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# The Schrödinger Equation

with a

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## Non-Smooth Magnetic Potential

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# Free Schrödinger Equation (in $\mathbb{R}^n$ )

$$\begin{cases} i\partial_t u(t, x) = -\Delta u(t, x) \\ u(0, x) = u_0(x) \end{cases}$$

$u(t, x) = e^{it\Delta} u_0$ . Unitary  $\Rightarrow L^2$  conservation.

Fourier Inversion:  $u(t, x) = \frac{1}{(4\pi i t)^{n/2}} \int_{\mathbb{R}^n} e^{i \frac{|x-y|^2}{4t}} u_0(y) dy$

Dispersive estimate  $\|u(t, \cdot)\|_p \lesssim |t|^{-n(\frac{1}{2} - \frac{1}{p})} \|u_0\|_p$ ,

$T T^*$  argument + Hardy-Littlewood-Sobolev



$$\underbrace{\|u(t, x)\|_{L_t^p L_x^q}_p}_{\leq \|u_0\|_2} \lesssim \|u_0\|_2 \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad \begin{matrix} 2 \leq p, q \leq \infty \\ \text{unless } n=2. \end{matrix}$$

Strichartz Inequalities

# Magnetic Schrödinger Equation (time-indep't)

$$\begin{aligned} i\partial_t u(t,x) &= \underbrace{\left(-\Delta + i(\vec{A}(x) \cdot \nabla + \nabla \cdot \vec{A}(x)) + V(x)\right)}_{= (-\Delta + L(x))} u(t,x) \\ &= Hu(t,x) \end{aligned}$$

Question: Is it true that  $\|e^{-itH} u_0\|_{L_t^p L_x^q} \lesssim \|u_0\|_2$ ?

Answer: Not always. If  $H$  has eigenvalue  $\lambda$ ,  
then  $u(t,x) = e^{-it\lambda} \psi(x)$  is solution eigenfunction  $\psi$

Strichartz inequality is only satisfied for  $p=\infty, q=2$ .

Better Question: For what class of  $V, \vec{A}$

can we say  $\|e^{-itH} P_{ac}(H) u_0\|_{L_t^p L_x^q} \lesssim \|u_0\|_2$ ?

Decay as  $x \rightarrow \infty$ ?

Local regularity?

Thm: Suppose  $V \in L^\infty$  and  $\lim_{|x| \rightarrow \infty} |x|^2 V(x) = 0$ ,

$\vec{A}$  is continuous, and  $\sum_{k=0}^{\infty} 2^{3k} \sup_{|x|=2^k} |\vec{A}(x)|^2 < \infty$ ,

and  $H$  does not have an eigenvalue or a resonance at  $\lambda=0$ .

Then  $\left\| e^{-itH} P_{ac}(H) u_0 \right\|_{L_t^p L_x^q} \lesssim \|u_0\|_2, \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad p > 2$ .

Remarks:

- Valid in dimensions  $n \geq 3$
- Lose  $p=2$  endpoint because of Christ-Kiselev lemma.
- Natural scaling is  $V(x) \propto |x|^{-2}$   
 $\vec{A}(x) \propto |x|^{-1}$
- Probably true for  $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$
- Can't commute derivatives in  $(\vec{A} \cdot \nabla + \nabla \cdot \vec{A})$ .

Method of Proof (Rodnianski-Schlag, '04):

Use Duhamel's solution formula.

$$u(t,x) = e^{it\Delta} u_0(tx) - i \int_0^t e^{i(t-s)\Delta} L u(s,x) ds$$

First term is already good.

Now factorize  $L = \sum_{j=1}^J Y_j^* Z_j$ .

That leaves  $\int_0^t e^{it\Delta} (e^{-is\Delta} Y_j^*) (Z_j e^{-isH} u_0) ds$ .

We would like to have  $\|Z_j e^{-isH} P_{ac}(H) u_0\|_{L^2(\mathbb{R} \times \mathbb{R}^n)} \lesssim \|u_0\|_2$

and  $\left\| \int_0^\infty e^{-is\Delta} Y_j^* g(s,x) ds \right\|_2 \lesssim \|g\|_{L^2(\mathbb{R} \times \mathbb{R}^n)}$ .

Finally,  $e^{it\Delta}$  maps  $L^2(\mathbb{R}^n)$  to  $L_t^p L_x^q$ .

[Need Christ-Kiselev lemma because domain is  $0 \leq s \leq t$ ]

Evaluating  $\|Y_j e^{is\Delta} f\|_{L^2(\mathbb{R} \times \mathbb{R}^n)}^2$

$T\bar{T}^*$  + Plancherel's Identity ( $s \leftrightarrow \lambda$ )

$$\|Y_j e^{is\Delta} f\|^2 = \sup_{\lambda \in [0, \infty]} \|Y_j (R_o^+(\lambda) - R_o^-(\lambda)) Y_j^*\|_{2 \rightarrow 2}$$

where  $R_o^\pm(\lambda) = \lim_{\epsilon \rightarrow 0} (-\Delta - (\lambda \pm i\epsilon))^{-1}$

• Fourier Multiplier  $\frac{1}{|\xi|^2 - \lambda} \neq \frac{\pi i}{\sqrt{\lambda}} d\sigma_{\{\xi = \sqrt{\lambda}\}}$

