Dirichlet to Neumann Properties for Fourier Restrictions

Michael Goldberg

University of Cincinnati

Talks given for Würzburg University
and AMS Central Section Meeting

March 9 and April 16, 2023

Basic Facts

Support from Simons Foundation grant #635369.

Collaboration with Dmitriy Stolyarov (St. Petersburg)

Basic Facts

Support from Simons Foundation grant #635369.
Collaboration with Dmitriy Stolyarov (St. Petersburg)

Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx, \;\; x, \xi \in \mathbb{R}^n$.

Basic Facts

Support from Simons Foundation grant #635369.
Collaboration with Dmitriy Stolyarov (St. Petersburg)

Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\,dx, \;\; x, \xi \in \mathbb{R}^n$.

Notation: $\xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$

Work with paraboloid $\{\xi_n = |\xi'|^2 = \xi_1^2 + \xi_2^2 + \ldots + \xi_{n-1}^2\}$.

Let Σ be a bounded region of the paraboloid.
Identify $\xi \in \Sigma$ with the corresponding $\xi' \in \mathbb{R}^{n-1}$.

Dirichlet-to-Neumann Properties

We're going to work with $f \in L^p(\mathbb{R}^n)$ where $\hat{f}\big|_\Sigma \in L^2(\Sigma)$ by the Stein-Tomas theorem.

I'll say there is a Dirichlet-to-Neumann property if additional smoothness of $\hat{f}\big|_\Sigma$ (e.g. belonging to $H^\ell(\Sigma)$ for some $\ell > 0$) implies that $\frac{\partial^k \hat{f}}{\partial \xi_n^k}\Big|_\Sigma$ is smoother than its usual *a priori* estimates.

A priori bounds

What are the *a priori* bounds for a restriction of $\frac{\partial^k \hat{f}}{\partial \xi_n^k}$?

Keep in mind that $\frac{\partial^k \hat{f}}{\partial \xi_n^k} \in W^{-k,p'}(\mathbb{R}^n)$ is very bad.

A priori bounds

What are the *a priori* bounds for a restriction of $\frac{\partial^k \hat{f}}{\partial \xi_n^k}$?

Keep in mind that $\frac{\partial^k \hat{f}}{\partial \xi_n^k} \in W^{-k,p'}(\mathbb{R}^n)$ is very bad.

Theorem

If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \frac{2n+2}{n+3+2k}$,
then $\left.\frac{\partial^k \hat{f}}{\partial \xi_n^k}\right|_{\Sigma}$ exists as an element of $H^{-k}(\Sigma)$.

A priori bounds

What are the *a priori* bounds for a restriction of $\frac{\partial^k \hat{f}}{\partial \xi_n^k}$?

Keep in mind that $\frac{\partial^k \hat{f}}{\partial \xi_n^k} \in W^{-k,p'}(\mathbb{R}^n)$ is very bad.

Theorem

If $f \in L^p(\mathbb{R}^n)$, $1 \le p \le \frac{2n+2}{n+3+2k}$,
then $\left.\frac{\partial^k \hat{f}}{\partial \xi_n^k}\right|_{\Sigma}$ exists as an element of $H^{-k}(\Sigma)$.

The case $k = 0$ is the Stein-Tomas theorem.
The case $k = \frac{n-1}{2}$ is proved by stationary phase.

The in-between cases are proved by interpolation.

A priori bounds

What are the *a priori* bounds for a restriction of $\frac{\partial^k \hat{f}}{\partial \xi_n^k}$?

Keep in mind that $\frac{\partial^k \hat{f}}{\partial \xi_n^k} \in W^{-k,p'}(\mathbb{R}^n)$ is very bad.

Theorem

If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \frac{2n+2}{n+3+2k}$,
then $\left.\frac{\partial^k \hat{f}}{\partial \xi_n^k}\right|_{\Sigma}$ exists as an element of $H^{-k}(\Sigma)$.

The case $k = 0$ is the Stein-Tomas theorem.
The case $k = \frac{n-1}{2}$ is proved by stationary phase.

The in-between cases are proved by interpolation.

In this talk I'll concentrate on the $k = 1$ case.

An extreme case

If $\hat{f}\big|_\Sigma \equiv 0$, that makes its restriction as smooth as possible.
What can we say about $\frac{\partial \hat{f}}{\partial \xi_n}$ in this case?

