Differentiability of Fourier Restrictions

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Basic Facts

Support from Simons Foundation grant #635369.
Collaboration with Dmitriy Stolyarov.

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Even when $f \in L^1$,there's no guarantee of Hölder continuity of $\hat{f}$, either on $\mathbb{R}^n$ or when restricted to surfaces.

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$$f \in L^p(\mathbb{R}^n) \;\Rightarrow\; \hat{f}\big|_\Sigma \in H^{-s}(\Sigma),\; \tfrac{2n+2}{n+3} \le p < \tfrac{2n}{n+1}. \qquad \text{[Cho-Guo-Lee, '15]}$$

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Loss of regularity in Cho-Guo-Lee is $s = \frac{n+3}{2} - \frac{n+1}{p}$.

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Proposition

If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \frac{2n+2}{n+5}$, then $\partial_n \hat{f}\big|_\Sigma \in H^{-1}(\Sigma)$.

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Proof: Interpolate between no derivatives (S-T or C-G-L) and the case of $\frac{n-1}{2}$ derivatives above.

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Theorem (G-Stolyarov, '20)

Let $1 \leq p \leq \frac{2n+2}{n+7}$. *If* $\hat{f}\big|_\Sigma \in H^\ell(\Sigma)$ *for sufficiently large* ℓ, *then*

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We believe "sufficiently large" means $\ell > \frac{(2n+2)-(n+3)p}{(2n+2)-(n+5)p}$.

This is proved in a few cases when integer arithmetic works out nicely.

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Regularity of $\hat{f}$ does not extend to a neighborhood of Σ.

Derivative exists in a functional analysis sense:
Let Σ_r be the translation $\Sigma + (0, \ldots, r)$. The map

$$r \mapsto \hat{f}\big|_{\Sigma_r} \in L^2(\Sigma)$$

is differentiable (in the L^2-norm topology) at $r = 0$.

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Pointwise derivatives need a bound with L^∞_r on the inside.

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so it is not close to being differentiable at $r = 0$.

This construction doesn't produce a counterexample when $p = 1$.
$\big($ $\hat{f}$ can't be any worse than continuous.$\big)$

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$\ell > \frac{n-1}{2}$ lets you reduce to the case $\hat{f}\big|_\Sigma \equiv 0$
by subtracting away a nice function $g \in L^1(\mathbb{R}^n)$.

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One way to do that: Take its Fourier transform in r and try to control $\mathcal{F}\Big(\frac{1}{r}\hat{f}\big|_{\Sigma_r}\Big)$ in terms of an $L^1(\rho)$ norm,

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In order to continue using T^*T methods, we look for a bound in $L^1(\rho; L^2(\Sigma))$, which is contained inside $L^2(\Sigma; L^1(\rho))$.

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$$\|T_\rho f\|^2_{L^2(\Sigma)} = \iint_{\mathbb{R}^{2n}} \int_\Sigma e^{i(x-y)\cdot(\xi',|\xi'|^2)} \mathrm{sgn}\big((\rho - x_n)(\rho - y_n)\big) f(x) f(y)\, d\xi'\, dydx$$

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Now take the square root and show that $\displaystyle\int_{\mathbb{R}} \|T_\rho f\|_{L^2(\Sigma)}\,d\rho \lesssim \|f\|_1$.

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3. Is there a pointwise-a.e. transverse derivative in dimensions $n = 3, 4, 5$? [Bochner-Riesz counterexample in $n = 2$.]
4. Is there a nontrivial $L^1 \to L^1$ bound for Bochner-Riesz multipliers on the space of functions with $\hat{f}\big|_{S^{n-1}} \equiv 0$?