Wave Propagation on Periodic Planar Graphs

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Motivated by Quantum Harmonic Lattice:

Put a harmonic oscillator at each vertex of a graph, with position q_x and momenutum p_x.

The local Hamiltonian is $H_x = p_x^2 + \omega^2 q_x^2$.

Introduce nearest-neighbor interactions to produce a global Hamiltonian

$$H = \sum_{x \in \Lambda} \left(p_x^2 + \omega^2 q_x^2 + \sum_{y \in \Lambda} \lambda_{xy} (q_y - q_x)^2 \right)$$

where $\omega > 0$ and $\lambda_{xy} \begin{cases} > 0 & \text{if } x, y \text{ are adjacent.} \\ = 0 & \text{otherwise.} \end{cases}$

The classical system has a discrete Klein-Gordon equation as its equation of motion:

$$u_{tt}(x,t) = -\omega^2 u(x,t) + \sum_{y\in\Lambda} \lambda_{xy}\Big(u(y,t) - u(x,t)\Big)$$

If (Λ, λ) is a periodic weighted graph in $\mathbf{R}^d$, eigenfunctions of the discrete Laplacian are built out of plane waves $e^{ik\cdot x}$, $k \in \mathbb{T}^d$.

Main Example: If $\Lambda = \mathbf{Z}^d$, and $\lambda_{x,x+e_j} = \lambda_j$, the discrete Klein-Gordon equation has plane-wave solutions

$$u_k(x,t) = e^{i(k\cdot x - \varphi(k)t)}$$

where $\varphi(k) = \sqrt{\omega^2 + 4\sum_{j=1}^{d} \lambda_j \sin^2\left(\frac{k_j}{2}\right)}$.

Soultions to K-G equation on $\mathbf{Z}^d$:

The fundamental solution for

$$\begin{cases} u_{tt} = -\omega^2 u(x,t) + \sum_{j=1}^{d} \lambda_j \big(u(x+e_j,t) + u(x-e_j,t) - 2u(x,t) \big) \\ u(x,0) = \delta_0 \\ u_t(x,0) = 0 \end{cases}$$

is $\Phi_1(x,t) = \int_{\mathbb{T}^d} \cos(t\varphi(k)) e^{ik\cdot x}\, dk$

and the fundamental solution for

$$\begin{cases} u_{tt} = -\omega^2 u(x,t) + \sum_{j=1}^{d} \lambda_j \big(u(x+e_j,t) + u(x-e_j,t) - 2u(x,t) \big) \\ u(x,0) = 0 \\ u_t(x,0) = \delta_0 \end{cases}$$

is $\Phi_2(x,t) = \int_{\mathbb{T}^d} \frac{\sin(t\varphi(k))}{\varphi(k)} e^{ik\cdot x}\, dk$

Lieb-Robinson bounds (finite propagation speed): $\Phi(x,t)$ decays exponentially for large $x \gg t$.

For the classical system, this follows from analyticity of $\varphi(k)$.

Dispersive estimates: How does $\sup_x |\Phi(x,t)|$ decay with t?

This requires control of oscillatory integrals like

$$\int_{\mathbb{T}^d} e^{\pm it\varphi(k)} e^{ik\cdot x}\, dk \quad \text{as } t \to \infty$$

If $x_0 = t\nabla\varphi(k_0)$ for some $k_0 \in \mathbb{T}^d$,
there is stationary phase at k_0.

Non-degenerate stationary phase estimate:

$$|\Phi(x_0, t)| \lesssim \frac{1}{t^{d/2}\sqrt{\det D^2\varphi(k_0)}}$$

If $\det D^2\varphi(k_0) = 0$, asymptotic decay depends on Taylor series of $\varphi(k)$ centered at k_0.

When $d = 1$, van der Corput Lemma implies $|\Phi_1(x,t)| \lesssim t^{-1/3}$.

Similarly, $|\Phi_2(x,t)| \lesssim t^{-1/3}$, with a constant depending on ω (comes from factor of $1/\varphi(k)$).

Degenerate stationary phase is difficult in $d > 1$.

Asymptotic decay depends on Taylor series expansion with respect to "adapted" coordinates (Varchenko, 1976).

