Bochner-Riesz Estimates for Functions with Vanishing Fourier Transform

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Motivation: When does Helmholtz equation $(-\Delta - 1)u = f$ have a solution $u \in L^2(\mathbf{R}^n)$?

(It must be unique, because $(-\Delta - 1)u = 0$ only has the trivial solution in L^2.)

The formal requirement is:

"f is orthogonal to the nullspace of $(-\Delta - 1)$."

or perhaps:

"$\widehat{f} = 0$ on the unit sphere."

But that ignores problems with unbounded operators, existence of Fourier restriction, etc.

What function space should f belong to?

It should permit a meaningful L^2-restriction of $\hat{f}$ to spheres.

Specifically, with $F(r) = \|\hat{f}\|_{L^2(r\mathbf{S}^{n-1})}$
we need to have $\frac{F(r)}{|r^2-1|} \in L^2([0,\infty))$.

Theorem (Agmon): Given f with $xf \in L^2(\mathbf{R}^n)$ and $\hat{f} = 0$ in $L^2(\mathbf{S}^{n-1})$, then $(\Delta + 1)u = f$ has a solution in $L^2(\mathbf{R}^n)$.
Moreover $\|u\|_{L^2} \lesssim \|xf\|_{L^2}$.

Method: Locally flatten $\mathbf{S}^{n-1}$ and shift to plane $\{\xi_n = 0\}$. Now the problem reduces to 1-dimensional Hardy inequality
$\| \int_x^\infty f\|_2 \lesssim \|xf\|_2$.

Why $f \in L^p(\mathbf{R}^n)$ shouldn't work:

If $f \in L^p(\mathbf{R}^n)$, that doesn't give $\widehat{f}$ any Sobolev regularity. For $1 \leq p \leq \frac{2n+2}{n+3}$, the restrictions $\widehat{f}\Big|_{r\mathbf{S}^{n-1}}$ are continuous in r but not necessarily Hölder continuous of any positive order.

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Why $f \in L^p(\mathbf{R}^n)$ might work anyway:

$F(r) = \|\widehat{f}\|_{L^2(r\mathbf{S}^{n-1})}$ can be smooth even when $\widehat{f}$ isn't.

Theorem 1. *Let* $n \geq 3$ *and* $\max(1, \frac{2n}{n+4}) \leq p \leq \frac{2n+2}{n+5}$.
Suppose $f \in L^p(\mathbf{R}^n)$ *and* $\hat{f}\Big|_{\mathbf{S}^{n-1}} = 0$.
Then there exists a unique $u \in L^2(\mathbf{R}^n)$ *such that* $-\Delta u - u = f$.

The result is also true for a range of Bochner-Riesz multipliers

$$S^\alpha f := \mathcal{F}^{-1}\Big[(1 - |\xi|^2)_+^\alpha \hat{f}\Big].$$

Theorem 2. *Let* $n \geq 2$ *and* $\frac{1}{2} < \alpha < \frac{3}{2}$, *and* $1 \leq p \leq \frac{2n+2}{n+1+4\alpha}$.
Suppose $f \in L^p(\mathbf{R}^n)$ *and* $\hat{f}\Big|_{\mathbf{S}^{n-1}} = 0$.
Then $\|S^{-\alpha} f\|_2 \leq C(n, \alpha, p)\|f\|_p$.

Application: embedded resonances of $-\Delta + V$

Let $V \in L^{n/2}(\mathbf{R}^n)$, and set $v = |V|^{1/2}$ and $w = |V|^{1/2}\text{sgn}(V)$.

Suppose $(I + v(-\Delta - z)^{-1}w)^{-1} \in \mathcal{B}(L^2)$ has a pole at $z = \lambda + i0$ for some $\lambda > 0$. Then $I + v(-\Delta - (\lambda + i0))^{-1}w$ has eigenfunction $\phi \in L^2$.

