Dispersive bounds for Polyharmonic Schrödinger Operators

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Preliminaries

Support from Simons Foundation grant #635369.

Collaborators: Burak Erdogan (Illinois) and William Green (Rose-Hulman)

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Schrödinger equation $\begin{cases} iu_t(x,t) = ((-\Delta)^m + V)u(x,t), \; x \in \mathbb{R}^n, \; m \in \mathbf{N} \\ u(x,0) = u_0(x) \end{cases}$

Want to prove same $L^1 \to L^\infty$ bounds as the $V \equiv 0$ case.

Free Dispersive Estimates

What happens when $V \equiv 0$:

$e^{-it(-\Delta)^m}$ is convolution with kernel $\mathcal{F}\big(e^{-it|\xi|^{2m}}\big) = t^{-\frac{n}{2m}} K\Big(\frac{|x|}{t^{1/2m}}\Big)$.

$K(|x|)$ is a bounded function,
and $K(|x|) \;\sim\; \frac{1}{|x|^{\frac{n(m-1)}{2m-1}}} e^{-i|x|^{(1+\frac{1}{2m-1})}}$ for large values of $|x|$.

This allows derivatives of K up to order $n(m-1)$ to be bounded as well.

Perturbed Dispersive Estimates

We'd like to find a condition on V so that the propagator e^{-itH} satisfies

$$\left\|e^{-itH}P_{ac}(H)u_0\right\|_\infty \lesssim |t|^{-\frac{n}{2m}}\|u_0\|_1$$

$$\text{and} \quad \left\||H|^{\frac{n(m-1)}{2m}}e^{-itH}P_{ac}(H)u_0\right\|_\infty \lesssim |t|^{-\frac{n}{2}}\|u_0\|_1.$$

The projection $P_{ac}(H)$ is needed to avoid steady-state solutions of the form $u(x,t) = e^{-it\lambda_0}\psi$.

We'll choose a condition on V so that H has no singular continuous spectrum.

Global Kato norms

Define $\|V\|_{K^\alpha} := \sup_{z\in\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|V(y)|}{|z-y|^\alpha}\,dy, \quad 0 \le \alpha < n.$

K^0 coincides with $L^1(\mathbb{R}^n)$.

Multiplication by $V \in K^{n-2m}$ has the same dilation scaling as $(-\Delta)^m$.

Let K_0 denote the K^{n-2m} closure of $C_c(\mathbb{R}^n)$ functions.
Then $H = (-\Delta)^m + V$ is a relatively compact perturbation of $(-\Delta)^m$.

[as long as $n > 2m$...]

Main Results

Theorem (Erdogan-G-Green)

In dimensions $2m < n < 4m$, both perturbed dispersive bounds are true provided $V \in K_0$ and $H = (-\Delta)^m + V$ has no threshold resonances or embedded eiganvalues in $[0, \infty)$.

The bounds are:

$$\left\| e^{-itH} P_{ac}(H) u_0 \right\|_\infty \lesssim |t|^{-\frac{n}{2m}} \|u_0\|_1$$

$$\text{and} \quad \left\| |H|^{\frac{n(m-1)}{2m}} e^{-itH} P_{ac}(H) u_0 \right\|_\infty \lesssim |t|^{-\frac{n}{2}} \|u_0\|_1.$$

Setup, Part 1

Stationary method: Use the functional calculus of H,

$$e^{-itH} P_{ac}(H) = \frac{1}{2\pi i} \int_0^\infty e^{-it\lambda} \big(R_V^+(\lambda) - R_V^-(\lambda)\big)\, d\lambda.$$

Resolvents: $R_V^\pm(\lambda) := \lim_{\epsilon \to 0^+} \big((-\Delta)^m + V - (\lambda \pm i\epsilon)\big)^{-1}.$

$$R_0^\pm(\lambda) := \lim_{\epsilon \to 0^+} \big((-\Delta)^m - (\lambda \pm i\epsilon)\big)^{-1}$$

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Useful identity:

$$R_V^+(\lambda) - R_V^-(\lambda) = (I + R_0^-(\lambda)V)^{-1}\big[R_0^+(\lambda) - R_0^-(\lambda)\big](I + VR_0^+(\lambda))^{-1}.$$

Setup, Part 2

After a change of variables $\lambda \mapsto \lambda^{2m}$, we have

$$e^{-itH}P_{ac}(H) = C\int_0^\infty e^{-it\lambda^{2m}}\lambda^{2m-1}J^*(\lambda)\big[R_0^+(\lambda^{2m}) - R_0^-(\lambda^{2m})\big]J(\lambda)\,d\lambda.$$

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$R_0^\pm(\lambda^{2m})$ = Fourier multiplication by $\displaystyle \frac{1}{|\xi|^{2m} - \lambda^{2m}} \pm \frac{\pi i}{m\lambda^{2m-1}} d\sigma\Big|_{|\xi|=\lambda}$.

