Differentiability of Fourier Restrictions

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Basic Facts

Support from Simons Foundation grant #635369.

Collaboration with Dmitriy Stolyarov (St. Petersburg)
and Chun Ho Lau (Cincinnati)

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Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx, \;\; x, \xi \in \mathbb{R}^n$.

Notation: $\xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$

Work with paraboloid $\{\xi_n = |\xi'|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_{n-1}^2\}$.

Let Σ be a bounded region of the paraboloid.
Identify $\xi \in \Sigma$ with the corresponding $\xi' \in \mathbb{R}^{n-1}$.

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Lebesgue Spaces:

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$$f \in L^p(\mathbb{R}^n) \;\Leftrightarrow\; \int_{\mathbb{R}^n} |f(x)|^p\,dx < \infty$$

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L^2-based Sobolev Spaces:

$$\hat{f} \in H^s(\Sigma) \text{ or } H^s(\mathbb{R}^{n-1}) \;\Leftrightarrow\; \left\{ \begin{matrix} \text{Derivatives of order } s \\ \text{exist as a function in } L^2(\mathbb{R}^{n-1}). \end{matrix} \right\}$$

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$$\hat{f} \in H^{-s}(\Sigma) \text{ or } H^{-s}(\mathbb{R}^{n-1}) \;\Leftrightarrow\; \left\{\begin{matrix}\text{``}\hat{f} \text{ is the derivative order } s \\ \text{of a function in } L^2(\mathbb{R}^{n-1})\text{.''}\end{matrix}\right\}$$

Most elements of $H^{-s}(\mathbb{R}^{n-1})$ are not functions. But smooth functions form a dense subspace.

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If $f(x) \in L^1(\mathbb{R}^n)$, then $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx$ is continuous.

The restriction to a surface $\hat{f}\big|_{\Sigma}$ is also continuous.

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If $f(x) \in L^p(\mathbb{R}^n)$, $1 \le p \le 2$, then $\hat{f}(\xi)$ exists as an element of $L^{\frac{p}{p-1}}(\mathbb{R}^n)$.

It's not clear if $\hat{f}\big|_\Sigma$ makes sense. Σ is a measure-zero set.

It's really not clear if $\frac{\partial \hat{f}}{\partial \xi_n}\big|_\Sigma$ makes sense.

Nothing good happens on flat surfaces

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There are many examples of $f(x) \in L^p(\mathbb{R}^n)$, $1 < p \le 2$, where $\hat{f}\big|_P$ is undefined. It's easy to build examples because the Fourier transform respects the separation of variables. That is,

$$\text{The Fourier transform of } f_1(x')f_2(x_n) \ \text{ is } \hat{f}_1(\xi')\hat{f}_2(\xi_n).$$

and a typical function $f_2(x_n) \in L^p(\mathbb{R})$ only has $\hat{f}_2(\xi_n)$ defined for almost every ξ_n. Create one where $\hat{f}_2(0) = \int_{\mathbb{R}} f_2(x_n)\, dx_n$ is infinite.

Good things happen on curved surfaces

Even though the paraboloid Σ is a measure-zero set, just like P, the fact that it is curved makes Fourier restrictions possible.

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Theorem (Stein-Tomas Restriction Theorem)

If $f \in L^p(\mathbb{R}^n)$ for any $1 \leq p \leq \frac{2n+2}{n+3}$,
then $\hat{f}\big|_\Sigma$ exists as an element of $L^2(\Sigma)$.

More specifically, there is a bound

$$\left\| \hat{f}\big|_\Sigma \right\|_{L^2(\Sigma)} \leq C \|f\|_{L^p(\mathbb{R}^n)}$$

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Let's shift Σ up and down by amount r by defining Σ_r as the corresponding region of the paraboloid $\{\xi_n = |\xi'|^2 + r\}$.

You can show the function which maps $r \in \mathbb{R}$
to the function $\hat{f}\big|_{\Sigma_r} \in L^2(\Sigma)$ is continuous.

It isn't differentiable because $\frac{\partial \hat{f}}{\partial \xi_n}$ is the Fourier transform of $-ix_n f(x)$, and we aren't assuming that $x_n f(x)$ is integrable in any useful way.

