The Schrödinger Equation on Triangular Lattices

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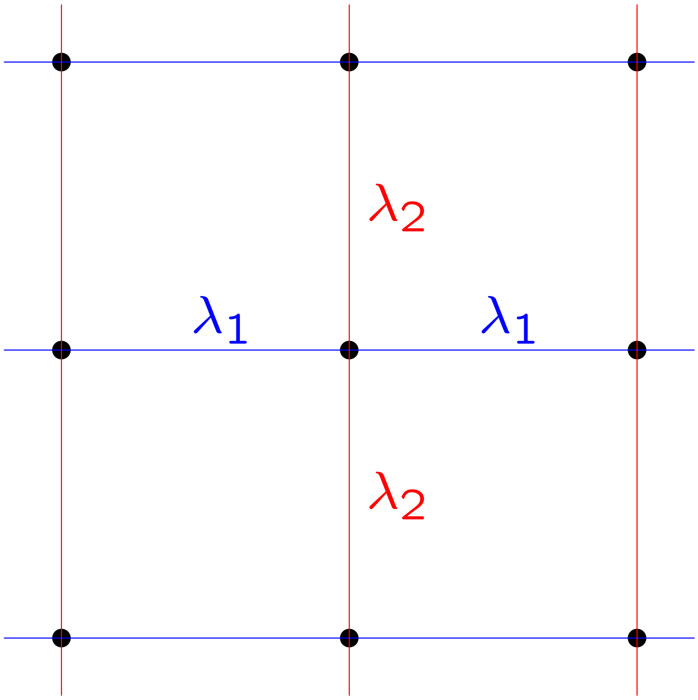
Warm-up: Fundamental solution of $e^{it\Delta_d}$ on $\mathbf{Z}^2$.



$$\Delta_d u(x) = \sum_{j=1}^{2} \lambda_j \Big(u(x+e_j) + u(x-e_j) - 2u(x) \Big)$$

The Schrödinger equation has plane-wave solutions

$$u_k(x,t) = e^{i(k\cdot x - \varphi(k)t)}$$

with the phase function

$$\varphi(k) = 4\left[\lambda_1 \sin^2\left(\frac{k_1}{2}\right) + \lambda_2 \sin^2\left(\frac{k_2}{2}\right) \right]$$

The fundamental solution for

$$\begin{cases} u_t = i \sum_{j=1}^{2} \lambda_j \big(u(x+e_j,t) + u(x-e_j,t) - 2u(x,t) \big) \\ u(x,0) = \delta_0 \end{cases}$$

is $\Phi(x,t) = \dfrac{1}{4\pi^2} \displaystyle\int_{\mathbb{T}^2} e^{-it\varphi(k)} e^{ik\cdot x}\, dk.$