Detailed analysis for $d = 2$:

$$\varphi(k) = \sqrt{\omega^2 + 2\lambda_1(1 - \cos k_1) + 2\lambda_2(1 - \cos k_2)}$$

$$\begin{aligned} &\det D^2\varphi(k) \\ &\qquad = \varphi^{-4}(k)\Big(\omega^2 ab - \lambda_1 b(1-a)^2 - \lambda_2 a(1-b)^2\Big) \end{aligned}$$

where $a = \cos k_1$ and $b = \cos k_2$.

Where is $\det D^2\varphi(k) = 0$?

- A closed curve Γ_1 around origin, corresponding to extremal propagation velocity.

- A closed curve Γ_2 around $(\pi,\pi) \in \mathbb{T}^2$ corresponding to ???

At all $k \in \mathbb{T}^2$, $D^2\varphi(k)$ has rank ≥ 1.

Among $k \in \Gamma_1$, the second and third-order directional derivatives of $\varphi(k)$ never vanish at the same time.

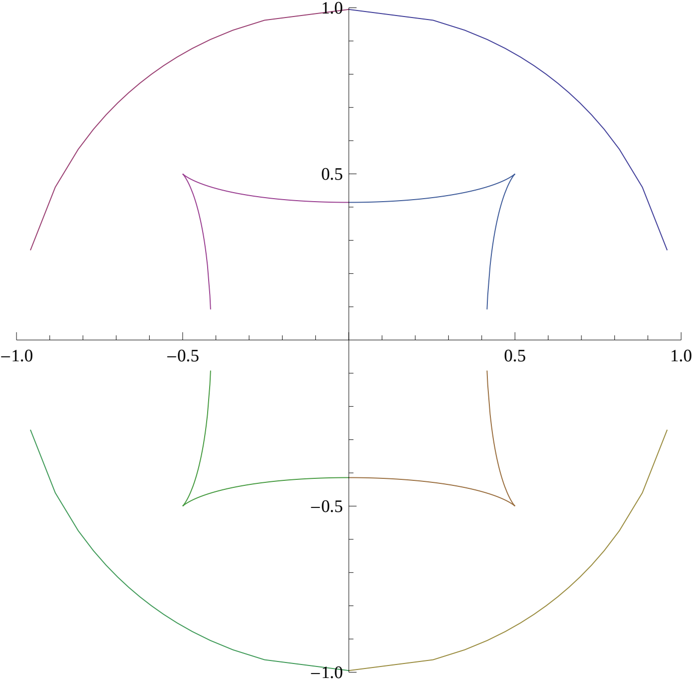
This leads to an estimate $|\Phi(x,t)| \lesssim t^{-5/6}$ when $\frac{x}{t}$ is near an extremal velocity.

Peculiar Results:

Among $k \in \Gamma_2$, there is a unique point (up to mirror symmetries) where both the second and third-order directional derivatives of $\varphi(k)$ vanish, but a relevant fourth-order quantity is nonzero.

Thus there is a unique velocity (again up to symmetry) where fundamental solutions of the discerete Klein-Gordon equation decay at the rate $t^{-3/4}$.

This region of least dispersion occurs in middle of the propagation pattern, not at its leading edge.