$\psi = (-\Delta - (\lambda + i0))^{-1}w\phi$ is an eigenfunction of $-\Delta + V$ but it isn't obviously in L^2.

However if V is real-valued, then $\mathcal{F}(w\phi)$ is required to vanish on the sphere radius $\sqrt{\lambda}$. Bootstrapping with Theorem 1 shows that in fact $\psi \in L^2(\mathbf{R}^n)$.

Sketch of proofs: The main case is $p = \frac{2n+2}{n+1+4\alpha}$.
[The lower bound on p in Theorem 1 is due to Sobolev embedding.]

By Plancherel's identity and monotone convergence,

$$\|S^{-\alpha} f\|_2^2 \leq \left\| |1 - |\xi|^2|^{-\alpha} \widehat{f} \right\|_2^2 = \lim_{\epsilon \to 0} \langle A_\epsilon \widehat{f}, \widehat{f} \rangle$$

where $A_\epsilon(\xi) = \Big((1 - |\xi|^2)^2 + \epsilon^2 \Big)^{-\alpha}$.

Let σ_r denote the surface measure of $r\mathbf{S}^{n-1} \subset \mathbf{R}^n$.
By assumption $\langle \sigma_1 \widehat{f}, \widehat{f} \rangle = 0$, so it suffices to estimate

$$\lim_{\epsilon \to 0} \langle (A_\epsilon - c_\epsilon \sigma_1) \widehat{f}, \widehat{f} \rangle$$

for a well-chosen constant c_ϵ.

We'll use $c_\epsilon = \int_0^\infty A_\epsilon(r)\, dr$.

Now we'd like to show that the multiplier $A_\epsilon(\xi) - c_\epsilon \sigma_1$ produces a bounded operator from $L^{\frac{2n+2}{n+1+4\alpha}}(\mathbf{R}^n)$ to its dual, uniformly in ϵ. It helps to write out

$$A_\epsilon(\xi) - c_\epsilon \sigma_1 = \int_0^\infty A_\epsilon(r)(\sigma_r - \sigma_1)\, dr.$$

Recall that $A_\epsilon(r) \lesssim |r-1|^{-2\alpha}$ over the range $0 \leq r \leq 2$.

So if $\alpha < 1$, then $(r-1)A_\epsilon(r)$ is uniformly integrable at $r = 1$. And if $\alpha = 1$, then the integral of $(r-1)A_\epsilon(r)$ remains bounded due to cancellations.

Using well-known properties of $\check{\sigma}_r$, one can show that the convolution kernel associated to $A_\epsilon(\xi) - c_\epsilon \sigma_1$ is bounded by:

$$|K_\epsilon(x)| \leq \frac{C}{|x|^{\frac{n+1-4\alpha}{2}}}$$

Thus $|x|^{\frac{n+1-4\alpha}{2}} K_\epsilon$ defines a bounded operator from L^1 to L^∞.

One can also show that the Fourier transform of $\dfrac{K_\epsilon}{|x|^{2\alpha+i\mu}}$ is bounded pointwise by $\log|\mu|$, so these define bounded operators from L^2 to itself (except if $\mu = 0$).

Both of the above estimates are independent of $\epsilon > 0$. Complex interpolation (imitating the sharp Stein-Tomas theorem) completes the proof.

What about $\alpha > 1$?

Basic idea: Continue Taylor expansion around $r = 1$.

We started out by assuming that $\langle \sigma_1 \widehat{f}, \widehat{f} \rangle = 0$.

Let σ' be the distribution $\left(\frac{d}{dr} \sigma_r \right) \Big|_{r=1}$.

It turns out that $\langle \sigma' \widehat{f}, \widehat{f} \rangle = 0$ as well, for a reason that is either deep, or completely trivial – I can't decide which.

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Back on slide 2 we introduced $F(r) = \| \widehat{f} \|_{L^2(r\mathbf{S}^{n-1})}$.