An extreme case

If $\hat{f}\big|_\Sigma \equiv 0$, that makes its restriction as smooth as possible.
What can we say about $\frac{\partial \hat{f}}{\partial \xi_n}$ in this case?

Theorem (G-Stolyarov, '20)

If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \frac{2n+2}{n+7}$, and $\hat{f}\big|_\Sigma \equiv 0$, then $\left\| \frac{\partial \hat{f}}{\partial \xi_n} \right\|_{L^2(\Sigma)} \lesssim \|f\|_p$.

We need $n \geq 5$ here, so that $\frac{2n+2}{n+7} \geq 1$.

An extreme case

If $\hat{f}\big|_\Sigma \equiv 0$, that makes its restriction as smooth as possible.
What can we say about $\frac{\partial \hat{f}}{\partial \xi_n}$ in this case?

Theorem (G-Stolyarov, '20)

If $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \frac{2n+2}{n+7}$, and $\hat{f}\big|_\Sigma \equiv 0$, then $\left\| \frac{\partial \hat{f}}{\partial \xi_n} \right\|_{L^2(\Sigma)} \lesssim \|f\|_p$.

We need $n \geq 5$ here, so that $\frac{2n+2}{n+7} \geq 1$.

$\frac{\partial \hat{f}}{\partial \xi_n} \in L^2(\Sigma)$ is much better than the *a priori* bound in $H^{-1}(\Sigma)$.

The main idea

Why does $\hat{f}$ vanishing on Σ have so much influence on $\frac{\partial \hat{f}}{\partial \xi_n}$?

The main idea

Why does $\hat{f}$ vanishing on Σ have so much influence on $\frac{\partial \hat{f}}{\partial \xi_n}$?

Look at $F(r) = \|\hat{f}\|^2_{L^2(\Sigma + r\bar{e}_n)} \; = \; \iint_{R^{2n}} f(x)\overline{f}(y)\widehat{\Sigma}(x-y)e^{ir(x_n - y_n)}dxdy.$

The main idea

Why does $\hat{f}$ vanishing on Σ have so much influence on $\frac{\partial \hat{f}}{\partial \xi_n}$?

Look at $F(r) = \|\hat{f}\|^2_{L^2(\Sigma + r\bar{e}_n)} = \iint_{R^{2n}} f(x)\overline{f}(y)\widehat{\Sigma}(x-y)e^{ir(x_n - y_n)}dxdy$.

Basic properties: $F(r) \geq 0$, and $F(0) = 0$ by assupmtion.

Show that $F''(r)$ is bounded by $\|f\|_p^2$. Then $0 \leq F(\delta) \lesssim \delta^2\|f\|_p^2$.

The main idea

Why does $\hat{f}$ vanishing on Σ have so much influence on $\frac{\partial \hat{f}}{\partial \xi_n}$?

Look at $F(r) = \|\hat{f}\|^2_{L^2(\Sigma + r\bar{e}_n)} \; = \; \iint_{R^{2n}} f(x)\overline{f}(y)\widehat{\Sigma}(x-y)e^{ir(x_n - y_n)}dxdy.$

Basic properties: $F(r) \geq 0$, and $F(0) = 0$ by assupmtion.

Show that $F''(r)$ is bounded by $\|f\|_p^2$. Then $0 \leq F(\delta) \lesssim \delta^2 \|f\|_p^2$.

So $\hat{f}\big|_\Sigma \equiv 0$, and on a nearby surface, $\big\|\hat{f}\big\|_{L^2(\Sigma + r\bar{e}_n)} \lesssim \delta \|f\|_p$.

The main idea

Why does $\hat{f}$ vanishing on Σ have so much influence on $\frac{\partial \hat{f}}{\partial \xi_n}$?

Look at $F(r) = \|\hat{f}\|^2_{L^2(\Sigma + r\bar{e}_n)} \; = \; \iint_{R^{2n}} f(x)\bar{f}(y)\hat{\Sigma}(x-y)e^{ir(x_n - y_n)}dxdy.$

Basic properties: $F(r) \geq 0$, and $F(0) = 0$ by assupmtion.

Show that $F''(r)$ is bounded by $\|f\|_p^2$. Then $0 \leq F(\delta) \lesssim \delta^2 \|f\|_p^2$.

So $\hat{f}\big|_\Sigma \equiv 0$, and on a nearby surface, $\big\|\hat{f}\big\|_{L^2(\Sigma + r\bar{e}_n)} \lesssim \delta \|f\|_p$.