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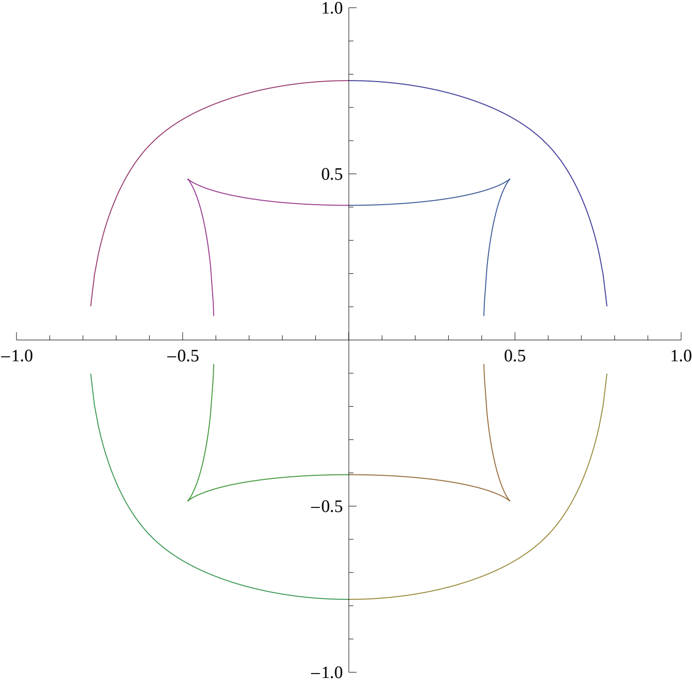
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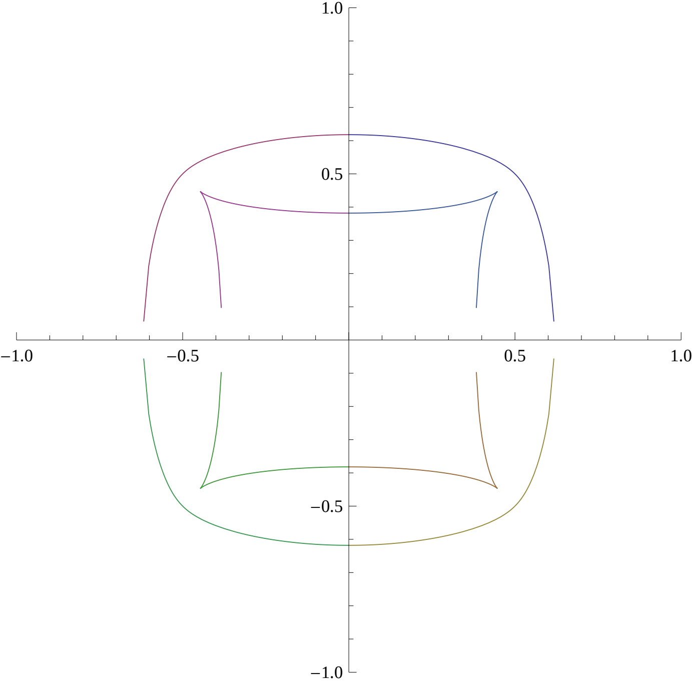
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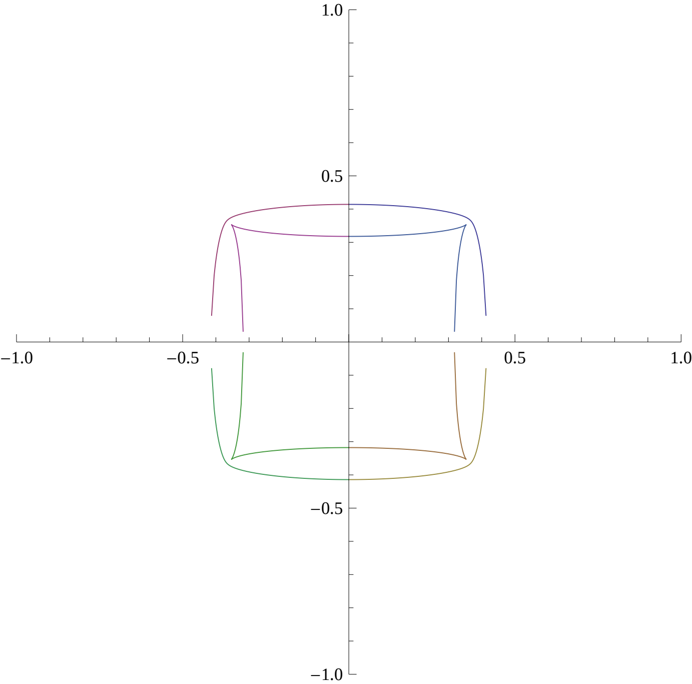

