The Schrödinger Equation with a Large Magnetic Potential

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Linear Schrödinger Equation in $\mathbf{R}^3$.

$$\begin{aligned}-iu_t &= (-\Delta + i(\mathbf{A}\cdot\nabla + \nabla\cdot\mathbf{A}) + V)u \\ &= (-\Delta + L)u \\ &= Hu\end{aligned}$$

If $\mathbf{A}, V \equiv 0$, some properties of H_0 include:

- Spectrum of H_0 is absolutely continuous, supported on $[0, \infty)$.

- Kato Smoothing bound:

$$\left\| \langle x\rangle^{-1-\varepsilon}(1-\Delta)^{\frac{1}{4}} e^{itH_0}\psi \right\|_{L^2_t L^2_x} \lesssim \|\psi\|_{L^2}$$

- Strichartz Inequalities:

$$\left\| e^{itH_0}\psi \right\|_{L^q_t L^r_x} \lesssim \|\psi\|_{L^2}, \qquad \frac{2}{q} = 3\left(\frac{1}{2} - \frac{1}{r}\right),$$

where $q \in [2, \infty)$

These statements are not generally true for e^{itH}.

If $\mathbf{A}$ or V is large, there may exist bound states which have no time-decay.

Questions:

- Are bound states the only problem?

- What happens if we remove them with the orthogonal projection $P_{ac}(H)$?

Theorem 1 (Erdoğan, G, Schlag) *Suppose* $\mathbf{A}$, div $\mathbf{A}$, *and* V *have rapid polynomial decay, meaning*

$$|\mathbf{A}(x)|,\ |\mathrm{div}\,\mathbf{A}(x)|,\ |V(x)|\ \leq C\langle x\rangle^{-\beta}.$$

Then the spectrum of H *is absolutely continuous on* $(0,\infty)$.

Furthermore, the propagator $e^{itH}P_{[0,\infty)}(H)$ *satisfies the same Kato smoothing and Strichartz estimates (except for* $q=2$*) as in the free case, provided there is no eigenvalue or resonance at zero.*

About the conditions:

- The potentials can be very large and/or negative.

- Our current value for β is near 8.

- We expect the theorem to be true for $\beta > 2$.

- The case $\beta = 3$ includes all bounded magnetic fields with compact support.

- Results about spectrum depend only on $\mathbf{A}$ and V, but not on $(\operatorname{div} \mathbf{A})$.

Outline of Proof

Step 1: Absence of Embedded Eigenvalues.

Step 2: Limiting Absoprtion Principle for H.

Step 3: Resolvent Estimates at Zero Energy.

Step 4: Resolvent Estimates at High Energy.

Step 5: Dynamical Consequences.

Step 1: Absence of Embedded Eigenvalues.

First show that any eigenfunction must have exponential decay.

Use Carleman inequalities to conclude that the eigenfunction is everywhere zero.

Best result due to Koch-Tataru ('05).

Applicable whenever $\mathbf{A}(x), V(x) = o((1 + |x|)^{-1})$. Some local singularities are also acceptable.

Step 2: Limiting Absorption Principle.

This is an operator estimate for the resolvent

$$R(\lambda^2) \;:=\; (H - (\lambda + i0)^2)^{-1}.$$

Important examples:
On any compact set $K \subset (0, \infty)$,

$$\left\| \langle x\rangle^{-\sigma} R(\lambda^2) \langle x\rangle^{-\sigma} f \right\|_{L^2} \;\leq\; \frac{C_K}{\lambda} \|f\|_{L^2}, \qquad \sigma > \frac{1}{2}$$

$$\left\| \langle x\rangle^{-\sigma} R(\lambda^2) \langle x\rangle^{-\sigma} f \right\|_{H^1} \;\leq\; \frac{C_K \langle \lambda \rangle}{\lambda} \|f\|_{L^2}.$$

The proof follows an argument by Agmon ('75), and is based on the perturbation identity

$$R(\lambda^2) = (I + R_0(\lambda^2) L)^{-1} R_0(\lambda^2)$$

Facts about the free resolvent

$$R_0(\lambda^2) = (-\Delta - (\lambda + i0)^2)^{-1}$$

The free resolvent can be seen in two ways:

- Multiplication of the Fourier transform by

$$\frac{1}{|\xi|^2 - \lambda^2} \; + \; \frac{\pi i}{\lambda} d\sigma_{\lambda \mathbf{S}^2}$$

  This is well-behaved except when $|\xi| \sim \lambda$.

- Convolution with the kernel $K(x) = \dfrac{e^{i\lambda|x|}}{4\pi|x|}$.

