A Schrödinger Dispersive Estimate in $\mathbf{R}^3$ With Singular Potentials

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Start with Wiener's Theorem:

If $f \in C(\mathbb{T})$ *has* $\widehat{f} \in \ell^1(\mathbf{Z})$ *and* $f(x) \neq 0$ $\forall x$*, then the Fourier coefficients of* $\frac{1}{f(x)}$ *also belong to* $\ell^1(\mathbf{Z})$*.*

There is also a version for $f \in C_0(\mathbf{R})$
with $\widehat{f} \in L^1(\mathbf{R})$.

Two methods of proof:

Fourier Analysis (Wiener)

Commutative Banach Algebras/Maximal Ideals (Gelfand)

Our version involves operator-valued functions $f \in C_0(\mathbf{R}; B(X))$.

Theorem 1 *If $\widehat{f}$ satisfies the "L^1-Strong" bound*

$$\int_{\mathbf{R}} \|\widehat{f}(\rho)\eta\|_X \, d\rho \lesssim C\|\eta\|_X \tag{2}$$

and $I + f(x)$ is invertible for each $x \in \mathbf{R}$,

[plus technical conditions on $\widehat{f}(\rho)$]

then $[I + f(x)]^{-1} \in C(\mathbf{R}; B(X))$ also satisfies (2).

Note: Inequality (2) says that $\widehat{f} : X \to L^1(\mathbf{R}; X)$. These maps embed isometrically in $B(L^1(\mathbf{R} : X))$.

Linear Schrödinger Equation in $\mathbf{R}^3$

$$\begin{cases} iu_t + Hu = 0, & H = -\Delta + V(x) \\ \quad u(0,x) = f(x) & \end{cases}$$

The search for Dispersive Estimates:

Is $\|e^{-itH} f\|_\infty \lesssim |t|^{-\frac{3}{2}} \|f\|_1$ for all (most?) $f \in L^1$?

It is true for $V \equiv 0$ by Fourier inversion.

Scaling considerations: $V_r(x) = r^{-2} V(rx)$ has same dynamical properties as V.

Spectral problem: If there is an eigenvector $(-\Delta + V)\Psi = \lambda\Psi$, then the solution $u(t,x) = e^{-i\lambda t}\Psi(x)$ does not decay as $t \to \infty$.

We will measure V with the global Kato norm

$$\|V\|_{\mathcal{K}} := \sup_{y} \int_{\mathbf{R}^3} \frac{|V(x)|}{|x-y|}\, dx$$

Theorem 3 (Rodnianski-Schlag, '04)
If $\|V\|_{\mathcal{K}} < 4\pi$, *then* $\|e^{-itH} f\|_\infty \lesssim \|f\|_1$

There are no eigenvalues, and constant is explicit.

Theorem 4 (Beceanu-G)
If V *belongs to* $\mathcal{K}$*-closure of* $C_c^b(\mathbf{R}^3)$
and H *has no eigenvalues or resonances in* $[0, \infty)$,

Then $\|e^{-itH} P_{ac}(H) f\|_\infty \lesssim \|f\|_1$ *for all* $f \in L^1$.

Idea of Proof: e^{-itH} is a spectral multiplier

$$e^{-itH}P_{ac}(H)f = C\int_0^\infty e^{-it\lambda}(R^+(\lambda) - R^-(\lambda))f\,d\lambda$$

where $\quad R^{\pm}(\lambda) := \lim_{\varepsilon\to 0}(H - (\lambda \pm i\varepsilon))^{-1}$

This is related to the case $V \equiv 0$ by the identity

$$R^{\pm}(\lambda) = R_0^{\pm}(\lambda)[I + VR_0^{\pm}(\lambda)]^{-1}$$

The factor $R_0^{\pm}(\lambda)$ is good. It describes $V \equiv 0$.
We need L^1 estimate on Fourier transform
of $[I + VR_0^{\pm}(\lambda)]^{-1}$ to complete the calculation.

Notice that $\|V\|_{\mathcal{K}}$ is well defined if V is a measure

$$\|V\|_{\mathcal{K}} = \sup_y \int_{\mathbf{R}^3} \frac{1}{|x-y|}\, dV(x)$$

Theorem 5 (Beceanu-G, in progress)
The dispersive estimate also holds if $V(x)$ is supported on a surface $\Sigma \subset \mathbf{R}^3$

Open Question 1: Is the "local Kato condition"

$$\lim_{R\to 0}\left[\sup_y \int_{|x-y|<R} \frac{1}{|x-y|}\, dV(x)\right] = 0$$

sufficient for dispersive estimates?

Open Question 2: The Wiener Theorem applies so long as V satisfies a "distal Kato condition"

$$\lim_{R\to\infty}\left[\sup_y \int_{|x-y|>R} \frac{1}{|x-y|}\,dV(x)\right] = 0.$$

Then V is not necessarily compact relative to $-\Delta$.

What is the correct spectral condition for such H?

Open Question 3: Are the technical conditions an essential part of Theorem 1?

Or is it sufficient for $f \in C_0(\mathbf{R}; B(X))$ to be strongly continuous, with $[I + f(x)]^{-1}$ uniformly bounded?

Are there other interesting noncommutative L^1-inversion theorems?

Open Question 4: Are we done yet?