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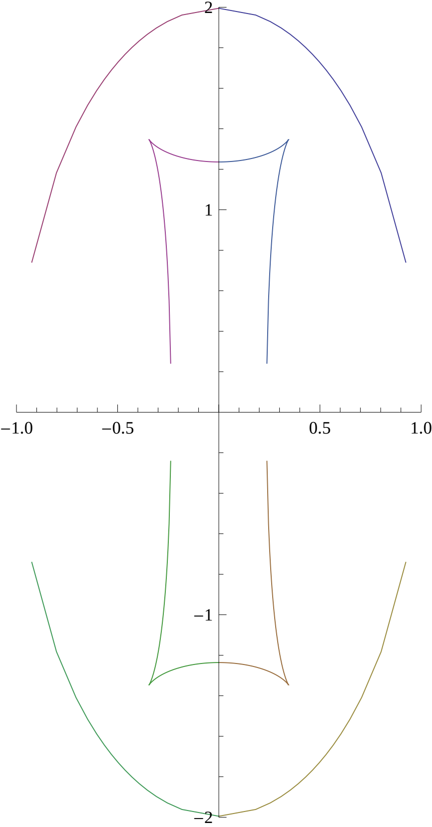
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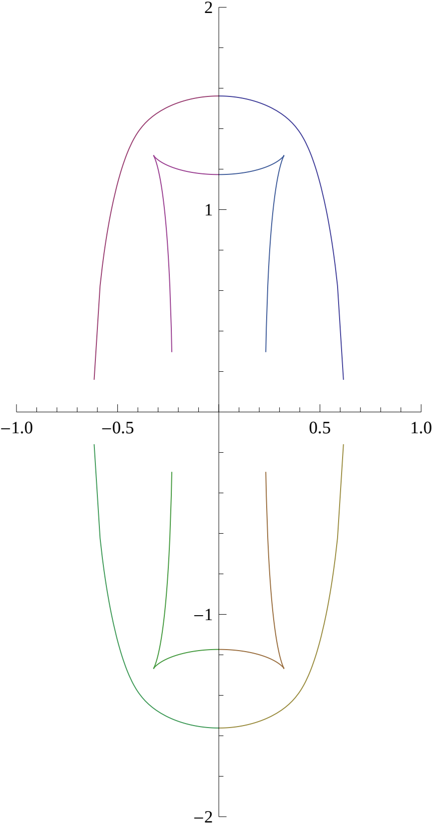

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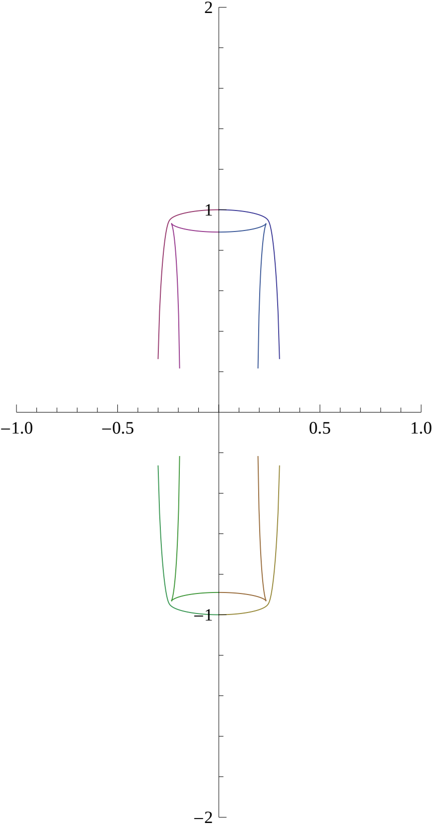