Then $\langle \sigma_r \widehat{f}, \widehat{f} \rangle = [F(r)]^2 \geq 0$, so any point where $\langle \sigma_r \widehat{f}, \widehat{f} \rangle = 0$ must be a local minimum. That forces $\frac{d}{dr} \langle \sigma_r \widehat{f}, \widehat{f} \rangle = 0$ as well.

To extend the Theorems to $1 < \alpha < \frac{3}{2}$, one establishes a uniform bound on the multipliers $\left[A_\epsilon(\xi) - c_\epsilon \sigma_1 - d_\epsilon \sigma'\right]$, again using the operator norm from $L^{\frac{2n+2}{n+1+4\alpha}}(\mathbf{R}^n)$ to its dual.

We choose the new constant to be $d_\epsilon = \int_0^\infty (r-1) A_\epsilon(r)\, dr$.

Thus $A_\epsilon(\xi) - c_\epsilon \sigma_1 - d_\epsilon \sigma' = \int_0^\infty A_\epsilon(r)(\sigma_r - \sigma_1 - (r-1)\sigma')\, dr$.

So long as $\alpha < \frac{3}{2}$, the functions $(r-1)^2 A_\epsilon(r)$ are uniformly integrable at $r = 1$. There is logarithmic divergence if $\alpha = \frac{3}{2}$, with no mitigating cancellation.

Corollaries via Interpolation:

The spaces $\{f \in L^p(\mathbf{R}^n), \widehat{f} = 0 \text{ on } \mathbf{S}^{n-1}\}$ are not well suited for interpolation because the vanishing-restriction property is not preserved by most actions. The dual space is also an awkward quotient of $L^{p'}$.

One can at least use complex interpolation of operators.
For example: given a general function $f \in L^1(\mathbf{R}^2)$, one knows that $\|S^0 f\|_q \lesssim \|f\|_1$ provided $q > \frac{4}{3}$.

However if $\widehat{f} = 0$ on the unit circle, one can interpolate between

$$\left.\begin{array}{r} \|S^{-\frac{3}{4}+i\mu} f\|_2 \lesssim \|f\|_1 \\ \|S^{\alpha+i\mu} f\|_1 \lesssim \|f\|_1, \alpha > \frac{1}{2} \end{array}\right\} \text{ to get that } \|S^0 f\|_q \lesssim \|f\|_1,\ q > \frac{5}{4}.$$

The $\alpha = \frac{1}{2}$ endpoint:

When $\alpha = \frac{1}{2}$, $p = \frac{2n+2}{n+1+4\alpha}$ is the Stein-Tomas exponent.

Of course $\left|1 - |\xi|^2\right|^{-1/2}$ fails to be square-integrable.

Does setting $\widehat{f}(\xi) = 0$ on the unit sphere allow $\left|1-|\xi|^2\right|^{-\frac{1}{2}} \widehat{f} \in L^2$?

In n=1, the answer is **no**.
Take $f(x) = \eta(x + N\pi) - \eta(x - N\pi)$ for some bump function η.
Then $S^{-1/2}f(x) \sim \frac{1}{\sqrt{|x-N\pi|}}$ over most of the interval $x \in [0, 2N\pi]$.

Cancellation of f is needed on more length scales,
similar to what occurs in a Hardy space.

[In fact, The correct condition may be $e^{\pm ix} f(x) \in H^1(\mathbf{R})$.]

The $n = 1$ counterexample doesn't work in higher dimensions. It is much harder to force $\widehat{f}$ to vanish on the entire unit sphere. Which brings us to...

Proposition: I don't know how to resolve the statement

$$\left\{ f \in L^{\frac{2n+2}{n+3}}(\mathbf{R}^n),\ \widehat{f}\,\Big|_{S^{n-1}} = 0 \right\} \overset{??}{\Longrightarrow} S^{-1/2} f \in L^2(\mathbf{R}^n).$$

except in one dimension. But it would be really nifty if something in Fourier Analysis was true when $n \geq 2$ but not $n = 1$.