Summary: $\hat{f}\big|_{\Sigma + r\bar{e}_n}$ is a continuous $L^2(\Sigma)$-valued function of $r \in [-1, 1]$.
But it is Lipschitz-continuous at $r = 0$ if $\hat{f}$ vanishes on Σ.

Prior Work

Why the condition $\hat{f}\big|_{\Sigma} \equiv 0$ is interesting (to me):

Spectral theory of Schrödinger operators $H = -\Delta + V(x)$ on $\mathbb{R}^n$.

Prior Work

Why the condition $\hat{f}\big|_{\Sigma} \equiv 0$ is interesting (to me):

Spectral theory of Schrödinger operators $H = -\Delta + V(x)$ on $\mathbb{R}^n$.

Free resolvent $R_0^+(\lambda^2) = \lim_{\epsilon \to 0^+} (-\Delta - (\lambda + i\epsilon)^2)^{-1}$.

$R_0^+(\lambda^2)$ is a Fourier multiplier with "symbol" $\frac{1}{|\xi|^2 - \lambda^2} + \frac{\pi i}{2\lambda} d\sigma_{|\xi| = \lambda}$.

Prior Work

Why the condition $\hat{f}\big|_\Sigma \equiv 0$ is interesting (to me):

Spectral theory of Schrödinger operators $H = -\Delta + V(x)$ on $\mathbb{R}^n$.

Free resolvent $R_0^+(\lambda^2) = \lim_{\epsilon \to 0^+} (-\Delta - (\lambda + i\epsilon)^2)^{-1}$.

$R_0^+(\lambda^2)$ is a Fourier multiplier with "symbol" $\frac{1}{|\xi|^2 - \lambda^2} + \frac{\pi i}{2\lambda} d\sigma_{|\xi| = \lambda}$.

A resonance is a function ψ "close to L^2" satisfying $\psi = -R_0^+(\lambda^2) V \psi$.

It's a eigenfunction of H if $\psi \in L^2$.

We'd like every resonance to be an eigenfunction.

S. Agmon's Argument

Agmon: If V is real-valued, then each resonance has $\widehat{V\psi}\big|_{|\xi|=\lambda} \equiv 0$.

Then $R_0^+(\lambda^2)V\psi$ is better than the *a priori* estimates for the free resolvent, and it is better than the initial assumptions on ψ.

Bootstrap until $\psi \in L^2$.

S. Agmon's Argument

Agmon: If V is real-valued, then each resonance has $\widehat{V\psi}\big|_{|\xi|=\lambda} \equiv 0$.

Then $R_0^+(\lambda^2)V\psi$ is better than the *a priori* estimates for the free resolvent, and it is better than the initial assumptions on ψ.

Bootstrap until $\psi \in L^2$.

Agmon worked with ψ and $V\psi$ in weighted $L^2(\mathbb{R}^n)$, so that $\widehat{V\psi} \in H^s(\mathbb{R}^n)$.

Then he reduced the special bound on $R_0^+(\lambda^2)V\psi$ to the Hardy inequality.

Work with W. Schlag

G-Schlag ('04) worked with $V\psi \in L^p(\mathbb{R}^3)$.

Lemma

If $\phi \in L^1(\mathbb{R}^3)$ *and* $\hat{\phi}\big|_{|\xi|=\lambda} \equiv 0$, *then* $R_0(\lambda^2)\phi \in L^2(\mathbb{R}^3)$.

Work with W. Schlag

G-Schlag ('04) worked with $V\psi \in L^p(\mathbb{R}^3)$.

Lemma

If $\phi \in L^1(\mathbb{R}^3)$ and $\hat{\phi}\big|_{|\xi|=\lambda} \equiv 0$, then $R_0(\lambda^2)\phi \in L^2(\mathbb{R}^3)$.

There is an $\mathbb{R}^n$ version of this lemma, with $\phi \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \frac{2n+2}{n+5}$.
(G. '16)

An open problem

Intuition suggests that if $\hat{f}$ vanishes at the place where a Fourier multiplier is most singular, there should be a qualitative improvement in the mapping bounds.

An open problem

Intuition suggests that if $\hat{f}$ vanishes at the place where a Fourier multiplier is most singular, there should be a qualitative improvement in the mapping bounds.

Example:
The ball multiplier in R^2 typically maps $L^1(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ for any $q > \frac{4}{3}$.