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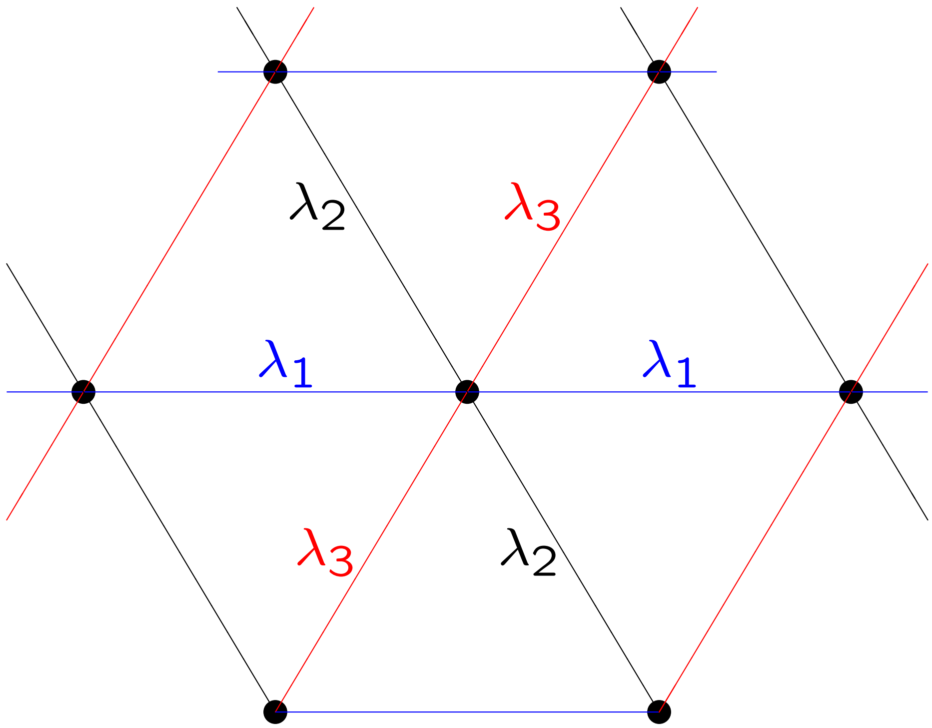
Example: Triangular lattice in $\mathbf{R}^2$.



The Schrödinger equation has plane-wave solutions

$$u_k(x,t) = e^{i(k\cdot x - \varphi(k)t)}$$

with the phase function

$$\varphi(k) = 4\Big[\lambda_1 \sin^2\left(\frac{k_1}{2}\right) + \lambda_2 \sin^2\left(\frac{-k_1+\sqrt{3}k_2}{2}\right) + \lambda_3 \sin^2\left(\frac{k_1+\sqrt{3}k_2}{2}\right)\Big]$$

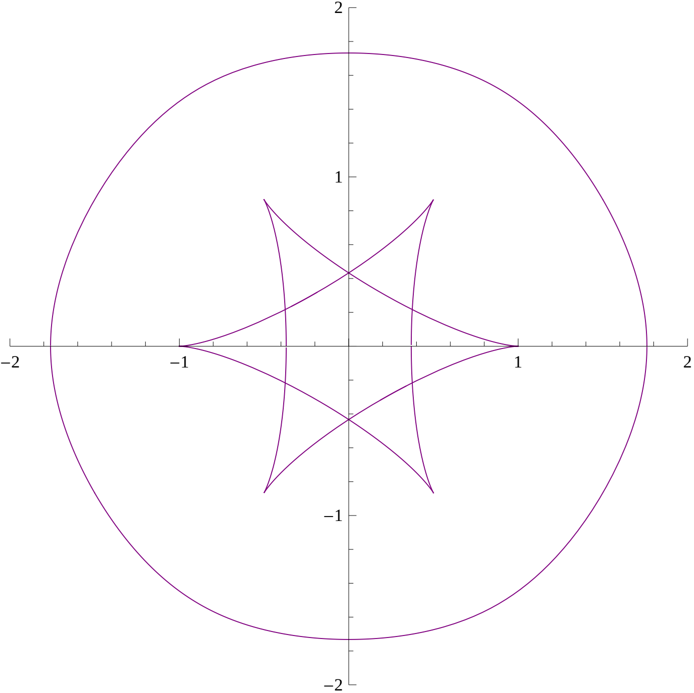
Observations:

The number and type of critical points of $\varphi(k)$ is variable, and depends on the parameters λ_j.

