Pointwise Bounds for the 3-Dimensional Wave Propagator

(and spectral multipliers)

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AMS Southeastern Sectional Meeting
Charleston, SC
March 11, 2017

Support provided by Simons Foundation grant #281057.

Perturbed wave equation in $\mathbb{R}^3$: $\begin{cases} u_{tt} - \Delta u + Vu = 0 \\ u(0,x) = 0 \\ u_t(0,x) = g(x) \end{cases}$ $(*)$

Potential $V(x)$ has finite Kato norm

$$\|V\|_{\mathcal{K}} := \sup_y \int_{\mathbb{R}^3} \frac{|V(x)|}{|x-y|} dx$$

and belongs to the norm-closure of $C_c(\mathbb{R}^3)$.

This has same scaling as $\frac{C}{|x|^2}$ or $L^{3/2}(\mathbb{R}^3)$, and is (just barely) sufficient to ensure that V is compact relative to $-\Delta$.

If $V(x) \equiv 0$, the fundamental solution of $\begin{cases} u_{tt} - \Delta u = 0 \\ u(0,x) = 0 \\ u_t(0,x) = g(x) \end{cases}$

is given by Kirchoff's formula :

$$K_0(t,x,y) = \frac{\delta_0(t - |x-y|) \pm \delta_0(t + |x-y|)}{4\pi(t \text{ or } |x-y|)},$$

which satisfies $\displaystyle\int_{-\infty}^{\infty} |K_0(t,x,y)|\, dt = \frac{1}{2\pi|x-y|}$.

Question: Does the fundamental solution of (*) also satisfy

$$\int_{-\infty}^{\infty} |K(t,x,y)|\, dt = \frac{C}{|x-y|} \text{ ?}$$

Answer: Not always. If $V(x)$ has large negative part, then $H = -\Delta + V$ may have finitely many negative eigenvalues $-\mu_j^2$.

Then (*) has solutions of the form

$$u(t,x) = \frac{\sinh(\mu_j t)}{\mu_j}\varphi_j(x),$$

where $\varphi_j(x)$ solves $(-\Delta + V)\varphi_j = -\mu_j^2 \varphi_j$.

Integrating $\int_{-\infty}^{\infty} \sinh(\mu_j t)\,dt$ will go badly...

Theorem (Beceanu - G.): If $\lambda = 0$ is not an eigenvalue or resonance of H, then

$$\int_{-\infty}^{\infty} \Big| K(t,x,y) - \sum_j \frac{\text{sinh}(\mu_j t)}{\mu_j} P_j(x,y) \Big| \, dt < \frac{C}{|x-y|}.$$

Also, $K(t,x,y)$ is supported inside the light cones $|t| \geq |x-y|$.

Corollary: The resolvents $R_V(z) := (H-z)^{-1}$ are integral operators whose kernels are bounded pointwise by $\frac{C}{|x-y|}$ for all z in a neighborhood of $\mathbb{R}^+$.

Sketch of Proof: Like many linear dispersive bounds, it starts with the Stone formula for spectral measure of H.

$$\begin{aligned}
\frac{\sin(t\sqrt{H})}{\sqrt{H}} - \sum_j \frac{\sinh(\mu_j t)}{\mu_j} P_j &= \frac{1}{2\pi i}\int_0^\infty \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}} (R_V^+(\lambda) - R_V^-(\lambda))\, d\lambda \\
&= \frac{1}{\pi i}\int_{-\infty}^\infty \sin(t\lambda) R_V^+(\lambda^2)\, d\lambda \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty (e^{-it\lambda} - e^{it\lambda}) R_V^+(\lambda^2)\, d\lambda.
\end{aligned}$$

Here $R_V^+(\lambda^2) := \lim_{\varepsilon\to 0}(H - (\lambda + i\varepsilon)^2)^{-1}$ has a meromorphic extension into the upper halfplane, with poles at $\lambda = i\mu_j$.

All we need is an L^1 estimate on the Fourier transform of $R_V^+(\lambda^2)(x,y)$.

If $V \equiv 0$, there is an exact formula: $R_0^+(\lambda^2)(x,y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}$.
Then $\mathcal{F}\Big(R_0^+(\lambda^2)\Big)(t,x,y) = \frac{\delta_0(t-|x-y|)}{4\pi|x-y|}$,
whose integral (in t) is bounded by $\frac{1}{4\pi|x-y|}$. So far, so good.

Now $$\begin{aligned} R_V^+(\lambda^2) &= [I + R_0^+(\lambda^2)V]^{-1} R_0^+(\lambda^2) \\ &= \qquad\quad G(\lambda) \qquad\quad R_0^+(\lambda^2). \end{aligned}$$

On the Fourier side, $\mathcal{F}\Big(R_V^+(\lambda^2)\Big)(t) = \mathcal{F}\Big(G(\lambda)\Big) * \mathcal{F}\Big(R_0^+(\lambda^2)\Big)(t)$.

It suffices to show that $I(x,w) = \int_{-\infty}^{\infty} \Big|\mathcal{F}\Big(G(\lambda)\Big)(t,x,w)\Big|\, dt$

is a bounded operator on the space $\dfrac{L^\infty(\mathbb{R}^3)}{|\cdot - y|}$, uniformly in y.

