The Schrödinger and Floquet Equations with $L^{n/2}$ Potentials

Michael Goldberg

AMS Western Regional Meeting,
Albuquerque, New Mexico,
October 13, 2007.

Partial support provided by NSF grant DMS-0600925

Linear Schrödinger Equation in $\mathbf{R}^n$, $n \geq 3$.

$$\begin{cases} iu_t + (-\Delta + V(t,x))u = 0 \\ \qquad\qquad u(0,\,\cdot\,) = f, \quad f \in L^2(\mathbf{R}^n) \end{cases}$$

Structural assumptions on V:

- Periodic in time — $V(t + 2\pi, x) = V(t,x)$.
- Short-range in space — $V \in L_x^{n/2} L_t^\infty$.
- Complex-valued — No L^2 conservation law!

The search for Strichartz Estimates:

Is $\|u\|_{L_t^2 L_x^q} \lesssim \|f\|_{L^2}$ for most $f \in L^2(\mathbf{R}^n)$?

$$\left[q = \tfrac{2n}{n-2}\right]$$

The main obstacles are solutions with

$$\phi(t+2\pi,x) = e^{2\pi i\lambda t}\phi(t,x).$$

These fall into two main classes:

Eigenvalues — where $e^{-i\lambda t}\phi \in L^2(\mathbb{T}\times\mathbf{R}^n)$.
These can be quasiperiodic if λ is real,
or grow/decay exponentially if λ is complex.

Resonances — where $e^{-i\lambda t}\phi \in L^q(\mathbf{R}^n; L^2(\mathbb{T}))$.
These can only occur for $\lambda \in \mathbf{R}$.

Eigenfunctions have well-defined initial data

$$\Phi(x) = \phi(0, x) \in L^2(\mathbf{R}^n).$$

At each eigenvalue λ, the collection of such Φ forms a subspace $X_\lambda \subset L^2(\mathbf{R}^n)$.

Eigenvalues/Resonances at $\bar{\lambda}$ of the adjoint operator (with $\overline{V(t,x)}$) are also a concern.

Their initial data forms a subspace $\tilde{X}_\lambda \subset L^2(\mathbf{R}^n)$.

Theorem 1 *Suppose that neither operator has any resonances, and at each eigenvalue* λ *the bound solutions* $\phi, \tilde{\phi}$ *satisfy*

$$\begin{cases} \langle x \rangle \phi \in L^2(\mathbb{T} \times \mathbf{R}^n) & or\ \phi \in L^{q'}(\mathbf{R}^n; L^2(\mathbb{T})) \\ \langle x \rangle \Phi \in L^2(\mathbf{R}^n) & or\ \Phi \in W^{1,q'}(\mathbf{R}^n) \end{cases}$$

Finally suppose that L^2*-orthogonal projection from* X_λ *to* $\tilde{X}_\lambda$ *is bijective.*

Then there are finitely many eigenvalues, counted with multiplicity, and

$$\|u\|_{L_t^2 L_x^q} + \|u\|_{C(\mathbf{R}; L^2(\mathbf{R}^n))} \lesssim \|f\|_2$$

for all $f \in L^2(\mathbf{R}^n)$ *orthogonal to* $\oplus_\lambda \tilde{X}_\lambda$.

Corollaries and Special cases:

- If $V = V(x)$ is time-independent, it suffices to check $\lambda \in [0, \infty)$.

- If $V = V(x)$ is real valued, it suffices to check the eigenvalues/resonances at $\lambda = 0$.

- If V is real-valued, and $|V(t,x)| \leq C\langle x\rangle^{-2-\epsilon}$, it suffices to check $\lambda \in \mathbf{Z}$, and only in dimensions $n \leq 6$.

Let U^+ represent the forward free propagator

$$U^+ f(t,x) = e^{-it\Delta} f(x),\ \ t \geq 0$$
$$U^+ g(t,x) = \int_{-\infty}^{t} e^{-i(t-s)\Delta} g(s,x)\, ds$$

which satisfies the mapping estimates

$$U^+ : L^2(\mathbf{R}^n) \to L_t^2 L_x^q,$$
$$L_t^2 L_x^{q'} \to L_t^2 L_x^q$$

Define $w(x) = \Big(\|V(\,\cdot\,,x)\|_\infty\Big)^{1/2} \in L^n(\mathbf{R}^n)$.

This gives a factorization $V = w^2(x) z(t,z)$ with z bounded and time-periodic.

Using Duhamel's method, a formal solution is

$$u = U^{+}f + i \underbrace{U^{+}wz}_{\substack{\text{maps to} \\ L_t^2 L_x^q}} (I - iwU^{+}wz)^{-1} \underbrace{wU^{+}f}_{\in L^2_{t,x}}$$

Problem: The operator $(I - iwU^{+}wz)^{-1}$ is unbounded on $L^2_{t,x}$ precisely because there are eigenvalues and resonances.

Can we show that $wU^{+}f$ belongs to its domain?

Computations for time-independent V:

Take Fourier transform in time variable.

$$\widehat{g}(\tau, x) = \int_{\mathbf{R}} e^{-i\tau t} g(t, x)\, dt$$

Each cross-section $L^2(\mathbf{R}^n)$ with fixed τ is an invariant space. Furthermore,

$$(wU^{+}wzg)\widehat{\ }(\tau, x) = i(wR^{-}(\tau)wz)\widehat{g}(\tau, x)$$

is a continuously varying (in τ) family of compact operators, whose norm decreases as $\tau \to \infty$. Here, we adopt the resolvent notation

$$R^{-}(\tau) = \lim_{\epsilon \downarrow 0}(-\Delta - (\tau - i\epsilon))^{-1}$$

Fredholm Alternative $\Rightarrow$ $(I + wR^{-}(\tau)wz)^{-1}$ exists unless τ is an eigenvalue or resonance.

