A Limiting Absorption Principle for Dirac Operators in Two and Higher Dimensions

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Background: The linear Schrödinger equation $iu_t = -\Delta u$ does not respect special relativity. Relationships between velocity, momentum, and energy follow Newtonian mechanics.

The Klein-Gordon equation $-u_{tt} = (-\Delta + m^2)u$ has the right relationship between these quantities, but it doesn't have a unitary evolution.

The equation $iu_t = \left(\sqrt{-\Delta + m^2}\right)u$ is unitary and encodes special relativity, but the operator is nonlocal. This makes it unclear how to add electromagnetic fields.

The Dirac System in $\mathbb{R}^n$: Let $\alpha_1, \ldots, \alpha_n$ and β be anti-commuting matrices with $\alpha_j^2 = I = \beta^2$.

When $n = 2$ it is convenient to use the Pauli spin matrices.

Define the Dirac operator $D_m := -\Big(i \sum_{j=1}^{n} \alpha_j \partial_j\Big) + m\beta.$

Thanks to the anti-commutation properties, $D_m^2 = -\Delta + m^2$.

Then the free Dirac equation is $\boxed{i\mathbf{u}_t = D_m \mathbf{u}}$

Not too surprisingly, solutions of the free Dirac equation satisfy the same Strichartz inequalities as the Klein-Gordon equation,

$$\left\| \langle \nabla \rangle^{-\theta} e^{-itD_m} \mathbf{u} \right\|_{L^p_t L^q_x} \lesssim \|\mathbf{u}\|_{L^2}$$

with the admissibility conditions

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad \theta \geq \frac{1}{2} + \frac{1}{p} - \frac{1}{q} \quad \text{when } m > 0.$$

[D'Ancona-Fanelli, Cacciafesta]

Or if $m = 0$, then the wave equation Strichartz estimates apply:

$$\left\| |\nabla|^{-\theta} e^{-itD_m} \mathbf{u} \right\|_{L^p_t L^q_x} \lesssim \|\mathbf{u}\|_{L^2}$$

with the admissibility conditions

$$\frac{2}{p} + \frac{n-1}{q} = \frac{n-1}{2}, \quad \theta = \frac{n}{2} - \frac{1}{p} - \frac{n}{q}.$$

We'd like to know if a perturbed Dirac operator $D_m + V(x)$ yields the same bounds. Here V is a Hermitian matrix with each entry bounded pointwise by $\langle x\rangle^{-2-\epsilon}$ ($\langle x\rangle^{-1-\epsilon}$ if $m = 0$).

Spectral properties: The essential spectrum of D_m is $(-\infty, -m] \cup [m, \infty)$. The spectrum of $D_m + V$ has no singular continuous part, or embedded eigenvalues or resonances [Georgescu-Mantoiu].

Threshold resonances and eigenvalues are possible, along with a finite point spectrum inside $(-m, m)$.

Theorem (Erdogan-G-Green)**:** *If* $V(x)$ *is continuous and has the specified pointwise decay, and there are no threshold resonances or eigenvalues, then the semigroup*

$$e^{-i(D_m+V)t}P_{ac}\,\mathbf{u}$$

satisfies the same Strichartz bounds as the corresponding free Dirac equation.

Short proof: First establish uniform bounds on the resolvent $(D_m + V - (\lambda \pm i\varepsilon))^{-1}$ for all $|\lambda| \in [m, \infty)$.

In particular, show that $\left\| |V|^{1/2}(D_m + V - (\lambda \pm i\varepsilon))^{-1}|V|^{1/2} \right\|_{2\to 2}$ has a uniform bound.

Kato smoothing arguments lead to a weighted bound in $L^2_{t,x}$ for both the free and perturbed Dirac evolution.

Then an argument due to Rodnianski-Schlag parlays these into Strichartz estimates for the perturbed equation.

Remark: Only the first part is new, and it turns out most of the work has been done before.

In fact, Georgescu-Mantoiu already proved uniform resolvent bounds on any compact interval inside of $|\lambda| \in (m, \lambda_1]$.
That leaves one big task.

Theorem(E-G-G): *If $V(x)$ has continuous entries bounded by $\langle x \rangle^{-1-\epsilon}$, then there exist constants $\lambda_1 < \infty$ and $\delta > 0$ so that the operator norm*

$$\left\| |V|^{1/2}(D_m + V - (\lambda \pm i\varepsilon))^{-1}|V|^{1/2} \right\|_{2\to 2}$$

is bounded uniformly over $|\lambda| > \lambda_1$ and $0 < \epsilon < \delta|\lambda|$.

It is not necessary for $V(x)$ to be Hermitian for this result.

And a smaller task to do the same in a neighborhood of $\lambda = \pm m$.

The uniform bound for $(D_m - \lambda)^{-1}$ is well known.

The perturbation identity

$$(D_m + V - \lambda)^{-1} = (D_m - \lambda)^{-1}\Big(I + V(D_m - \lambda)^{-1}\Big)^{-1}$$

would be immediately useful if the operator norm of $(D_m - \lambda)^{-1}$ decayed as $\lambda \to \infty$. It doesn't.

Using the fact that $D_m^2 = -\Delta + m^2$, we can rewrite the last factor as

$$\Big(I + \underbrace{V(D_m + \lambda)}_{1^{\text{st}}\text{-order}}\underbrace{\Big(-\Delta - (\lambda^2 - m^2)\Big)^{-1}}_{\text{Schrödinger resolvent}}\Big)^{-1}.$$

This has a lot in common with magnetic Schrödinger operators! (with magnetic potential $V \cdot \nabla$)

Uniform resolvent bounds for magnetic Schrödinger operators:

Positive commutator methods [Robert, D'Ancona-Fanelli-Cacciafesta].
Straightforward integration by parts.
Needs some differentiabilty of $V(x)$.
Also need $n - 3 \geq 0$.

Directional decomposition of the resolvent [E-G-G].
Constructs the operator inverse via convergent power series.
Complicated estimates of iterated integrals.
May sometimes need differentiability of $V(x)$ when $n = 2$
(but not this time).

Remarks about the $\lambda = m$ threshold:

If $m > 0$, this regime is identical a Schrödinger operator near $\lambda = 0$. Resolvent expansions [Jensen, Kato, Nenciu] are known.

The $m = 0$ case is quite different. For example, the resolvent of $(-\Delta)$ in $\mathbb{R}^2$ has a resonance at $\lambda = 0$. The resolvent of a massless Dirac operator D_0 doesn't.

For future study:

- Low-energy resolvent expansions when $m = 0$.
- Classification of threshold obstructions.
- Pointwise dispersive estimates.
- L^p-boundedness of the wave operators?