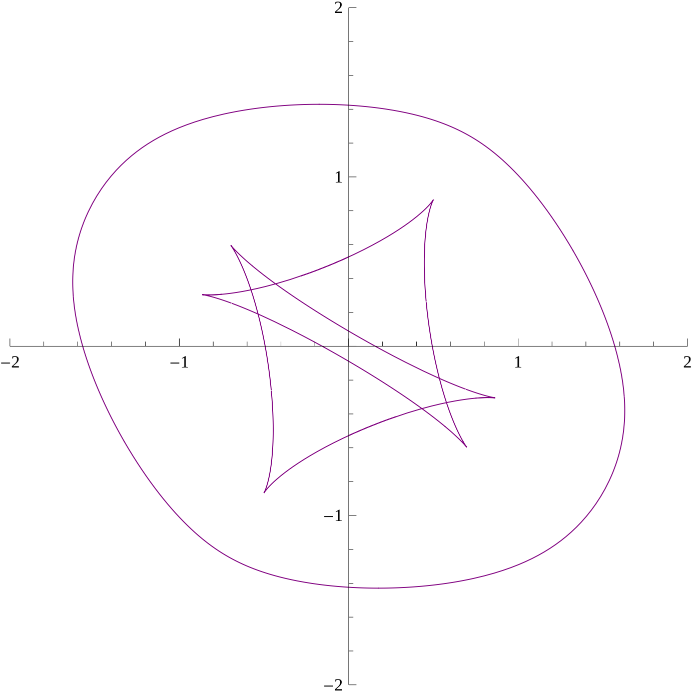
Bifurcations involving the number of caustics (e.g. $\lambda_1 = \lambda_2 = 1, \lambda_3 = \frac{1}{2}$) do not affect the power-law decay $|\Phi(x,t)| \leq C|t|^{-3/4}$.

Several cusps interact when $\lambda_1 = \lambda_2 = 1, \lambda_3 = \frac{1}{\sqrt{8}}$ in what is known as a "butterfly singularity." For this choice of λ_j the dispersive bound is only $|\Phi(x,t)| \leq C|t|^{-2/3}$.

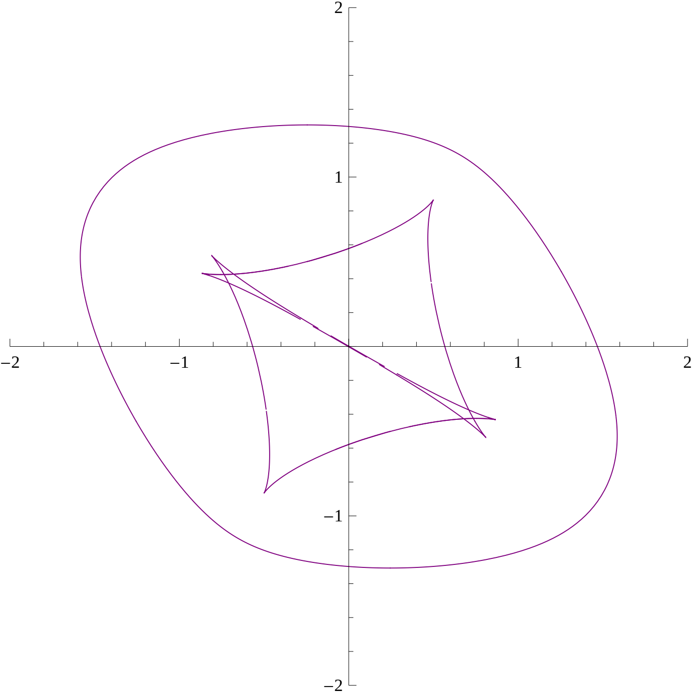
Other time-decay exponents may be possible.



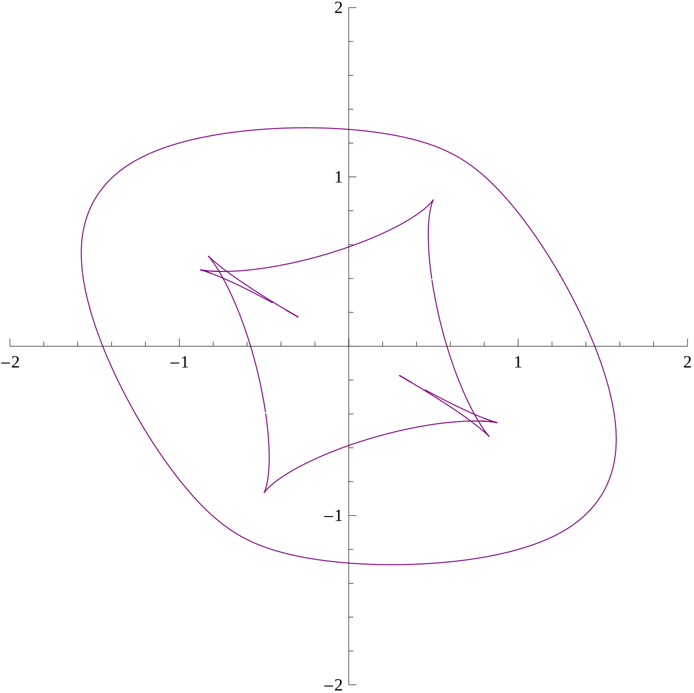
$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 1$