An open problem

Intuition suggests that if $\hat{f}$ vanishes at the place where a Fourier multiplier is most singular, there should be a qualitative improvement in the mapping bounds.

Example:
The ball multiplier in R^2 typically maps $L^1(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ for any $q > \frac{4}{3}$.

If $\hat{f}$ vanishes on the unit circle, then $M_{ball} f \in L^q(\mathbb{R}^2)$ for any $q > \frac{5}{4}$.
(G. '16)

An open problem

Intuition suggests that if $\hat{f}$ vanishes at the place where a Fourier multiplier is most singular, there should be a qualitative improvement in the mapping bounds.

Example:
The ball multiplier in R^2 typically maps $L^1(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$ for any $q > \frac{4}{3}$.

If $\hat{f}$ vanishes on the unit circle, then $M_{ball} f \in L^q(\mathbb{R}^2)$ for any $q > \frac{5}{4}$.
(G. '16)

Conjecture: This is also true for any $q > 1$.

Proving Dirichlet-to-Neumann Bounds

Assume $\hat{f}\big|_\Sigma \in H^\ell(\Sigma)$.

Proving Dirichlet-to-Neumann Bounds

Assume $\hat{f}\big|_{\Sigma} \in H^{\ell}(\Sigma)$.

Idea: We can control $\frac{\partial \hat{f}}{\partial \xi_n}$ by interpolating between

- $k = 0$ assumption $\quad \hat{f}\big|_{\Sigma} \in H^{\ell}(\Sigma)$.
- $k = \kappa_p = \frac{n+1}{p} - \frac{n+3}{2}$ *a priori* bound $\quad \frac{\partial^k \hat{f}}{\partial \xi_n^k}\Big|_{\Sigma} \in H^{-k}(\Sigma)$.

Note: $k = \kappa_p$ is the same as $p = \frac{2n+2}{n+3+2k}$. It's the maximum number of derivatives allowed in our *a priori* bound.

Proving Dirichlet-to-Neumann Bounds

Assume $\hat{f}\big|_{\Sigma} \in H^{\ell}(\Sigma)$.

Idea: We can control $\frac{\partial \hat{f}}{\partial \xi_n}$ by interpolating between

- $k = 0$ assumption $\quad \hat{f}\big|_{\Sigma} \in H^{\ell}(\Sigma)$.
- $k = \kappa_p = \frac{n+1}{p} - \frac{n+3}{2}$ *a priori* bound $\quad \frac{\partial^k \hat{f}}{\partial \xi_n^k}\Big|_{\Sigma} \in H^{-k}(\Sigma)$.

Note: $k = \kappa_p$ is the same as $p = \frac{2n+2}{n+3+2k}$. It's the maximum number of derivatives allowed in our *a priori* bound.

Then we expect to see $\frac{\partial \hat{f}}{\partial \xi_n} \in L^2(\Sigma)$ if $\ell = \frac{\kappa_p}{\kappa_p - 1}$.

Proving Dirichlet-to-Neumann Bounds

The idea hints at the right answer despite being badly flawed.

Proving Dirichlet-to-Neumann Bounds

The idea hints at the right answer despite being badly flawed.

- The assumption $\hat{f}\big|_{\Sigma} \in H^{\ell}(\Sigma)$ is much too fragile for Complex interpolation.

Proving Dirichlet-to-Neumann Bounds

The idea hints at the right answer despite being badly flawed.

- The assumption $\hat{f}\big|_{\Sigma} \in H^{\ell}(\Sigma)$ is much too fragile for Complex interpolation.
- Standard counterexamples (radial, translated, Knapp) introduce other constraints on ℓ, k, and p.

Method 1 (G-Stolyarov '20)

Instead of interpolation, we use the Leibniz rule.

$$\frac{\partial^2}{\partial r^2}\langle \hat{f}, \hat{g}\rangle_{L^2(\Sigma + r\bar{e}_n)} = \langle \hat{f}, \frac{\partial^2 \hat{g}}{\partial \xi_n^2}\rangle + \langle \frac{\partial^2 \hat{f}}{\partial \xi_n^2}, \hat{g}\rangle + 2\langle \frac{\partial \hat{f}}{\partial \xi_n}, \frac{\partial \hat{g}}{\partial \xi_n}\rangle.$$

Method 1 (G-Stolyarov '20)

Instead of interpolation, we use the Leibniz rule.

$$\frac{\partial^2}{\partial r^2}\langle \hat{f}, \hat{g}\rangle_{L^2(\Sigma + r\bar{e}_n)} = \langle \hat{f}, \frac{\partial^2 \hat{g}}{\partial \xi_n^2}\rangle + \langle \frac{\partial^2 \hat{f}}{\partial \xi_n^2}, \hat{g}\rangle + 2\langle \frac{\partial \hat{f}}{\partial \xi_n}, \frac{\partial \hat{g}}{\partial \xi_n}\rangle.$$

The left-hand side can be controlled by $\|f\|_p \|g\|_p$, similar to the $F''(r)$ claim earlier.