The integral separates into variables k_1, k_2 so that

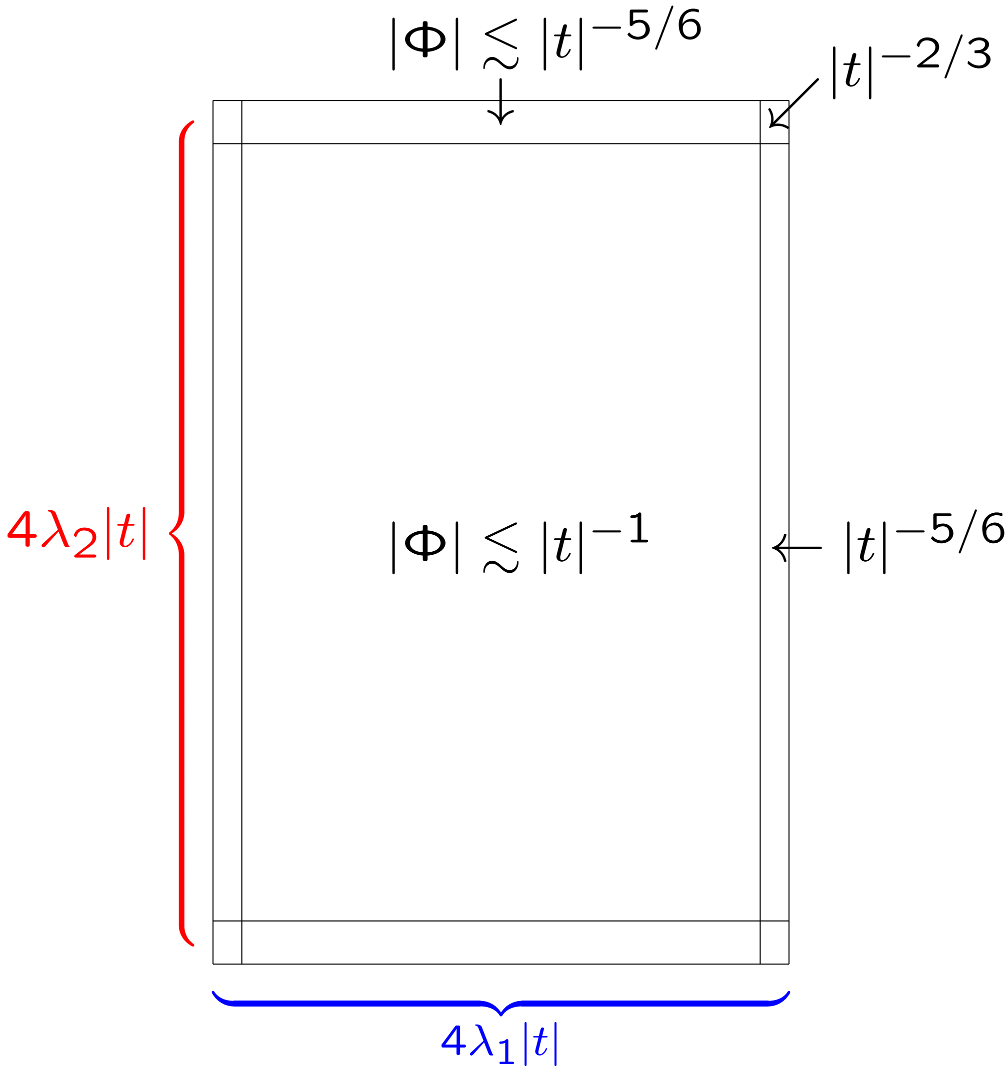
$$\Phi(x,t) = \prod_{j=1}^{2} e^{2i\lambda_j t} J_{x_j}(2\lambda_j t)$$

where x_j are coordinates of x and J_α are Bessel functions of the first kind.

Asymptotics for Bessel functions:

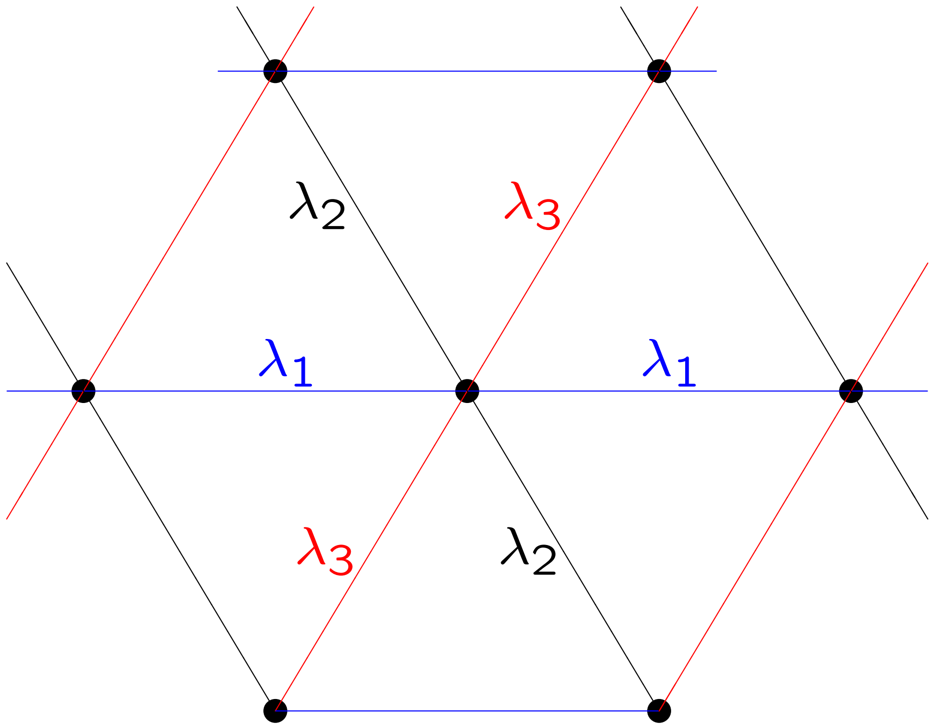
$$|J_x(2\lambda t)| \lesssim \begin{cases} C|t|^{-1/2} & \text{if } |x| < (2-\epsilon)\lambda|t| \\ C|t|^{-1/3} & \text{if } |x| \sim 2\lambda|t| \\ \frac{C|2\lambda t|^{|x|}}{|x|!} & \text{if } |x| > (2+\epsilon)\lambda|t| \end{cases}$$

As a result, the “light cone” of $\Phi(x,t)$
is a box with side length $4\lambda_j|t|$,
and the size of $\Phi(x,t)$ has this profile:



Outside of the light cone there is rapid decay.