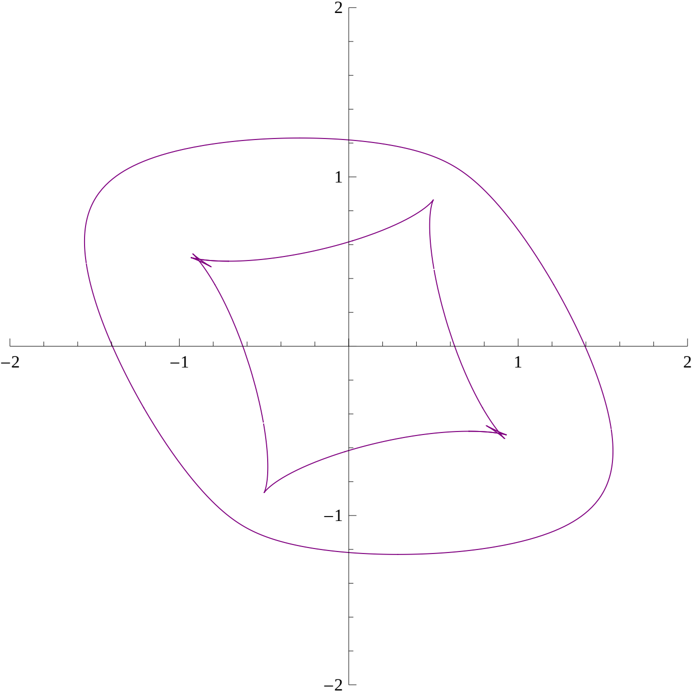
$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 0.65$



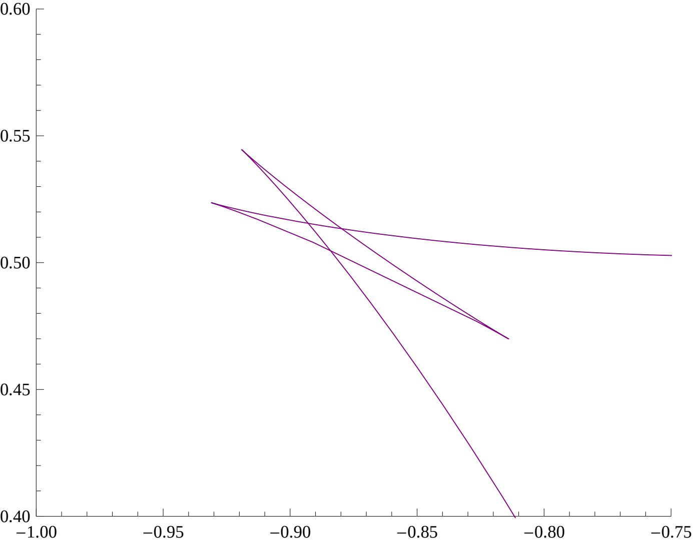
$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 0.51$