If λ is a "good" eigenvalue, then we have a local estimate

$$\|(I+wR^-(\tau)wz)^{-1}\psi\|_2 \lesssim \begin{cases} |\tau-\lambda|^{-1}\|\psi\|_2, & \psi \in \bar{z}\bar{w}\tilde{X}_\lambda \\ \|\psi\|_2, & \psi \perp \bar{z}\bar{w}\tilde{X}_\lambda \end{cases}$$

even though the family of operators $(wR^-(\tau)wz)$ is not differentiable in τ.

Thus for each $\tilde{\Phi} \in \tilde{X}_\lambda$ we need to know whether

$$\begin{aligned} &\left\|\left(1+\tfrac{1}{|\tau-\lambda|}\right)\langle (wU^+f)\hat{}\,(\tau), \bar{z}\bar{w}\tilde{\Phi}\rangle\right\|_{L^2_\tau} \\ &= \left\|\left(1+\tfrac{1}{|\tau-\lambda|}\right)\langle (U^+f)\hat{}\,(\tau), \overline{V}\tilde{\Phi}\rangle\right\|_{L^2_\tau} \end{aligned}$$

is controlled by $\|f\|_2$.

Kato Smoothing estimates:

Since 1 is bounded, it causes no difficulties.

$$\begin{aligned}\left\|1 \cdot \langle (U^{+} f)\hat{}(\tau), \overline{V}\tilde{\Phi}\rangle\right\|_{L^2_\tau} &= \left\|\langle e^{-it\Delta} f, \overline{V}\tilde{\Phi}\rangle\right\|_{L^2(\mathbf{R}^+)} \\ &\lesssim \|f\|_2 \|\overline{V}\tilde{\Phi}\|_{q'} \\ &\lesssim \|f\|_2 \|\tilde{\Phi}\|_q\end{aligned}$$

The singularity $|\tau - \lambda|^{-1}$ creates the main terms.

$$\begin{aligned}&\left\|\tfrac{1}{|\tau-\lambda|}\langle (U^{+} f)\hat{}(\tau), \overline{V}\tilde{\Phi}\rangle\right\|_{L^2_\tau} \\ &\qquad = \left\|\langle R^{-}(\lambda)(e^{-it(\Delta+\lambda)} - 1) f, \overline{V}\tilde{\Phi}\rangle\right\|_{L^2(\mathbf{R}^+)} \\ &\qquad = \left\|\langle (e^{-it(\Delta+\lambda)} - 1) f, \tilde{\Phi}\rangle\right\|_{L^2(\mathbf{R}^+)}\end{aligned}$$

To make this finite requires $\langle f, \tilde{\Phi}\rangle = 0$ and also a Kato smoothing bound for $\langle e^{-it\Delta} f, \tilde{\Phi}\rangle$.

Sufficient conditions include $\tilde{\Phi} \in L^{q'}$ or $\langle x\rangle\tilde{\Phi} \in L^2$.

Time support issues: since Duhamel's formula should yield a solution $u(t,x)$ supported in the time interval $t \in [0,\infty)$, we need to verify that

$$e^{\mu t}(I + wU^{+}wz)^{-1}wU^{+}f$$

still belongs to $L^2_{t,x}$ for any $\mu \leq 0$.

Response: Repeat the computations over the translated domain $\tau \in i\mu + \mathbf{R}$ and beware of complex eigenvalues.

What changes if V is periodic in time?

Fundamental domain for eigenfunctions becomes $[0, 2\pi] \times \mathbf{R}^n$, often considered as $\mathbb{T} \times \mathbf{R}^n$.

Fourier transform $\widehat{z}(\tau, x)$ is supported in $\tau \in \mathbf{Z}$.

Invariant subspaces of $wU^{+}wz$ are found by restricting τ to an equivalence class $[\tau] \in \mathbf{R}/\mathbf{Z}$.

Plancherel's identity is taken over $\tau \in [0, 1] \sim \mathbf{R}/\mathbf{Z}$.

More changes for periodic V:

As an operator on $e^{i\tau t}L^2(\mathbb{T} \times \mathbf{R}^n)$, the norm of $(I - iwU^+ wz)^{-1}\psi$ is controlled by

$$\begin{cases} \Big(1+|\cot(\tau-\lambda)|\Big)\|\psi\|, & \text{for eigenvectors } \psi \\ \|\psi\|, & \text{otherwise} \end{cases}$$

Since this function is periodic in τ, the main "Kato smoothing" estimate becomes discrete in t.

$$\sum_{k\in\mathbf{Z}} \left|\langle e^{-2\pi i k\Delta} f, \tilde{\Phi}\rangle\right|^2 \lesssim \|f\|_2^2 \|\tilde{\Phi}\|^2$$

Such a bound is true for $\tilde{\Phi} \in W^{1,q'}(\mathbf{R}^n)$ or else $\langle x\rangle\tilde{\Phi} \in L^2(\mathbf{R}^n)$, among other spaces.

Proof uses Fourier restriction properties of $\tilde{\Phi}$.

Applications and possible extensions:

- Orbital stability for ground state (or excited states) of NLS.

- Similar problems for semi-linear wave equation.

- Time-periodic magnetic potentials.

- Schrödinger/wave equation on other manifolds?

- Generalization to all $V \in L_x^{n/2} L_t^{\infty}$?

- ______(your suggestion here)______