Problem: Fundamental solution of $e^{it\Delta_d}$ on triangular lattice in $\mathbf{R}^2$.



The Schrödinger equation has plane-wave solutions

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with the phase function

$$\varphi(k) = 4\Big[\lambda_1 \sin^2\left(\frac{k_1}{2}\right) + \lambda_2 \sin^2\left(\frac{-k_1+\sqrt{3}k_2}{2}\right) + \lambda_3 \sin^2\left(\frac{k_1+\sqrt{3}k_2}{2}\right)\Big]$$

Once again we can write out

$$\Phi(x,t) = \fint e^{-it\varphi(k)} e^{ik\cdot x}\,dk$$

averaged over a fundamental domain in k-space.

Dispersive bounds: The decay of $\sup_x |\Phi(x,t)|$ depends on asymptotics for the oscillatory integral $\fint e^{-it\varphi(k)} e^{ik\cdot x}\,dk$ as $t \to \infty$.

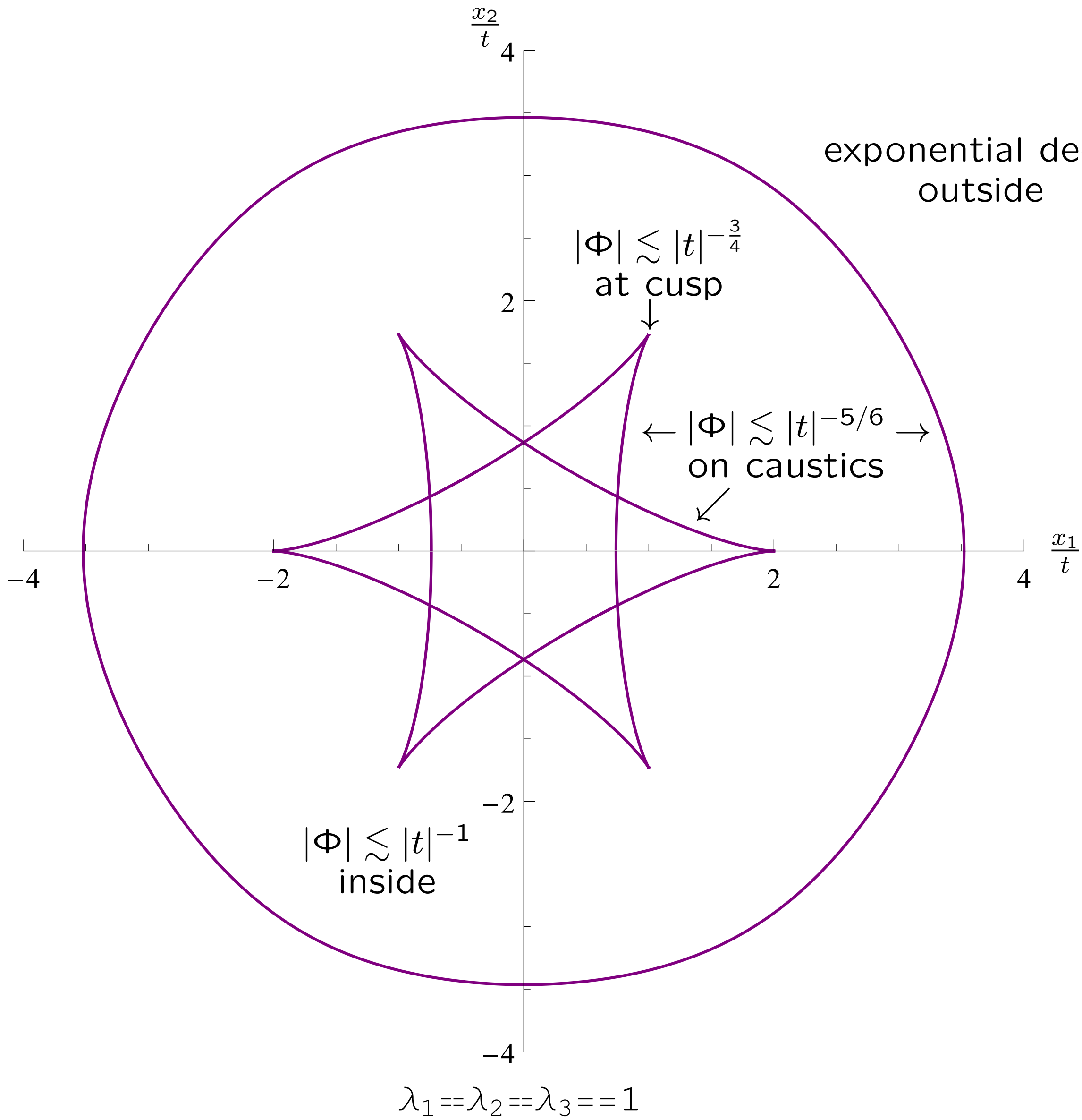
If $x_0 = t\nabla\varphi(k_0)$ for some $k_0 \in \mathbb{T}^d$, the integral defining $\Phi(x_0,t)$ has stationary phase at k_0.