Method 1 (G-Stolyarov '20)

Instead of interpolation, we use the Leibniz rule.

$$\frac{\partial^2}{\partial r^2}\langle \hat{f}, \hat{g}\rangle_{L^2(\Sigma + r\bar{e}_n)} = \langle \hat{f}, \frac{\partial^2 \hat{g}}{\partial \xi_n^2}\rangle + \langle \frac{\partial^2 \hat{f}}{\partial \xi_n^2}, \hat{g}\rangle + 2\langle \frac{\partial \hat{f}}{\partial \xi_n}, \frac{\partial \hat{g}}{\partial \xi_n}\rangle.$$

The left-hand side can be controlled by $\|f\|_p \|g\|_p$, similar to the $F''(r)$ claim earlier.

The first term on the right side is controlled by $\|\hat{f}\|_{H^\ell(\Sigma)} \|\frac{\partial^2 \hat{g}}{\partial \xi_n^2}\|_{H^{-\ell}(\Sigma)}$. The second term is similar.

Method 1 (G-Stolyarov '20)

Instead of interpolation, we use the Leibniz rule.

$$\frac{\partial^2}{\partial r^2}\langle \hat{f}, \hat{g}\rangle_{L^2(\Sigma + r\bar{e}_n)} = \langle \hat{f}, \frac{\partial^2 \hat{g}}{\partial \xi_n^2}\rangle + \langle \frac{\partial^2 \hat{f}}{\partial \xi_n^2}, \hat{g}\rangle + 2\langle \frac{\partial \hat{f}}{\partial \xi_n}, \frac{\partial \hat{g}}{\partial \xi_n}\rangle.$$

The left-hand side can be controlled by $\|f\|_p\|g\|_p$, similar to the $F''(r)$ claim earlier.

The first term on the right side is controlled by $\|\hat{f}\|_{H^\ell(\Sigma)}\|\frac{\partial^2 \hat{g}}{\partial \xi_n^2}\|_{H^{-\ell}(\Sigma)}$. The second term is similar.

The last term on the right is the one we want to control.

Method 1 (G-Stolyarov '20)

The end result is a bound

$$\left\|\frac{\partial\hat{f}}{\partial\xi_n}\right\|^2_{L^2(\Sigma)} \lesssim \|f\|^2_p + \|\hat{f}\|_{H^\ell(\Sigma)} + \left\|\frac{\partial^2\hat{f}}{\partial\xi_n^2}\right\|_{H^{-\ell}(\Sigma)}$$

Method 1 (G-Stolyarov '20)

The end result is a bound

$$\left\|\frac{\partial \hat{f}}{\partial \xi_n}\right\|^2_{L^2(\Sigma)} \lesssim \|f\|_p^2 + \|\hat{f}\|_{H^\ell(\Sigma)} + \left\|\frac{\partial^2 \hat{f}}{\partial \xi_n^2}\right\|_{H^{-\ell}(\Sigma)}$$

If you use the *a priori* estimate for $k = 2$, the conclusion is that $\ell = 2$ is sufficient.

Iterating the Leibniz rule gets us to a bound in terms of $\left\|\frac{\partial^k \hat{f}}{\partial \xi_n^k}\right\|_{H^{-(k-1)\ell}(\Sigma)}$. If κ_p is an integer, we can obtain the optimal $\ell = \frac{\kappa_p}{\kappa_p - 1}$ this way.

Method 2 (work in progress)

Let's consider the difference quotients $F(r) = \frac{1}{r}\big[\hat{f}\big|_{\Sigma + r\bar{e}_n} - \hat{f}\big|_{\Sigma}\big]$.

For $r \neq 0$ this is a continuous $L^2(\Sigma)$-valued function of r.
We would like to show it is continuous at $r = 0$ as well.