  This is easy to control in the limit $\lambda \to 0$.

Overview of Agmon's method:

$$R(\lambda^2) \;=\; (I + R_0(\lambda^2)L)^{-1}R_0(\lambda^2)$$

Prove the desired mapping bounds for the free resolvent $R_0(\lambda^2)$, using the Fourier transform description.

The operator $(I + R_0(\lambda^2)L)$ is a compact perturbation of the identity. Apply the Fredholm Alternative theorem to find its inverse.

Show that any eigenfunction with $\langle x\rangle^{-\sigma} f \in L^2$ is a true L^2-eigenfunction.

There are no embedded eigenvalues, so the inverse must exist.

Step 3: Resolvent estimates at zero energy.

The method is essentially the same as before.

This time the desired mapping properties for the free resolvent are obtained by comparison to a convolution with $\frac{1}{|x|}$.

Stronger weights are required in this case. For example, when $|\lambda| < 1$, the estimate

$$\left\| \langle x \rangle^{-\sigma} R(\lambda^2) \langle x \rangle^{-\sigma} f \right\|_{H^1} \; \leq \; C_\sigma \lambda \|f\|_{L^2}$$

is only valid for $\sigma > 1$.

To apply the Fredholm alternative at zero energy, one must assume that there is no eigenvalue or resonance here.

Step 4: Resolvent estimates at high energy.

Theorem 2 *The estimates in the Limiting Absorption Principle continue to be valid as* $\lambda \to \infty$. *Most importantly,*

$$\left\| \langle x \rangle^{-\sigma} R(\lambda^2) \langle x \rangle^{-\sigma} f \right\|_{H^1} \le C_\sigma \|f\|_{L^2}$$

for all $|\lambda| > 1$ *and* $\sigma > \frac{1}{2}$.

The Fredholm alternative shows that $R(\lambda^2)$ exists pointwise in λ. More delicate estimates are needed to obatain a uniform bound.

The minimum decay and regularity requirements appear to be that $|\mathbf{A}(x)|, |V(x)| \le C\langle x \rangle^{-1-\varepsilon}$ and that $\mathbf{A}$ is continuous.

Remark 3 *D. Robert ('92) proved a similar result for* C^∞ *perturbations with symbol-like decay, using the method of Mourre commutators.*

If $\mathbf{A}$ and V are small, then $(I + R_0(\lambda^2)L)^{-1}$ can be written explicitly as

$$\sum_{k=0}^{\infty} (-1)^k \Big(R_0(\lambda^2)L\Big)^k$$

because $\|R_0(\lambda^2)L\| < 1$.

Remark 4 *Many strong results exist for small magnetic potentials.*

Example: Georgiev, Stefanov, and Tarulli ('06) have proved Strichartz inequalties (including the endpoint) for small, rough, and time-dependent *perturbations.*

If $\mathbf{A}$ and V are large, the power series

$$\sum_{k=0}^{\infty}(-1)^k\Big(R_0(\lambda^2)L\Big)^k$$

is still convergent for all $\lambda > \lambda_A$ because of the following fact.

Lemma 5 *There exists a constant $C < \infty$ so that*

$$\limsup_{\lambda\to\infty}\left\|\Big(R_0(\lambda^2)L\Big)^m\right\| \leq \frac{C^m C_L^m}{(m!)^{\varepsilon/2}}$$

The quantity C_L is defined as $\sup_x |\langle x\rangle^{1+\varepsilon}\mathbf{A}(x)|$.

How Lemma 5 implies Theorem 2:

If we choose $m \gg C_L^{(2/\varepsilon)}$, then

$$\left\| \left(R_0(\lambda^2) L \right)^m \right\| < \tfrac{1}{2}$$

for all λ sufficiently large.

This makes it possible to sum the power series

$$\left\| \left(I + R_0(\lambda^2) L \right)^{-1} \right\| \;\leq\; \sum_{j=0}^{\infty} \sum_{k=0}^{m-1} \left\| (R_0(\lambda^2) L)^{mj+k} \right\|$$

$$\leq\; C^m C_L^m$$

for a fixed $m \gg C_L^{(2/\varepsilon)}$ and all $\lambda > \lambda_A$.

Inspiration for Lemma 5.

Consider the operator $R_0(\lambda^2) V R_0(\lambda^2)$.

This is an integral operator with kernel

$$K(x,z) = \int \frac{e^{i\lambda(|x-y|+|y-z|)}}{|x-y|\,|y-z|} V(y)\, dy$$

The phase function $(|x-y| + |y-z|)$ has critical points only where x, y, and z are collinear, and in order.