• Convolution with Kernel

$$K(\lambda, x) = |x|^{-n+2} K(\sqrt{\lambda} x) \sim \begin{cases} |x|^{-n+2} & \text{if } |\sqrt{\lambda} x| \leq 1 \\ \frac{e^{\pm i\sqrt{\lambda} |x|}}{|x|^{\frac{n-1}{2}}} & \text{if } |\sqrt{\lambda} x| > 1. \end{cases}$$

Facts:  $R_o(\lambda)$  maps  $\langle x \rangle^{-\frac{1}{2}-\epsilon} L^2(\mathbb{R}^n)$  to  $\langle x \rangle^{\frac{1}{2}+\epsilon} L^2(\mathbb{R}^n)$

with operator norm  $\sim \lambda^{\frac{1}{2}}$

Powers of  $\nabla$  increase norm by a factor of  $\lambda^{\frac{1}{2}}$

Typical  $Y_j$  are:  $\langle x \rangle^{-\frac{1}{2}-\epsilon} |\nabla|^{\frac{1}{2}}$   
or  $\langle x \rangle^{-1-\epsilon}$

Analysis of  $Z_j$  is the same.

except we use  $R_L^*(\lambda) = (H - (\lambda \pm i0))^{-1}$   
 $= R_o(\lambda) \underbrace{[I + LR_o(\lambda)]^{-1}}$   
uniformly bounded?

For  $\lambda \in [0, \lambda_1]$ , use Fredholm Alternative (Agmon, '75).

1<sup>st</sup> Problem:  $L$  is first-order.

Operator norm of  $\nabla R_o(\lambda)$  does not decay  
in the limit  $\lambda \rightarrow \infty$ .

Uniform control of  $[I + LR_o(\lambda)]^{-1}$  via power series?

Thm (Erdogan - G - Schlag '06-'07):

Given any  $\epsilon > 0$ , there exists  $m_\epsilon < \infty$  so that

$$\limsup_{\lambda \rightarrow \infty} \left\| (LR_o(\lambda))^m \right\|_{X \rightarrow X} < \epsilon^m.$$

(i.e. spectral radius vanishes as  $\lambda \rightarrow \infty$ )

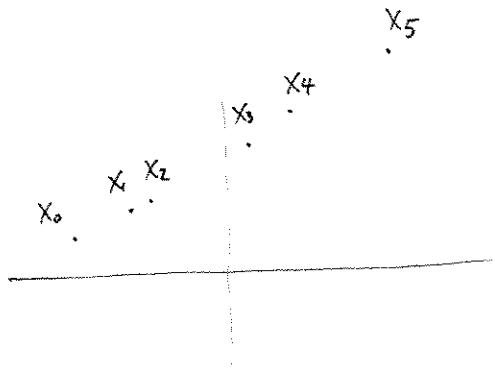
Idea of Proof:

$(LR_0(\lambda))^m$  is an  $m$ -fold integral operator.

Its Kernel oscillates like  $e^{\pm i\sqrt{\lambda}(|x_0-x_1|+|x_1-x_2|+\dots+|x_{m-1}-x_m|)}$

There is stationary phase whenever

$x_0, x_1, x_2, \dots, x_m$  are colinear and ordered.



Eventually  $|x_k|$  is large enough  
that  $\vec{A}(k)$  is small.

(cf. Volterra operators)

Since  $\vec{A}(x)$  isn't smooth, we can't use IBP  
in the complementary region.

Riemann-Lebesgue lemma is good enough.

2<sup>nd</sup> Problem: How to factorize  $\vec{A}(x) \cdot \nabla = Y_j^* Z_j$

If both  $Y_j, Z_j$  are (function of  $x$ ) ( $\frac{1}{2}$  derivative)

then you need control of  $|\nabla|^{\frac{1}{2}} A(x) |\nabla|^{\frac{1}{2}}$ .

$$\Rightarrow A \in W^{\frac{1}{2}, 2n} \cap L^\infty.$$

New Idea:  $Y_j = \cancel{\langle \partial_t \rangle^{\frac{1}{4}}} \langle \partial_t \rangle^{\frac{1}{4}} \langle x \rangle^{-\frac{1}{2}}$

$$Z_j = \langle \partial_t \rangle^{\frac{1}{4}} \langle x \rangle \vec{A}(x) \cdot \nabla$$

This works because  $\langle \partial_t \rangle^{\beta}$  behaves like  $\langle D_x \rangle^{2\beta}$   
but it commutes with  $\vec{A}(x)$ .

Minor Complication: Return to Duhamel formula

$$\int_0^t e^{i(t-s)\Delta} \underbrace{\langle x \rangle^{-\frac{1}{2}} \langle \partial_s \rangle^{\frac{1}{4}}}_{Y_j^*} \underbrace{\langle \partial_s \rangle^{\frac{1}{4}} \langle x \rangle \vec{A}(x) \cdot \nabla}_{Z_j} e^{-isH} P_{ac}(H) u_0 ds$$

Domain of integration  
is not  $\{t \geq s \geq 0\}$

Because Fractional Derivatives  
Are Non-Local.