Wave Propagation on Square Lattices

Vita Borovyk and Michael Goldberg

SIAM Conference on Analysis of PDEs
San Diego, California
November 14, 2011.

Support provided by NSF grant DMS-1002515

Motivated by Quantum Harmonic Lattice:

Put a harmonic oscillator at each vertex of $\mathbf{Z}^d$, with position q_x and momenutum p_x.

The local Hamiltonian is $H_x = p_x^2 + \omega^2 q_x^2$.

Introduce nearest-neighbor interactions to produce a global Hamiltonian

$$H = \sum_{x \in \mathbf{Z}^d} \left(p_x^2 + \omega^2 q_x^2 + \sum_{j=1}^{d} \lambda_j (q_{x+e_j} - q_x)^2 \right)$$

where $\omega > 0$ and each $\lambda_j > 0$.

The classical system has a discrete wave (Klein-Gordon) equation as its equation of motion:

$$u_{tt}(x,t) = -\omega^2 u(x,t) + \sum_{j=1}^{d} \lambda_j \Big(u(x+e_j,t) + u(x-e_j,t) - 2u(x,t) \Big)$$

This has a convenient basis of plane-wave solutions. For each $k \in \mathbb{T}^d$,

$$u_k(x,t) = e^{i(k \cdot x - \varphi(k)t)}$$

where $\varphi(k) = \sqrt{\omega^2 + 4 \sum_j \lambda_j \sin^2(k_j/2)}$.

The fundamental solution for

$$\begin{cases} u_{tt} = -\omega^2 u(x,t) + \sum_{j=1}^{d} \lambda_j \big(u(x+e_j,t) + u(x-e_j,t) - 2u(x,t)\big) \\ u(x,0) = \delta_0 \\ u_t(x,0) = 0 \end{cases}$$

is $\Phi_1(x,t) = \int_{\mathbb{T}^d} \cos(t\varphi(k)) e^{ik\cdot x}\, dk$

and the fundamental solution for

$$\begin{cases} u_{tt} = -\omega^2 u(x,t) + \sum_{j=1}^{d} \lambda_j \big(u(x+e_j,t) + u(x-e_j,t) - 2u(x,t)\big) \\ u(x,0) = 0 \\ u_t(x,0) = \delta_0 \end{cases}$$

is $\Phi_2(x,t) = \int_{\mathbb{T}^d} \frac{\sin(t\varphi(k))}{\varphi(k)} e^{ik\cdot x}\, dk$

Lieb-Robinson bounds (finite propagation speed): $\Phi(x,t)$ decays exponentially for large $x \gg t$.

For the classical system, this follows from analyticity of $\phi(k)$.

Dispersive estimates: How does $\sup_x |\Phi(x,t)|$ decay with t?

This requires control of oscillatory integrals like

$$\int_{\mathbb{T}^d} e^{\pm it\varphi(k)} e^{ik\cdot x}\, dk \quad \text{as } t \to \infty$$

If $x_0 = t\nabla\varphi(k_0)$ for some $k_0 \in \mathbb{T}^d$, there is stationary phase at k_0.

Non-degenerate stationary phase estimate:

$$|\Phi(x_0, t)| \lesssim \frac{1}{t^{d/2}\sqrt{\det D^2\varphi(k_0)}}$$

If $\det D^2\varphi(k_0) = 0$, asymptotic decay depends on Taylor series of $\varphi(k)$ centered at k_0.

When $d = 1$, van der Corput Lemma implies $|\Phi_1(x,t)| \lesssim t^{-1/3}$.

Similarly, $|\Phi_2(x,t)| \lesssim t^{-1/3}$, with a constant depending on ω (thanks to factor of $1/\varphi(k)$).

Degenerate stationary phase is difficult in $d > 1$.

Asymptotic decay depends on Taylor series expansion with respect to "adapted" coordinates (Varchenko, 1976).

Detailed analysis for $d = 2$:

$$\varphi(k) = \sqrt{\omega^2 + 2\lambda_1(1 - \cos k_1) + 2\lambda_2(1 - \cos k_2)}$$

$$\begin{aligned}
&\det D^2\varphi(k) \\
&\qquad = \varphi^{-4}(k)\Big(\omega^2 ab - \lambda_1 b(1-a)^2 - \lambda_2 a(1-b)^2\Big)
\end{aligned}$$

where $a = \cos k_1$ and $b = \cos k_2$.

Where is $\det D^2\varphi(k) = 0$?

- A closed curve Γ_1 around origin, corresponding to extremal propagation velocity.

- A closed curve Γ_2 around $(\pi, \pi) \in \mathbb{T}^2$ corresponding to ???

At all $k \in \mathbb{T}^2$, $D^2\varphi(k)$ has rank ≥ 1.

Among $k \in \Gamma_1$, the second and third-order directional derivatives of $\varphi(k)$ never vanish at the same time.

This leads to an estimate $|\Phi(x,t)| \lesssim t^{-5/6}$ when $\frac{x}{t}$ is near an extremal velocity.

Peculiar Results:

Among $k \in \Gamma_2$, there is a unique point (up to mirror symmetries) where both the second and third-order directional derivatives of $\varphi(k)$ vanish, but a relevant fourth-order quantity is nonzero.

Thus there is a unique velocity (again up to symmetry) where fundamental solutions of the discerete Klein-Gordon equation decay at the rate $t^{-3/4}$.

This region of least dispersion occurs in middle of the propagation pattern, not at its leading edge.