L^p Bounds for Wave Operators for the Schrödinger Equation with a Threshold Eigenvalue

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Wave operators for $H = -\Delta + V(x)$

Dynamical definition: $W_\pm \psi = \lim_{t \to \pm\infty} e^{itH} e^{it\Delta} \psi$.

Time-independent definition: $W_\pm \underbrace{(e^{i\xi\cdot x})}_{\substack{\text{eigenfunction} \\ \text{of } -\Delta}} = \underbrace{e^{i\xi\cdot x} + (\text{scattered wave})}_{\substack{\text{eigenfunction} \\ \text{of } H}}$.

where the eigenfunction of H satisfies outgoing(+)/incoming(-) radiation condition.

$W_\pm$ is an isometry on $L^2(\mathbb{R}^n)$.
range($W_\pm$) = absolutely continuous subspace of H,
so $W_\pm W_\pm^* = P_{ac}(H)$.

Intertwining relations for wave operators:

For any measurable $f : [0, \infty] \to \mathbb{C}$,

$$f(H)W_\pm = W_\pm f(-\Delta)$$
$$f(H)P_{ac}(H) = f(H)W_\pm(W_\pm)^* = W_\pm f(-\Delta)W_\pm^*$$

Thus if $W_\pm$ is bounded on both $L^{p'}(\mathbb{R}^n)$ and $L^q(\mathbb{R}^n)$,
then every estimate $f(-\Delta) : L^p \to L^q$ is also true
(up to a constant) for the operator $f(H)P_{ac}(H)$.

Example: $\|e^{-it\Delta}\psi\|_p \leq C|t|^{n(\frac{1}{p}-\frac{1}{2})}\|\psi\|_{p'}$ for all $2 \leq p \leq \infty$.

We'd like same bounds for $e^{itH}P_{ac}(H)$ over a wide range of p.

Theorems in dimensions $n \geq 5$:

General assumptions on V are that $|V(x)| \leq C(1+|x|)^{-\beta}$, and the Fourier transform of $(1+|x|^2)^\sigma V$ belongs to $L^{\frac{n-1}{n-2}}$.
[for some large enough β and σ]

Yajima ('95) – if H doesn't have an eigenvalue at zero, then $W_\pm$ are bounded on all $L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$.

Yajima ('05), **Finco-Yajima** ('06), **Yajima** ('15) – if H does have an eigenvalue to zero, then $W_\pm$ are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \frac{n}{2}$.

Even and odd dimensions are treated separately.
The exponent range $p \in (1, \frac{n}{n-2}]$ is more recent; the original theorems only cover the range $p \in (\frac{n}{n-2}, \frac{n}{2})$.

Theorem (Green-G): With the same assumptions on $V(x)$,

If H has an eigenvalue at zero,
then $W_\pm$ are bounded on $L^p(\mathbb{R}^n)$ for $1 \le p < \frac{n}{2}$.

If the zero-energy eigenfunctions are orthogonal to $V(x)$,
then $W_\pm$ are bounded on $L^p(\mathbb{R}^n)$ for $1 \le p < n$.

If the zero-energy eigenfunctions are orthogonal to all $(a+bx_j)V(x)$,
then $W_\pm$ are bounded on $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$.

Spectral representation of wave operators.

Resolvent notation: $R^{\pm}(\lambda^2) = (H - (\lambda \pm i0)^2)^{-1}$
$R_0^{\pm}(\lambda^2) = (-\Delta - (\lambda \pm i0)^2)^{-1}$

$$W_{\pm}\psi = \frac{1}{2\pi i}\int_0^{\infty} \underbrace{(I + R_0^{\pm}(\lambda^2)V)}_{\substack{\text{converts to}\\ \text{eigenfunctions}\\ \text{of } H}}{}^{-1}[\underbrace{R_0^{+}(\lambda^2) - R_0^{-}(\lambda^2)}_{\substack{\text{projects onto}\\ \text{plane waves}\\ \text{frequency } |\lambda|}}]\psi\, d(\lambda^2)$$

Rewrite the operator inverse as

$$(I + R_0^{\pm}(\lambda^2)V)^{-1} = I \;-\; R_0^{\pm}(\lambda^2)v(I + wR_0^{\pm}(\lambda^2)v)^{-1}w$$

where $vw = V$. Both v and w can decay as fast as $\sqrt{V(x)}$.

This gives the integral formula

$$W_{\pm} = I - \frac{1}{2\pi i}\int_0^{\infty} R_0^{\pm}(\lambda^2)v(I + wR_0^{\pm}(\lambda^2)v)^{-1}w \\ [R_0^{+}(\lambda^2) - R_0^{-}(\lambda^2)]\, d(\lambda^2)$$

Steps toward our proof:

1) Break into high-energy and low-energy cases.

2) Use existing results (by Yajima and Jensen) when these are already strong enough.

3) Isolate a leading-order contributor to $W_{\pm}$, describe it as an integral operator, and control its kernel $K(x, y)$ pointwise.

Borrowed results:

Yajima ('95) – If one inserts high-energy cutoff $[1 - \eta(\lambda)]$, the contribution $W_{\pm}^{\text{high}}$ is bounded on all $L^p(\mathbb{R}^n)$, $p \in [1, \infty]$.