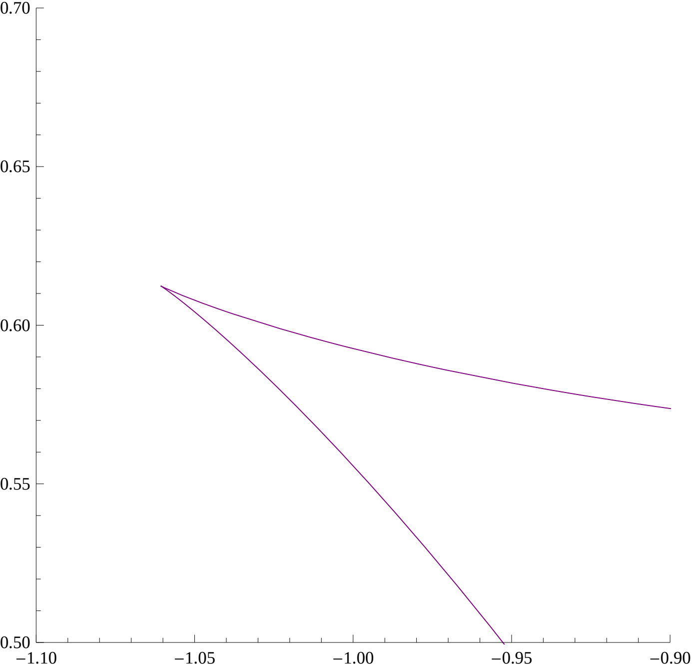
$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 0.49$



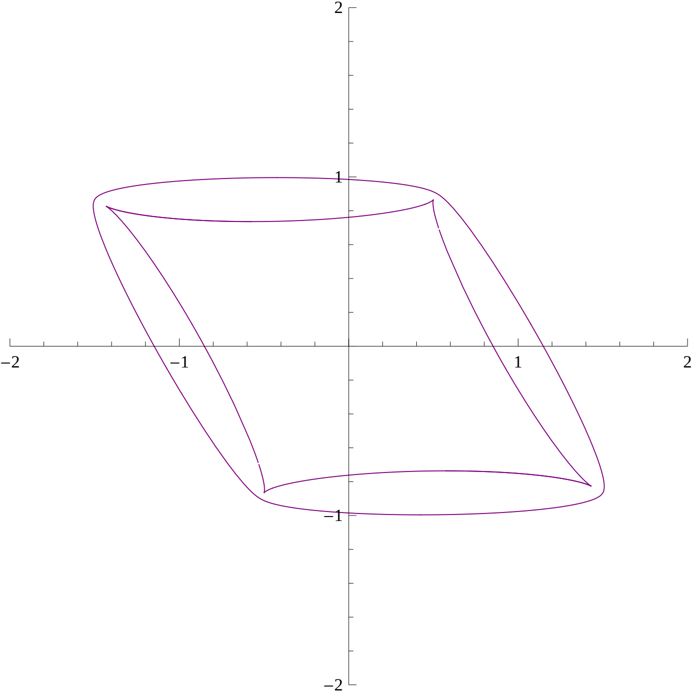
$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 0.42$



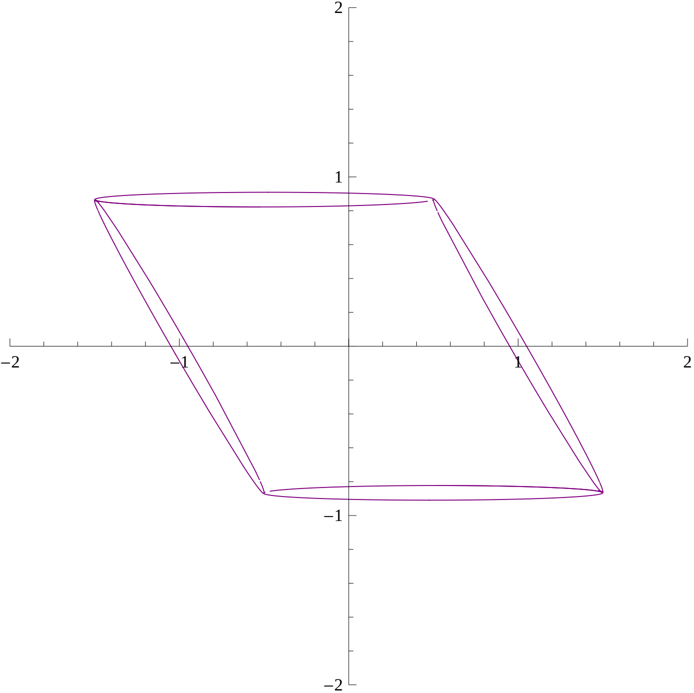
$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 0.42$



$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == \frac{1}{2\sqrt{2}}$



$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 0.15$



$\lambda_1 == \lambda_2 == 1, \quad \lambda_3 == 0.05$