Non-degenerate stationary phase estimate:

$$|\Phi(x_0,t)| \lesssim \frac{1}{|t|\sqrt{\det D^2\varphi(k_0)}}$$

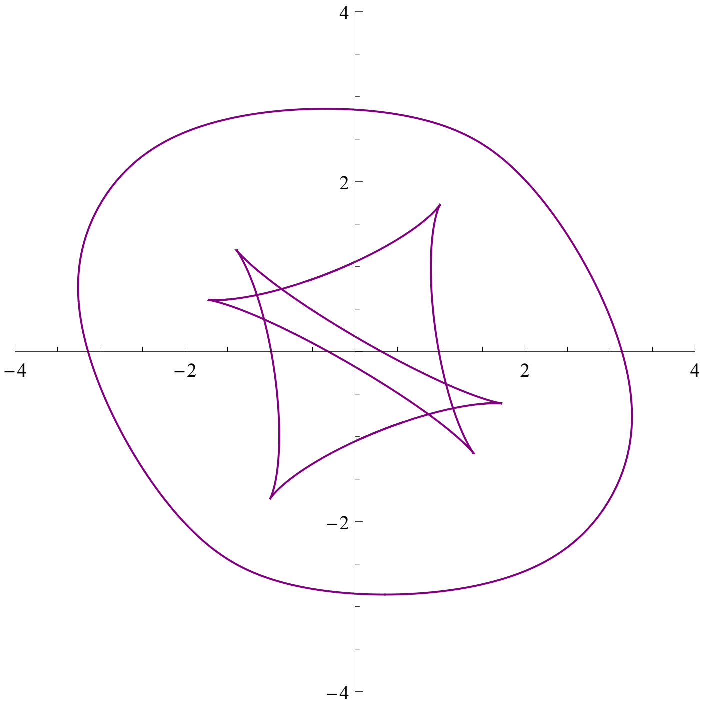
If $\det D^2\varphi(k_0) = 0$, asymptotic decay depends on more terms of Taylor series of $\varphi(k)$ centered at k_0.

[Varchenko (1976), also Greenblatt and Collins-Greenleaf-Pramanik]

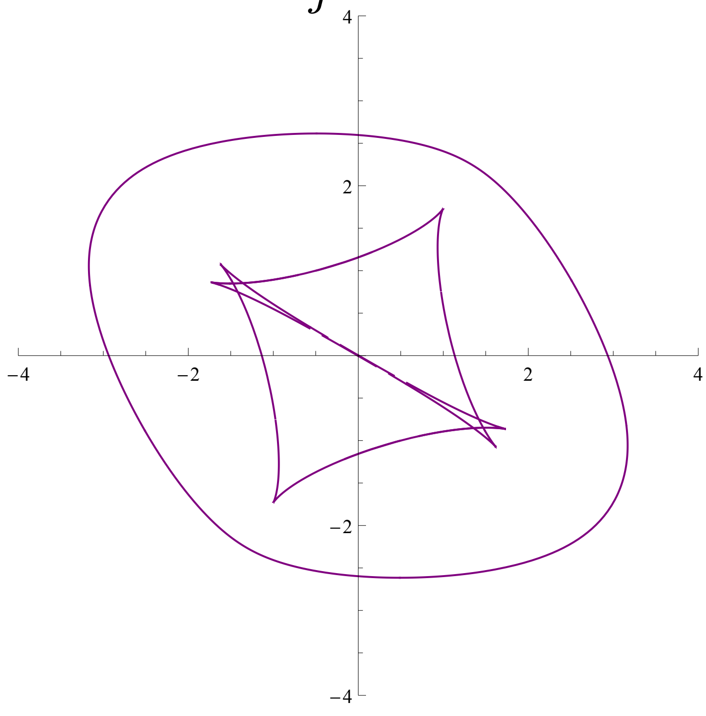


The dispersion relation as a function of $\frac{x}{t}$, in the case where all $\lambda_j = 1$.