Jensen ('80) – Asymptotic series expansions for $(I + wR_0^{\pm}(\lambda^2)v)^{-1}$. If zero is an eigenvalue, then the leading term is $\lambda^{-2}(wP_0v)$.

Yajima ('05, '15) – Control of most higher-order (in λ) terms. In dimensions $n = 5, 6, 8, 10$ some extra steps are required. Especially when $n = 6$.

That leaves us to compute

$$\iint \int_0^\infty \eta(\lambda) R_0^+(\lambda^2)(x,z) \frac{V(z)P_0(z,\tilde{z})V(\tilde{z})}{\lambda^2} [R_0^+(\lambda^2) - R_0^-(\lambda^2)](\tilde{z},y)\, d(\lambda^2) dz\, d\tilde{z}$$

Tackle the λ integral first. The free resolvent $R_0^+(\lambda^2)(x,z)$ has an exact formula in terms of Hankel function $H^{(1)}_{\frac{n-2}{2}}(|x-z|\lambda)$. Its complex conjugate is $R_0^-(\lambda^2)$.

So we have power-law behavior when $|x-z|\lambda$, $|\tilde{z}-y|\lambda < 1$, and oscillation when $|x-z|\lambda$, $|\tilde{z}-y|\lambda > 1$.

After integrating by parts in λ many times, one is left to bound

$$\iint \frac{V(z)P_0(z,\tilde{z})V(\tilde{z})}{|x-z|^{n-2}\langle |x-z|+|\tilde{z}-y|\rangle\langle |x-z|-|\tilde{z}-y|\rangle^{n-3}}\,dz d\tilde{z}$$

We assumed $|V(z)| \le C(1+|z|)^{-\beta}$ for a large β.
P_0 projects onto eigenfunctions, which also decay for large z.

Since $z, \tilde{z}$ are localized near the origin, the integral above is not bigger than

$$K(x,y) := \frac{C}{\langle x\rangle^{n-2}\langle |x|+|y|\rangle\langle |x|-|y|\rangle^{n-3}}$$

Finally, an integral operator whose kernel is controlled by

$$K(x, y) := \frac{C}{\langle x\rangle^{n-2}\langle |x| + |y|\rangle\langle |x| - |y|\rangle^{n-3}}$$

is bounded on $L^p(\mathbb{R}^n)$ over the range $1 \le p < \frac{n}{2}$.

The bottleneck occurs where $|y| > 2|x|$, because $|x|^{-(n-2)}|y|^{-(n-2)}$ only belongs to $L^{p'}(dy)$ for $p < \frac{n}{2}$.

Extra cancellation conditions: Suppose we know that

$$\int P_0(z,\tilde{z})V(\tilde{z})\,d\tilde{z} = 0$$

Then the integral below vanishes:

$$\iint \int_0^\infty \eta(\lambda) R_0^+(\lambda^2)(x,z) \frac{V(z)P_0(z,\tilde{z})V(\tilde{z})}{\lambda^2} [R_0^+(\lambda^2) - R_0^-(\lambda^2)](\mathbf{0},y)\, d(\lambda^2) dz\, d\tilde{z}$$

Which allows us to work with differences

$$[R_0^+(\lambda^2) - R_0^-(\lambda^2)](\tilde{z},y) - [R_0^+(\lambda^2) - R_0^-(\lambda^2)](0,y)$$

If $|\tilde{z}| < \frac{1}{2}|y|$, this gains a factor of $\frac{|\tilde{z}|}{|y|}$ in the estimates.

After splicing together regions $\{|\tilde{z}| < \frac{1}{2}|y|\}$ and $\{|\tilde{z}| > \frac{1}{2}|y|\}$, we can show that the estimates when $|x| \gtrsim |y|$ are unaffected, but now

$$K(x,y) \sim \langle x \rangle^{n-2} \langle y \rangle^{-(n-1)} \text{ when } \; |y| > 2|x|.$$

The faster decay in y means it will be a bounded operator on $L^p(\mathbb{R}^n)$ for $1 \leq p < n$.

If $\int P_0(z,\tilde{z})(a + b\tilde{z}_j)V(\tilde{z})\,d\tilde{z} = 0$, then we can introduce second-order differences to gain another factor of $\frac{|\tilde{z}|}{|y|}$.

Ultimately that allows an upper bound of the form

$$K(x, y) \sim \langle x\rangle^{n-2}\langle y\rangle^{-n} \text{ when } \ |y| > 2|x|.$$

which is almost integrable in y, so this operator is bounded on $L^p(\mathbb{R}^n)$ for $p \in [1, \infty)$.

It is reasonable to expect that $p = \infty$ can also be obtained with more cancellation conditions...

All sort of new issues arise if $n \leq 4$:

- Possibility of resonances instead of eigenvalues.
- Even $-\Delta$ has a resonance if $n \leq 2$.
- $\langle x \rangle^{-(2n-4)}$ isn't integrable for large x.
- $\langle |x| - |y| \rangle^{n-3}$ doesn't integrate well over $\{|x| \sim |y|\}$.
- More leading-order terms to consider.

There are results in $n = 1$ (D'Ancona-Fanelli), 3 (Yajima), and 4 (Jensen-Yajima).

[What about $n = 2$?]