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$$J(\lambda) = \big[I + VR_0^+(\lambda^{2m})\big]^{-1} \text{ for } \lambda > 0.$$

Setup, Part 3

We're trying to establish time decay of

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Use Plancherel to gain the last fractional power of t.

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Use Plancherel to gain the last fractional power of t.

We'll need $\widehat{J}(\rho)$ to be integrable in a suitable sense.
Recall that $J(\lambda) = \big[I + VR_0^+(\lambda^{2m})\big]^{-1}$.

L^1 Inversion Lemmas

Wiener's Lemma:

If $\hat{f} \in L^1(\mathbb{R})$ and $1 + f(\lambda) \neq 0$ for all $\lambda \in \mathbb{R}$,
then $\mathcal{F}\big(\frac{1}{1+f(\lambda)}\big) - \delta_{\rho=0} \in L^1(\mathbb{R})$.

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Operator-Valued version:

Theorem (Beceanu)

If $K(x,y) := \int_{\mathbb{R}} \big|\widehat{T}(\rho, x, y)\big|\, d\rho$ *is the kernel of an operator in* $\mathcal{B}(X)$*,*
and $I + T(\lambda)$ *is invertible in* $\mathcal{B}(X)$ *for all* $\lambda \in \mathbb{R}$*,*
then $\Big\|\mathcal{F}\big[(I + T(\lambda))^{-1}\big] - \delta_{\rho=0} I\Big\|_{L^1(\rho)}$ *is also an operator in* $\mathcal{B}(X)$*.*
...provided T *is the norm-limit of* $C_c(\mathbb{R})$ *operator-valued functions.*

Notes on Assembling The Pieces

We'd like to apply the inversion theorem to $I + VR_0^+(\lambda^{2m})$.
Mostly in $\mathcal{B}(L^1(\mathbb{R}^n))$. But perhaps in $\mathcal{B}(K^\alpha)$ as needed.

The kernel of $R_0^+(\lambda^{2m})$ has the form $|x-y|^{n-2m} f(\lambda|x-y|)$.

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More specifically, the kernel of $R_0^+(\lambda^{2m})$ looks like $\dfrac{e^{i\lambda|x-y|}}{\lambda^{\frac{4m-(n+1)}{2}}|x-y|^{\frac{n-1}{2}}}$.

Can't IBP more than $\frac{n-1}{2}$ times.
Be careful where you distribute powers of λ.

More Notes on Assembling the Pieces

There's a small problem that if $m > 1$, then the analytic extension of $R_0^+(\lambda^{2m})$ grows exponentially on the negative half-line $\lambda < 0$.

We should pick a different extension. The integral we care about is supported on $[0, \infty)$.

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Kato norms K^α, $0 \leq \alpha \leq n - 2m$, don't seem to interpolate well. It would simplify some arguments if they did.

Even More Notes on Assembling the Pieces

How to verify that $I + VR_0^+(\lambda^{2m})$ is invertible in $\mathcal{B}(L^1)$ for all $\lambda_0 > 0$.

It's a compact perturbation of the identity. If it's not invertible, there must be a function $\phi \in L^1(\mathbb{R}^n)$ in its null space. Then $\psi = R_0^+(\lambda_0^{2m})\phi$ is a (generalized) eigenfunction of $H = (-\Delta)^m + V$.

By an argument due to Agmon, $\widehat{\phi}(\xi) = 0$ for all $|\xi| = \lambda_0$.

That is sufficient to imply $\psi \in L^2(\mathbb{R}^n)$. [G, '16]
And we assumed H had no embedded eigenvalues.

About the Range of Dimensions

What happens when $n \leq 2m$:

Free resolvent $R_0^+(\lambda^{2m})$ behaves badly near $\lambda = 0$.
This probably can be handled in some way. $m = 1$, $n = 1$ case done by (Hill '20).

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What happens when $m > 4m$:

$R_0^+(\lambda^{2m})$ grows like $\lambda^{\frac{(n+1)-4m}{2}}$ as $\lambda \to \infty$.

One can generate counterexamples to $L^1 \to L^\infty$ PDE bounds unless $V(x)$ is more regular in some way.