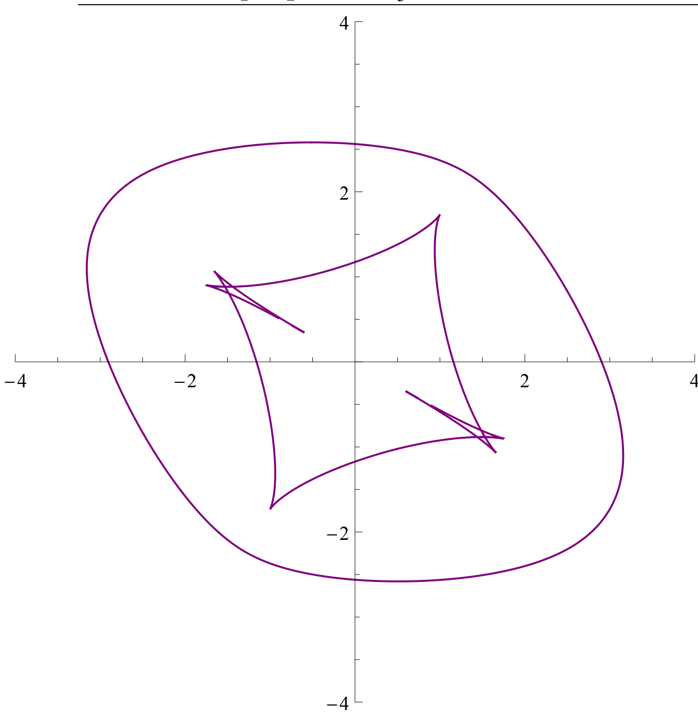
The picture changes as values of λ_j are varied.



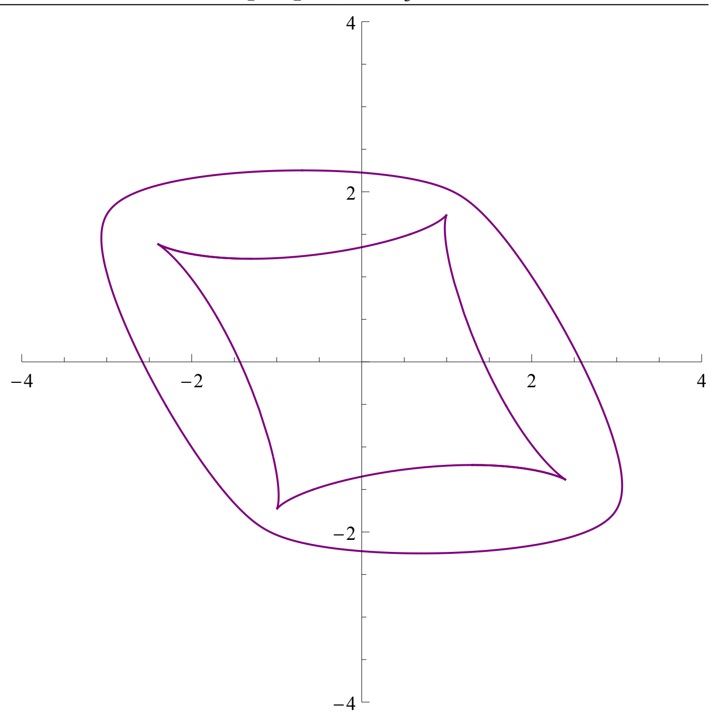
$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 0.65$



$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 0.51$



$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 0.49$



$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 0.3$

The inner caustic(s) have two types of transition:

1) From a pair of interlocking triangles to a single closed curve. (e.g. $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \frac{1}{2}$)

2) From a curve with 3-cusped "butterfly turns" to a simple closed curve. (e.g. $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = \frac{1}{\sqrt{8}}$)

The dispersive estimate is still $|t|^{-3/4}$
if λ_j are chosen at a Type 1 transition.
[Taylor series of φ reduces to $k_1^2 + 0k_1k_2^2 + k_2^4 +$ (higher order)]

The dispersive estimate has slower decay
if λ_j are chosen at a Type 2 transition.
[Taylor series of φ reduces to $(k_1 + k_2^2)^2 +$ (higher order)]

Trigonometric relations:

Let $\theta_1 = k_1$, $\theta_2 = \frac{-k_1+\sqrt{3}k_2}{2}$, $\theta_3 = \frac{-k_1-\sqrt{3}k_2}{2}$.

[Note that $\theta_1 + \theta_2 + \theta_3 = 0$.]