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Theorem

If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \frac{2n+2}{n+3+2k}$,
then $\frac{\partial^k \hat{f}}{\partial \xi_n^k}\Big|_\Sigma$ exists as an element of $H^{-k}(\Sigma)$.

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The case $k = 0$ is the Stein-Tomas theorem.
The case $k = \frac{n-1}{2}$ is proved by stationary phase.
[take an oscillatory integral and integrate by parts a lot.]

The in-between cases are proved by interpolation.

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The principle that

[smooth values along a surface] $\Rightarrow$ [smooth values for the normal derivative]

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Does $\hat{f}(\xi)$ have a Dirichlet-to-Neumann property along surfaces? No.

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Does $\hat{f}(\xi)$ have a Dirichlet-to-Neumann property along surfaces? No.

Does $\hat{f}(\xi)$ have a Dirichlet-to-Neumann property along curved surfaces? Funny you should ask. . . .

Nothing good happens on flat surfaces, Part II

Take the plane $P = \{\xi \in \mathbb{R}^n : \xi_n = 0\}$ again.

Build a function $f(x) = f_1(x')f_2(x_n)$ again, this time with $\hat{f}_2(0) = 0$.

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The Fourier transform of $f(x)$ is $\hat{f}(\xi) = \hat{f}_1(\xi')\hat{f}_2(\xi_n)$,
which gives it a value of zero everywhere along P.
A constant zero function is <u>very</u> smooth.

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A constant zero function is <u>very</u> smooth.

But $\frac{\partial \hat{f}}{\partial \xi_n}$ is basically the derivative of $\hat{f}_2(\xi_n)$, which typically doesn't exist.

The fact that $\hat{f}\big|_P \equiv 0$ didn't help the last derivative at all.

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Theorem (G. - Stolyarov, 2020)

If $f \in L^p(\mathbb{R}^n)$, *with* $1 \leq p \leq \frac{2n+2}{n+7}$, *and* $\hat{f}\big|_\Sigma \equiv 0$,
then $\frac{\partial \hat{f}}{\partial \xi_n}\Big|_\Sigma$ *exists as an element of* $L^2(\Sigma)$.

[Note: That's better than being an element of $H^{-1}(\Sigma)$*.]*

It actually suffices for $\hat{f}\big|_\Sigma$ *to belong to* $H^\ell(\Sigma)$ *for a large enough* ℓ*.*

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Theorem (G. - Lao, 2024)

In the above result, "large enough" means precisely $\ell \ge \frac{2n+2-(n+3)p}{2n+2-(n+5)p}$.

Interpreting $\frac{\partial \hat{f}}{\partial \xi_n}$ in context

The theorem said: If $f \in L^p(\mathbb{R}^n)$, with $1 \leq p \leq \frac{2n+2}{n+7}$, and $\hat{f}\big|_\Sigma \equiv 0$, then $\frac{\partial \hat{f}}{\partial \xi_n}\Big|_\Sigma$ exists as an element of $L^2(\Sigma)$.

For compactly supported functions $f(x)$, the Fourier transform $\hat{f}(\xi)$ is guaranteed to be smooth, so $\frac{\partial \hat{f}}{\partial \xi_n}$ is well defined on Σ by any definition.

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Let's look at our surfaces Σ_r, wheren the ξ_n coordinate has been shifted up or down by r.

We prove that the norm of difference quotients $\left\| \frac{1}{r}\left(\hat{f}\big|_{\Sigma_r} - 0\right)\right\|_{L^2(\Sigma)}$ is bounded as $r \to 0$.

What about the "real" partial derivative $\frac{\partial \hat{f}}{\partial \xi_n}$?

The definition of a partial derivative is
$\frac{\partial \hat{f}}{\partial \xi_n}(\xi', \xi_n) = \lim_{r \to 0} \frac{1}{r}\Big(\hat{f}(\xi', \xi_n + r) - \hat{f}(\xi', \xi_n)\Big)$.

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Since we're assuming $\hat{f}(\xi', \xi_n) = 0$ at points on Σ,
the main thing we need is for $\frac{1}{r}\Big(\hat{f}(\xi', |\xi'|^2 + r) - 0\Big)$ to be bounded as $r \to 0$.