Method 2 (work in progress)

Let's consider the difference quotients $F(r) = \frac{1}{r}\big[\hat{f}\big|_{\Sigma + r\bar{e}_n} - \hat{f}\big|_{\Sigma}\big]$.

For $r \neq 0$ this is a continuous $L^2(\Sigma)$-valued function of r.
We would like to show it is continuous at $r = 0$ as well.

Keeping the Riemann-Lebesgue Lemma in mind, it would suffice to take the Fourier transform in r and then show that

$$\int_{-\infty}^{\infty} \|\hat{F}(\rho)\|_{L^2(\Sigma)}\, d\rho \lesssim \|f\|_p + \|\hat{f}\|_{H^\ell(\Sigma)}.$$

Method 2 (work in progress)

Let's consider the difference quotients $F(r) = \frac{1}{r}\big[\hat{f}\big|_{\Sigma + r\bar{e}_n} - \hat{f}\big|_{\Sigma}\big]$.

For $r \neq 0$ this is a continuous $L^2(\Sigma)$-valued function of r.
We would like to show it is continuous at $r = 0$ as well.

Keeping the Riemann-Lebesgue Lemma in mind, it would suffice to take the Fourier transform in r and then show that

$$\int_{-\infty}^{\infty} \|\hat{F}(\rho)\|_{L^2(\Sigma)}\, d\rho \lesssim \|f\|_p + \|\hat{f}\|_{H^\ell(\Sigma)}.$$

In fact, that condition would suffice to show that the partial derivative $\frac{\partial \hat{f}}{\partial \xi_n}$ exists pointwise almost everywhere on Σ. Which is false for $f \in L^p$, $p > 1$, even if $\hat{f}$ vanishes on Σ.

Method 2 (work in progress)

Let's consider the difference quotients $F(r) = \frac{1}{r}\big[\hat{f}\big|_{\Sigma + r\bar{e}_n} - \hat{f}\big|_{\Sigma}\big]$.

For $r \neq 0$ this is a continuous $L^2(\Sigma)$-valued function of r.
We would like to show it is continuous at $r = 0$ as well.

Keeping the Riemann-Lebesgue Lemma in mind, it would suffice to take the Fourier transform in r and then show that

$$\int_{-\infty}^{\infty} \|\hat{F}(\rho)\|_{L^2(\Sigma)}\, d\rho \lesssim \|f\|_p + \|\hat{f}\|_{H^\ell(\Sigma)}.$$

In fact, that condition would suffice to show that the partial derivative $\frac{\partial \hat{f}}{\partial \xi_n}$ exists pointwise almost everywhere on Σ. Which is false for $f \in L^p$, $p > 1$, even if $\hat{f}$ vanishes on Σ.

So we need to use a weaker notion of integrability for $\hat{F}(\rho)$ which still implies that $F(r)$ is continuous.

Questions for Further Study

Questions for Further Study

- That Ball Multiplier Conjecture.

Questions for Further Study

- That Ball Multiplier Conjecture.
- Differentiability of $\hat{f}$ pointwise a.e. on Σ.

Questions for Further Study

- That Ball Multiplier Conjecture.
- Differentiability of $\hat{f}$ pointwise a.e. on Σ.
- Bounds on $\frac{\partial \hat{f}}{\partial \xi_n}$ in a non-Hilbert space.
  i.e. not using T^*T methods.

Questions for Further Study

- That Ball Multiplier Conjecture.
- Differentiability of $\hat{f}$ pointwise a.e. on Σ.
- Bounds on $\frac{\partial \hat{f}}{\partial \xi_n}$ in a non-Hilbert space.
  i.e. not using T^*T methods.
- what happens when $n = 4$? Or $n = 3$?
  (If the derivative of $\hat{f}$ exists, it will be in L^q for some $q < 2$.)

References

S. Agmon, *Spectral properties of Schrödinger operators and scattering theory.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **2** (1975), no. 2, 151–218.

M. Goldberg and W. Schlag, *A Limiting absorption principle for the three-dimensional Schrödinger equation with L^p potentials.* Intl. Math. Res. Not. **2004:75** (2004), 4049–4071.

M. Goldberg, *The Helmholtz equation with L^p data and Bochner-Riesz multipliers.* Math. Res. Lett. **23** (2016), no. 6, 1665–1679.

M. Goldberg and D. Stolyarov, *Restrictions of higher derivatives of the Fourier transform.* Trans. Amer. Math. Soc. Ser. B **7** (2020), 46–96.