Then $\mathrm{Det}\,(D^2\varphi) = 0$ when

$$\boxed{\lambda_1\lambda_2 \cos\theta_1 \cos\theta_2 + \lambda_1\lambda_3 \cos\theta_1 \cos\theta_3 + \lambda_2\lambda_3 \cos\theta_2 \cos\theta_3 = 0}$$

Cusps occur where, in addition,

$$\boxed{\lambda_1 \cos^2\theta_1 \sin\theta_1 + \lambda_2 \cos^2\theta_2 \sin\theta_2 + \lambda_3 \cos^2\theta_3 \sin\theta_3 = 0}$$

Type 1 Transitions occur if $\prod \cos\theta_j = 1$ at a cusp.
[i.e. $\theta_j = 0, \pm\pi$ in some order, so $\lambda_1\lambda_2 - \lambda_1\lambda_3 - \lambda_2\lambda_3 = 0$, modulo permutation of indices.]

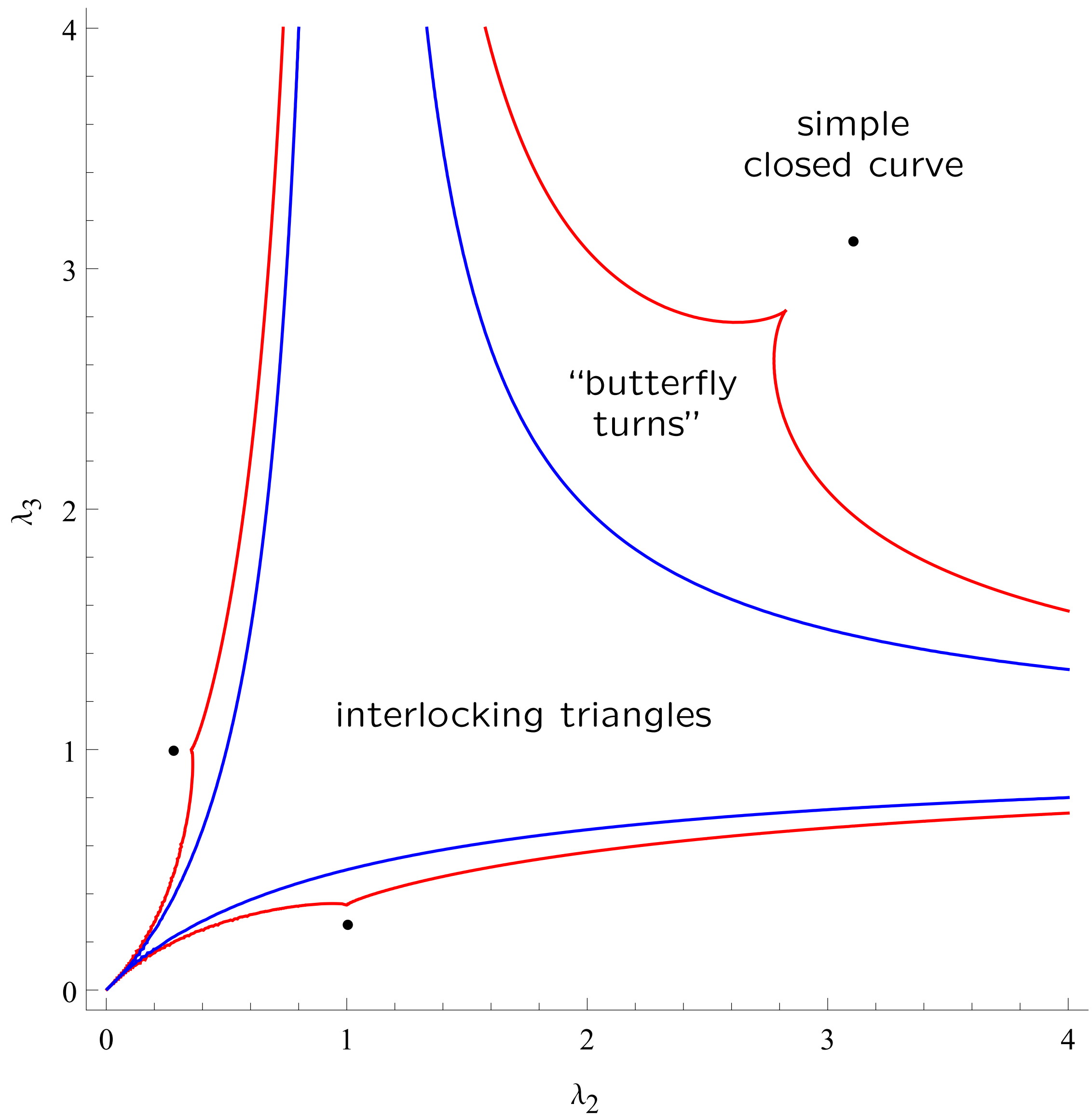
Type 2 Transitions occur if $\prod \cos\theta_j = \frac{1}{2}$ at a cusp.
This takes place if . . .