If $\angle xyz$ is bounded away from zero, we can use integration by parts to gain a factor of λ^{-1}.

More detailed inspiration for proof of Lemma 5

If we expand $(R_0(\lambda^2)L)^m$ in the same way, it will be an integral over $m-1$ variables. There are two main regions to consider:

- The region where every angle $\angle x_{k-1}x_kx_{k+1}$ is smaller than $\frac{1}{m}$. There is no oscillation here. Instead, there is a specific direction of motion.

  By treating this like a Volterra integral, one gains a factor involving $m!$.

- The complement of this region. If any angle $\angle x_{k-1}x_kx_{k+1}$ is large, then the integral over dx_k has nonstationary phase with gradient at least $\frac{\lambda}{m}$.

  Such an integral goes to zero as $\lambda \to \infty$, by applying a suitable Riemann-Lebesgue lemma.

Step 5: Dynamical Consequences

It is time to extract results from our understanding of the spectrum of H.

Theorem 6 *(Rodnianski, Schlag '04) Consider $H = -\Delta + L$ with $L = \sum_j Y_j^* Z_j$. Suppose each of the operators Y_j is Δ-smooth and each $Z_j P_\Omega(H)$ is H-smooth. Then the semigroup associated to H, projected onto the spectral set Ω, satisfies the following bounds:*

- *Kato Smoothing bound:*

$$\left\| \langle x \rangle^{-1-\varepsilon} (1-\Delta)^{\frac{1}{4}} e^{itH} P_\Omega(H)\psi \right\|_{L_t^2 L_x^2} \lesssim \|\psi\|_{L^2}$$

- *Strichartz Inequalities:*

$$\left\| e^{itH} P_\Omega(H)\psi \right\|_{L_t^q L_x^r} \lesssim \|\psi\|_{L^2}, \qquad \frac{2}{q} = 3\left(\frac{1}{2} - \frac{1}{r}\right),$$

where $q \in (2, \infty)$

Verifying that $L = i(\mathbf{A}\cdot\nabla + \nabla\cdot\mathbf{A}) + V$ satisfies the hypotheses:

Observe that L is self-adjoint, and is a bounded operator from $\langle x\rangle^{\beta} H^1$ to L^2.

Using the functional calculus and interpolation, it follows that $|L|^{\frac{1}{2}}$ is bounded from $\langle x\rangle^{\frac{\beta}{2}} H^{\frac{1}{2}}$ to L^2. The same is true for the operator $\operatorname{sgn}(L)|L|^{\frac{1}{2}}$. These will be our decomposition $L = YZ$.

We will use the criteria: An operator $ZP_\Omega(H)$ is H-smooth if

$$\sup_{\lambda\in\Omega} \|ZR(\lambda^2)Z^*\|_{2\to 2} < \infty.$$

For the resolvents, our estimates imply that both $R_0(\lambda^2)$ and $R(\lambda^2)$ are bounded as operators from $\langle x\rangle^{-\frac{\beta}{2}}H^{-\frac{1}{2}}$ to $\langle x\rangle^{\frac{\beta}{2}}H^{\frac{1}{2}}$, with norm independent of λ.

Meanwhile, the operators $|L|^{\frac{1}{2}}$ and $\mathrm{sgn}(L)|L|^{\frac{1}{2}}$ are both bounded from $\langle x\rangle^{\frac{\beta}{2}}H^{\frac{1}{2}}$ to L^2. Their adjoints must map L^2 into $\langle x\rangle^{-\frac{\beta}{2}}H^{-\frac{1}{2}}$, by duality.

It follows immediately that $|L|^{\frac{1}{2}}$ and $\mathrm{sgn}(L)|L|^{\frac{1}{2}}$ are Δ-smooth, and also H-smooth over the spectral set $\Omega = [0, \infty)$.

Summary of Results

- Understanding the spectrum of a Schrödinger operator with large magnetic potential.

  For this, we only assume pointwise decay of $\mathbf{A}$ and V, and also that $\mathbf{A}$ is continuous.

- Kato Smoothing and Strichartz estimates for the absolutely continuous portion of H.

  Stronger regularity conditions are required. For example we may need to assume that $\langle x \rangle^{\beta} \mathbf{A}$ is a bounded multiplier on $H^{\frac{1}{2}}$.

Parting Questions:

1) What are the ideal assumptions for $\mathbf{A}$ and V?

2) Is the endpoint Strichartz estimate true?

3) Are these results valid in other dimensions?