N. Wiener (1932)**:** Suppose $g(\lambda) \in C(\mathbb{T})$ has $\mathcal{F}g \in \ell^1(\mathbb{Z})$, and $g(\lambda) \neq 0$ pointwise over $\lambda \in \mathbb{T}$.

Then $\mathcal{F}\Big(\frac{1}{g(\lambda)}\Big) \in \ell^1(\mathbb{Z})$.

Beceanu (2010)**:** Similar theorems for operator-valued functions $G(\lambda)$ on the real line. In this case it's operators in $\mathcal{B}\Big(\frac{L^\infty}{|\cdot - y|}\Big)$.

One condition is that $G(\lambda) = I + R_0^+(\lambda^2)V$ should be invertible for each $\lambda \in \mathbb{R}$. This corresponds to the fact/our assumption that H has no eigenvalues in $[0, \infty)$.

A secondary issue is to get continuity with respect to $y \in \mathbb{R}^3$ and some sort of limit as $|y| \to \infty$.

Brief Summary: The Fourier transform of $R_V^+(\lambda^2)$ is [almost] the "forward solution" of wave equation (*).

It satisfies an integrability condition

$$\int_{-\infty}^{\infty} \left| \mathcal{F}\Big(R_V^+(\lambda^2)\Big)(t,x,y) \right| dt \le \frac{C}{|x-y|}$$

and a support condition

$$\mathcal{F}\Big(R_V^+(\lambda^2)\Big)(t,x,y) = \sum_j \frac{e^{\mu_j t}}{2\mu_j} P_j(x,y) \quad \text{for all } t < |x-y|.$$

Fourier Multipliers: Given a function $m : [0, \infty) \to \mathbb{C}$, one can define $m(\sqrt{-\Delta})$ to be the Fourier multiplier with symbol $m(|\xi|)$.

Operators of this type are well studied. In particular, we note the Hörmander-Mikhlin condition: Choose a smooth bump function ϕ supported on $[\frac{1}{2}, 2]$. Then if

$$\sup_{k \in \mathbb{Z}} \|\phi(\lambda) m(2^{-k}\lambda)\|_{H^s(\mathbb{R})}$$

for some $s > \frac{3}{2}$, then $m(|\xi|)$ is a Calderón-Zygmund operator.

If the condition holds for $s > 2$ then the integral kernel of $m(|\xi|)$ is bounded pointwise by $|x - y|^{-3}$.

Given the same function $m : [0, \infty) \to \mathbb{C}$, one can also define the spectral multiplier $m(\sqrt{H})$ in the functional calculus of H.

Theorem (Beceanu - G.)**:** If m satisfies the Hörmander-Mikhlin condition with $s > \frac{3}{2}$, then $m(\sqrt{H})$ is bounded on $L^p(\mathbb{R}^3)$ for $1 < p < \infty$.

If the condition holds for $s > 2$ then the integral kernel of $m(\sqrt{H})$ is bounded pointwise by $|x - y|^{-3}$.

In particular $H^{i\sigma}$ is well behaved, which is enough to deduce endpoint Strichartz estimates for (*).

It is not clear that the integral kernel of $m(\sqrt{H})$ satisfies

$$\int_{|x-y|>2|y-y'|} |K(x,y) - K(x,y')| \, dx < C$$

so our results do not include weak $(1,1)$ bounds for now.

Why these theorems are closely related:

The Stone formula for spectral measure gives us

$$
\begin{aligned}
m(\sqrt{H}) &= \frac{1}{2\pi i}\int_0^\infty m(\sqrt{\lambda})(R_V^+(\lambda) - R_V^-(\lambda))\,d\lambda \\
&= \frac{1}{\pi i}\int_{-\infty}^\infty \lambda m(|\lambda|) R_V^+(\lambda^2)\,d\lambda \\
&= \frac{1}{\pi i}\int_{-\infty}^\infty \mathcal{F}^{-1}\Big(\lambda m(|\lambda|)\Big)(t)\ \mathcal{F}\Big(R_V^+(\lambda^2)\Big)(t)\,dt.
\end{aligned}
$$

If we assume $s > 2$, then the Fourier transform of $\lambda m(\lambda)$ decays like $|t|^{-2}$. And $\mathcal{F}(R_V^+(\lambda^2))$ is [mostly] supported where $t > |x-y|$.

When $t < |x-y|$ there is an explicit description of $\mathcal{F}(R_V^+(\lambda^2))$ as a sum of exponential functions. This makes it easier to handle the $|t|^{-2}$ singularity near $t = 0$.

Questions for further study:

- Can you integrate the solution of (*) along time-like paths $(t, x(t))$ and still get a bound in terms of $\frac{1}{|x(0)-y|}$?
- Does $m(\sqrt{H})$ satisfy a weak $(1,1)$ bound, even for really nice multipliers m?
- Is there a Hardy space theory of these multipliers?
- What happens for $s \leq \frac{3}{2}$? Is there a robust L^p theory for Bochner-Riesz spectral multipliers?

- Do you have any good questions to contribute?