$$P(\lambda_1,\lambda_2,\lambda_3) =$$

$$
\begin{gathered}
\lambda_2^{24}\lambda_3^{24}+8\lambda_1^{2}\lambda_2^{24}\lambda_3^{22}+8\lambda_1^{2}\lambda_2^{22}\lambda_3^{24}+26\lambda_1^{4}\lambda_2^{24}\lambda_3^{20}-1004\lambda_1^{4}\lambda_2^{22}\lambda_3^{22}+26\lambda_1^{4}\lambda_2^{20}\lambda_3^{24}+ \\
40\lambda_1^{6}\lambda_2^{24}\lambda_3^{18}-156\lambda_1^{6}\lambda_2^{22}\lambda_3^{20}-156\lambda_1^{6}\lambda_2^{20}\lambda_3^{22}+40\lambda_1^{6}\lambda_2^{18}\lambda_3^{24}+15\lambda_1^{8}\lambda_2^{24}\lambda_3^{16}+ \\
6408\lambda_1^{8}\lambda_2^{22}\lambda_3^{18}+297466\lambda_1^{8}\lambda_2^{20}\lambda_3^{20}+6408\lambda_1^{8}\lambda_2^{18}\lambda_3^{22}+15\lambda_1^{8}\lambda_2^{16}\lambda_3^{24}-48\lambda_1^{10}\lambda_2^{24}\lambda_3^{14}+ \\
3664\lambda_1^{10}\lambda_2^{22}\lambda_3^{16}-2061504\lambda_1^{10}\lambda_2^{20}\lambda_3^{18}-2061504\lambda_1^{10}\lambda_2^{18}\lambda_3^{20}+3664\lambda_1^{10}\lambda_2^{16}\lambda_3^{22}- \\
48\lambda_1^{10}\lambda_2^{14}\lambda_3^{24}-84\lambda_1^{12}\lambda_2^{24}\lambda_3^{12}-13016\lambda_1^{12}\lambda_2^{22}\lambda_3^{14}+6033644\lambda_1^{12}\lambda_2^{20}\lambda_3^{16}+ \\
8913328\lambda_1^{12}\lambda_2^{18}\lambda_3^{18}+6033644\lambda_1^{12}\lambda_2^{16}\lambda_3^{20}-13016\lambda_1^{12}\lambda_2^{14}\lambda_3^{22}-84\lambda_1^{12}\lambda_2^{12}\lambda_3^{24}- \\
48\lambda_1^{14}\lambda_2^{24}\lambda_3^{10}-13016\lambda_1^{14}\lambda_2^{22}\lambda_3^{12}-8814408\lambda_1^{14}\lambda_2^{20}\lambda_3^{14}-8588000\lambda_1^{14}\lambda_2^{18}\lambda_3^{16}- \\
8588000\lambda_1^{14}\lambda_2^{16}\lambda_3^{18}-8814408\lambda_1^{14}\lambda_2^{14}\lambda_3^{20}-13016\lambda_1^{14}\lambda_2^{12}\lambda_3^{22}-48\lambda_1^{14}\lambda_2^{10}\lambda_3^{24}+ \\
15\lambda_1^{16}\lambda_2^{24}\lambda_3^{8}+3664\lambda_1^{16}\lambda_2^{22}\lambda_3^{10}+6033644\lambda_1^{16}\lambda_2^{20}\lambda_3^{12}-8588000\lambda_1^{16}\lambda_2^{18}\lambda_3^{14}- \\
13825990\lambda_1^{16}\lambda_2^{16}\lambda_3^{16}-8588000\lambda_1^{16}\lambda_2^{14}\lambda_3^{18}+6033644\lambda_1^{16}\lambda_2^{12}\lambda_3^{20}+ \\
3664\lambda_1^{16}\lambda_2^{10}\lambda_3^{22}+15\lambda_1^{16}\lambda_2^{8}\lambda_3^{24}+40\lambda_1^{18}\lambda_2^{24}\lambda_3^{6}+6408\lambda_1^{18}\lambda_2^{22}\lambda_3^{8}- \\
2061504\lambda_1^{18}\lambda_2^{20}\lambda_3^{10}+8913328\lambda_1^{18}\lambda_2^{18}\lambda_3^{12}-8588000\lambda_1^{18}\lambda_2^{16}\lambda_3^{14}- \\
8588000\lambda_1^{18}\lambda_2^{14}\lambda_3^{16}+8913328\lambda_1^{18}\lambda_2^{12}\lambda_3^{18}-2061504\lambda_1^{18}\lambda_2^{10}\lambda_3^{20}+6408\lambda_1^{18}\lambda_2^{8}\lambda_3^{22}+ \\
