ON THE L^P BOUNDEDNESS OF WAVE OPERATORS FOR FOUR-DIMENSIONAL SCHRÖDINGER OPERATORS WITH A THRESHOLD EIGENVALUE

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ABSTRACT. Let $H = -\Delta + V$ be a Schrödinger operator on $L^2(\mathbb{R}^4)$ with real-valued potential V, and let $H_0 = -\Delta$. If V has sufficient pointwise decay, the wave operators $W_{\pm} = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}$ are known to be bounded on $L^p(\mathbb{R}^4)$ for all $1 \leq p \leq \infty$ if zero is not an eigenvalue or resonance, and on $\frac{4}{3} < p < 4$ if zero is an eigenvalue but not a resonance. We show that in the latter case, the wave operators are also bounded on $L^p(\mathbb{R}^4)$ for $1 \leq p \leq \frac{4}{3}$ by direct examination of the integral kernel of the leading terms. Furthermore, if $\int_{\mathbb{R}^4} xV(x)\psi(x)\,dx = 0$ for all zero energy eigenfunctions ψ, then the wave operators are bounded on L^p for $1 \leq p < \infty$.

1. INTRODUCTION

This work is inspired by a conjecture of Jensen and Yajima in [16] on the range of p for which the Schrödinger wave operators are $L^p(\mathbb{R}^4)$ bounded in the presence of a threshold eigenvalue. In [16] it was proven that the wave operators are bounded on $L^p(\mathbb{R}^4)$ for $\frac{4}{3} < p < 4$ in the presence of a threshold eigenvalue and conjectured that the boundedness is true on $1 \leq p \leq \frac{4}{3}$ as well. Recent works of Yajima, [25] and the authors [12] prove a similar results extending the lower range of p for which $L^p(\mathbb{R}^n)$ bounds hold in dimensions $n > 4$. In this article we prove the conjectured bounds in four dimensions, and also show that the range of p can be extended upwards under certain orthogonality conditions on the zero energy eigenspace.

Let $H = -\Delta + V$ be a Schrödinger operator with a real-valued potential V and $H_0 = -\Delta$. If V satisfies $|V(x)| \lesssim \langle x \rangle^{-2-}$, the spectrum of H is the absolutely continuous spectrum on $[0, \infty)$ and a finite collection of non-positive eigenvalues, [19]. The wave operators are defined by the strong limits on $L^2(\mathbb{R}^n)$

$$(1) \qquad W_{\pm}f = \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} f.$$

Such limits are known to exist and be asymptotically complete for a wide class of potentials V. Furthermore, one has the identities

$$(2) \qquad\qquad W_\pm^* W_\pm = I, \qquad W_\pm W_\pm^* = P_{ac}(H),$$

with $P_{ac}(H)$ the projection onto the absolutely continuous spectral subspace associated with the Schrödinger operator H.

We say that zero energy is regular if there are no zero energy eigenvalues or resonances. There is a zero energy eigenvalue if there is a solution to $H\psi = 0$ with $\psi \in L^2(\mathbb{R}^n)$, and a resonance if $\psi \notin L^2(\mathbb{R}^n)$ instead belongs to a nearby space whose precise definition depends on the spatial dimension $n \leq 4$. In dimension $n = 4$, a resonance satisfies $\langle \cdot \rangle^{-\epsilon}\psi \in L^2(\mathbb{R}^4)$ for any $\epsilon > 0$.

There is a long history of results on the existence and boundedness of the wave operators. Yajima has established L^p and $W^{k,p}$ boundedness of the wave operators for the full range of $1 \leq p \leq \infty$ in [22, 23] in all dimensions $n \geq 3$, provided that zero energy is regular under varying assumptions on the potential V. The sharpest result for $n = 3$ was obtained in [3] by Beceanu .

If zero is not regular, in general, the range of p on which the wave operators are bounded shrinks. When $n > 4$ it was shown in [24, 8] that the wave operators are bounded on $L^p(\mathbb{R}^n)$ when $\frac{n}{n-2} < p < \frac{n}{2}$. Independent results of Yajima [25, 26] and the authors [12] then brought the lower bound on p down to $1 < p < \frac{n}{2}$ (with the authors obtaining $p = 1$ as well), and found conditions under which the upper bound may be raised. When $n = 3$ Yajima [27] showed that the wave operators are bounded on $1 \leq p < 3$ in the case of a zero energy eigenvalue and on $1 < p < 3$ but not $p = 1$ in the case of a zero energy resonance. This extended the range $\frac{3}{2} < p < 3$ proven in [24]. Finally, with conditions as in [12, 26], the full range of $1 \leq p \leq \infty$ is recovered when $n = 3$.

Results are also known in one dimension. In [21], Weder showed that the wave operators are bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, and that the endpoint $p = 1$ is possible under certain conditions on the Jost solutions, but is weak-type in general. Further work in one dimension was done by D'Ancona and Fanelli in [4]. To the best of the authors' knowledge, there are no results in the literature when zero is not regular and $n = 2$.

