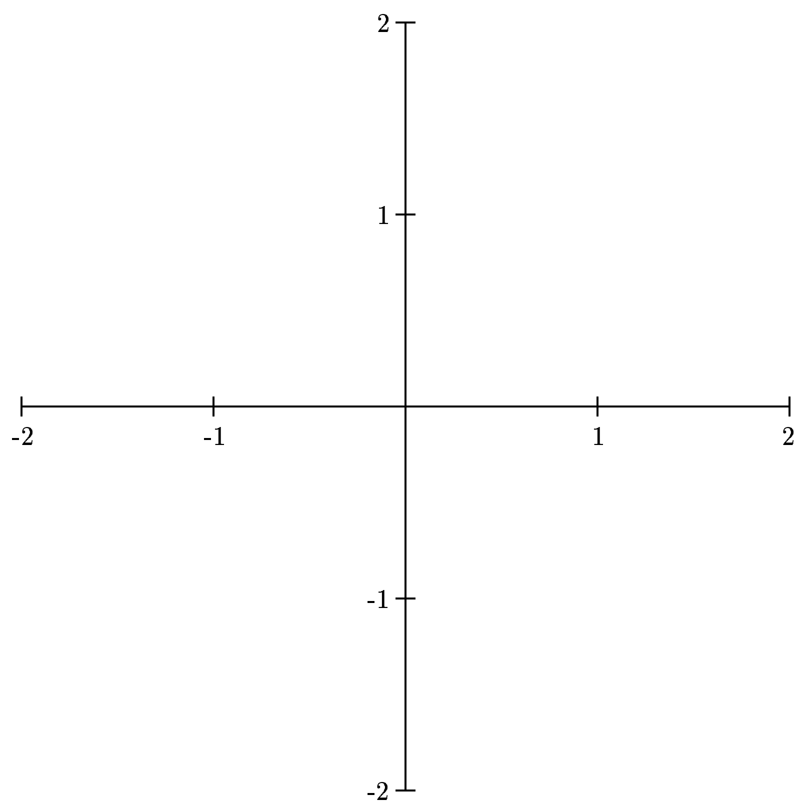
Section/TA ______________________

We are still considering the system of equations $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} y \\ x - x^2 - xy \end{bmatrix}$.

6a. It is known that every function of the form $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = Ce^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a solution.

Draw these trajectories on the axes below.

Also use your answers to Problem 5 to sketch phase portraits near each of the critical points.



6b. Let $\vec{x}(t)$ be the solution which satisfies the initial condition $\vec{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

What is $\vec{x}'(0)$? What is $\lim_{t\to\infty} \vec{x}(t)$?

Sketch the trajectory of $\vec{x}(t)$ on the axes provided.

Name ______________________________

Section/TA ___________________________

7a. Suppose an autonomous system of equations $\begin{cases} x'(t) = f(x(t), y(t)) \\ y'(t) = g(x(t), y(t)) \end{cases}$

has the property that $x\,f(x,y) + y\,g(x,y) < 0$ at every point except the origin.

Show that for every solution $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$, the quantity $\|\vec{x}(t)\|^2 = [x(t)]^2 + [y(t)]^2$ is always decreasing.

7b. Explain why the origin must be a stable critical point,
and why there cannot be any other critical points or limit cycles.

Name ______________________________

Section/TA ___________________________

8a. Suppose $f(x)$ is represented by the power series $\sum_{n=0}^{\infty} a_n x^n$.

Write out the power series representations for $f'(x)$ and $xf(x)$.

8b. Find a power series solution for the differential equation $f'(x) = xf(x)$.

What is the pattern for the coefficients a_n?

Do not solve the differential equation using other methods first – that would be cheating...

Name ____________________

Section/TA ____________________

9a. Suppose we wanted to rewrite the function $f(x) = \begin{cases} \cos x, & \text{if } |x| < \frac{\pi}{2} \\ 0, & \text{if } \frac{\pi}{2} \le |x| \le \pi \end{cases}$, with $f(x + 2\pi) = f(x)$,

as the Fourier series $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$.

Use the Euler-Fourier formulas to express each of the coefficients a_n and b_n as a definite integral, then evaluate a_0 exactly.

9b. True or false: $b_n = 0$ for every n.

If it is true, give a brief explanation why.
If it is false, find a specific value of n for which $b_n \neq 0$.

Name ____________________

Section/TA ____________________

10a. The *wave equation* $\begin{cases} \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \\ u(0,t) = 0 \quad \text{for all } t \geq 0 \\ u(L,t) = 0 \quad \text{for all } t \geq 0 \end{cases}$ has solutions in the form $u_n(x,t) = \sin\left(\frac{n\pi}{L}x\right) T_n(t)$.

What differential equation must $T_n(t)$ solve?
Based on this, what is the general form of the function $T_n(t)$?

10b. If a function $u(x,t) = \sum_{n=1}^{\infty} u_n(x,t)$ is a solution to the wave equation,
what relationship exists between $u(x, t + 2L)$ and $u(x,t)$?