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Does the theorem on the last slide prove it?

In a word: No.

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The theorem on the last slide said we can control

$$\lim_{r \to 0} \left\| \frac{1}{r} \big(\hat{f}(\xi', |\xi'|^2 + r) \right\|_{L^2(\Sigma)}.$$

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Taking the integral in that $L^2(\Sigma)$ norm smoothes a lot of things out before we take the $r \to 0$ limit.

That doesn't guarantee the $r \to 0$ limit exists at any individual point $(\xi', |\xi'|^2) \in \Sigma$.

Counterexamples

If $p > 1$, there exists a function $g(x_n) \in L^p(\mathbb{R})$ where $\hat{g}(\xi_n)$ is infinite whenever ξ_n is rational.

There is also a function $h(x)$ which is compactly supported and has $\hat{h}(\xi) = 0$ along the surface $\xi \in \Sigma$.

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There is also a function $h(x)$ which is compactly supported and has $\hat{h}(\xi) = 0$ along the surface $\xi \in \Sigma$.

The convolution $f(x) = \int_{\mathbb{R}} h(x', x_n - y_n) g(y_n)\, dy_n$ belongs to $L^p(\mathbb{R}^n)$ and has the property that $\hat{f}(\xi) = \hat{h}(\xi)\hat{g}(\xi_n)$.

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So $\hat{f}(\xi)$ is zero along the surface Σ,
but it's infinite everywhere else that ξ_n is rational.

That's not even bounded as you move in the ξ_n direction, much less continuous, much less differentiable.

A result for $f \in L^1(\mathbb{R}^n)$

The counterexample doesn't work when $g \in L^1(\mathbb{R})$, because then $\hat{g}(\xi_n)$ must be continuous. That still seems pretty far from being differentiable, but it turns out...

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Theorem

If $n > 5$ and $f \in L^1(\mathbb{R}^n)$ has the property $\hat{f}\big|_\Sigma \equiv 0$,

then the partial derivative $\frac{\partial \hat{f}}{\partial \xi_n}(\xi', |\xi'|^2)$

exists at almost every point $(\xi', |\xi'|^2) \in \Sigma$.

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We prove that the difference quotients are controlled by a bound

$$\Big\| \sup_{r \neq 0} \frac{1}{r}\Big(\hat{f}(\xi', |\xi'|^2 + r) - 0\Big)\Big\|_{L^2(\Sigma)} \leq C\|f\|_{L^1(\mathbb{R}^n)}.$$

Summary

- The Fourier transform of an integrable function is just continuous. The Fourier transform of $f \in L^p(\mathbb{R}^n)$ is defined almost everywhere.
- On a curved surface Σ, and for a range of p, we can make sense of
  - The values of $\hat{f}(\xi)$ along Σ.
  - The transverse partial derivative $\frac{\partial \hat{f}}{\partial \xi_n}$ along Σ.
    Usually this is an element of $H^{-1}(\Sigma)$, not a function.
- If the values of $\hat{f}(\xi)$ along Σ are all zero, or are smooth enough, then the things we're calling $\frac{\partial \hat{f}}{\partial \xi_n}$ gets nicer. In dimensions $n \geq 5$ it can improve all the way up to becoming a function in $L^2(\Sigma)$.
- In the case $p = 1$ and $n \geq 6$, we can show that the derivative $\frac{\partial \hat{f}}{\partial \xi_n}$ literally exists at almost every point on Σ.

Open Questions

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2. Is $\hat{f}(\xi)$ differentiable at almost every point $\xi \in \Sigma$?

3. Can we say anything about the restrictions of $\hat{f}(\xi)$ and its derivatives in any other function space besides $L^2(\Sigma)$ and $H^{-s}(\Sigma)$?

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3. Can we say anything about the restrictions of $\hat{f}(\xi)$ and its derivatives in any other function space besides $L^2(\Sigma)$ and $H^{-s}(\Sigma)$?

4. There are counterexamples in $n = 2$ where $\frac{\partial \hat{f}}{\partial \xi_2}$ just doesn't exist.
   What happens in dimensions 3 and 4 (and/or 5)?