40\lambda_1^{18}\lambda_2^{6}\lambda_3^{24}+26\lambda_1^{20}\lambda_2^{24}\lambda_3^{4}-156\lambda_1^{20}\lambda_2^{22}\lambda_3^{6}+297466\lambda_1^{20}\lambda_2^{20}\lambda_3^{8}- \\
2061504\lambda_1^{20}\lambda_2^{18}\lambda_3^{10}+6033644\lambda_1^{20}\lambda_2^{16}\lambda_3^{12}-8814408\lambda_1^{20}\lambda_2^{14}\lambda_3^{14}+ \\
6033644\lambda_1^{20}\lambda_2^{12}\lambda_3^{16}-2061504\lambda_1^{20}\lambda_2^{10}\lambda_3^{18}+297466\lambda_1^{20}\lambda_2^{8}\lambda_3^{20}-156\lambda_1^{20}\lambda_2^{6}\lambda_3^{22}+ \\
26\lambda_1^{20}\lambda_2^{4}\lambda_3^{24}+8\lambda_1^{22}\lambda_2^{24}\lambda_3^{2}-1004\lambda_1^{22}\lambda_2^{22}\lambda_3^{4}-156\lambda_1^{22}\lambda_2^{20}\lambda_3^{6}+6408\lambda_1^{22}\lambda_2^{18}\lambda_3^{8}+ \\
3664\lambda_1^{22}\lambda_2^{16}\lambda_3^{10}-13016\lambda_1^{22}\lambda_2^{14}\lambda_3^{12}-13016\lambda_1^{22}\lambda_2^{12}\lambda_3^{14}+3664\lambda_1^{22}\lambda_2^{10}\lambda_3^{16}+ \\
6408\lambda_1^{22}\lambda_2^{8}\lambda_3^{18}-156\lambda_1^{22}\lambda_2^{6}\lambda_3^{20}-1004\lambda_1^{22}\lambda_2^{4}\lambda_3^{22}+8\lambda_1^{22}\lambda_2^{2}\lambda_3^{24}+\lambda_1^{24}\lambda_2^{24}+8\lambda_1^{24}\lambda_2^{22}\lambda_3^{2}+ \\
26\lambda_1^{24}\lambda_2^{20}\lambda_3^{4}+40\lambda_1^{24}\lambda_2^{18}\lambda_3^{6}+15\lambda_1^{24}\lambda_2^{16}\lambda_3^{8}-48\lambda_1^{24}\lambda_2^{14}\lambda_3^{10}-84\lambda_1^{24}\lambda_2^{12}\lambda_3^{12}- \\
48\lambda_1^{24}\lambda_2^{10}\lambda_3^{14}+15\lambda_1^{24}\lambda_2^{8}\lambda_3^{16}+40\lambda_1^{24}\lambda_2^{6}\lambda_3^{18}+26\lambda_1 624\lambda_2^{4}\lambda_3^{20}+8\lambda_1^{24}\lambda_2^{2}\lambda_3^{22}+ \\
\lambda_1^{24}\lambda_3^{24} \qquad = \; 0.
\end{gathered}
$$

$\ldots\ \lambda_j^2$ satisfy a homogenous 24$^{\text{th}}$-order polynomial.

With $\lambda_1 = 1$, the three regimes are:



Note: $P(1, \lambda_2, \lambda_3)$ also vanishes at the three marked points, but the corresponding values of θ_j are imaginary.

Observed: When $\lambda_1 = 1$, $\lambda_2 = \lambda_3 = \sqrt{8}$,
the dispersive bound is $\sup_x |\Phi(x,t)| \lesssim |t|^{-2/3}$.

Reasonable suspicion: That is atypical behavior.
For generic (all other?) solutions of
$P(\lambda_1, \lambda_2, \lambda_3) = 0$, the dispersive bound is

$$\sup_x |\Phi(x,t)| \lesssim |t|^{-7/10}.$$

(which comes from $\frac{1}{2} + \frac{1}{5} = \frac{7}{10}$.)

Open Questions: Is the wave equation this messy?
What if Δ_d is replaced by another operator?
What about less regular periodic structures?
What happens if there are lattice defects?
Does discrete NLS depend this much on λ_j?