An important property of the wave operators is the intertwining identity,

$$(3) \qquad\qquad f(H)P_{ac}(H) = W_\pm f(-\Delta)W_\pm^*,$$

which is valid for any Borel function f. Using this, one can deduce properties of the operator $f(H)$ from the simpler operator $f(-\Delta)$, provided one has control on mapping properties of the wave operators $W_\pm$ and $W_\pm^*$. In four dimensions, boundedness of the wave operators on $L^p(\mathbb{R}^4)$ for a given $p \geq 2$ imply the dispersive estimates

$$(4) \qquad \|e^{itH} P_{ac}(H)\|_{L^{p'} \to L^p} \lesssim |t|^{-2+\frac{4}{p}}.$$

Here p' is the conjugate exponent satisfying $\frac{1}{p} + \frac{1}{p'} = 1$.

There has been much work on dispersive estimates for the Schrödinger evolution with zero energy obstructions in recent years by Erdoğan, Schlag, Toprak and the authors in various combinations, see [7, 9, 6, 5, 10, 11, 13] in which $L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ were studied for all $n > 1$. Estimates in $L^p(\mathbb{R}^n)$ are obtained by interpolating these results with the L^2 conservation law. These works have roots in previous work of [17], and also in [15, 18] where the dispersive estimates were studied as operators on weighted $L^2(\mathbb{R}^n)$ spaces.

Our main result confirms the conjecture of Jensen and Yajima, as well as extending the range of p upward under certain conditions on the zero energy eigenspace.

Theorem 1.1. *Let $\sigma > \frac{2}{3}$. Assume that $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for $\beta > 6$,*

$$(5) \qquad \mathcal{F}\big(\langle \cdot \rangle^{2\sigma} V\big) \in L^{\frac{3}{2}}(\mathbb{R}^4),$$

and $H = -\Delta + V$ has an eigenvalue at zero with no resonance.

i) *The wave operators extend to bounded operators on $L^p(\mathbb{R}^4)$ for all $1 \leq p < 4$.*

ii) *If $\int_{\mathbb{R}^4} x V(x) \psi(x)\,dx = 0$ for all zero energy eigenfunctions ψ, then the wave operators extend to bounded operators on $L^p(\mathbb{R}^4)$ for all $1 \leq p < \infty$.*

We expect that the endpoint case $p = \infty$ holds if one has the additional cancellation $\int_{\mathbb{R}^4} x^2 V(x) \psi(x)\,dx = 0$ and slightly more decay on the potential, see Remark 3.4 below. The corresponding dispersive estimate from $L^1(\mathbb{R}^4)$ to $L^\infty(\mathbb{R}^4)$ was recently shown to be true even without this extra hypothesis by the second author and Toprak in [13].

We prove Theorem 1.1 for $W = W_-$, the proof for W_+ is identical up to complex conjugation. Define $R_0^\pm(\lambda^2) := \lim_{\varepsilon \to 0^+} (H_0 - (\lambda \pm i\varepsilon)^2)^{-1}$ and $R_V^+(\lambda^2) := \lim_{\varepsilon \to 0^+} (H - (\lambda + i\varepsilon)^2)^{-1}$ as the free and perturbed resolvents, respectively. These operators are well-defined on polynomially weighted $L^2(\mathbb{R}^4)$ spaces due to the limiting absorption principle of Agmon, [2]. We

use the stationary representation of the wave operator

$$
\begin{aligned}
Wu &= u - \frac{1}{\pi i} \int_0^\infty \lambda R_V^+(\lambda^2) V [R_0^+(\lambda^2) - R_0^-(\lambda^2)]\, u\, d\lambda \\
&= u - \frac{1}{\pi i} \int_0^\infty \lambda \big[R_0^+(\lambda^2) - R_0^+(\lambda^2) V R_V^+(\lambda^2) \big] V [R_0^+(\lambda^2) - R_0^-(\lambda^2)]\, u\, d\lambda.
\end{aligned}
$$

(6)

The resolvents are related by the identity $R_V^+(\lambda^2) = R_0^+(\lambda^2) - R_0^+(\lambda^2) V R_V^+(\lambda^2)$, which justifies the equality in (6). In dimension $n = 4$, the free resolvent operators $R_0^\pm(\lambda^2)$ are bounded as $\lambda \to 0$, as are the perturbed resolvents $R_V^\pm(\lambda^2)$ if zero is regular. When zero is not regular, the perturbed resolvent becomes singular as $\lambda \to 0$.

We divide the representation for W in (6) into 'high' and 'low' energy parts, by writing $W = W\Phi^2(H_0) + W\Psi^2(H_0)$ with $\Phi, \Psi \in C_0^\infty(\mathbb{R})$ smooth cut-off functions that satisfy $\Phi^2(\lambda) + \Psi^2(\lambda) = 1$ with $\Phi(\lambda^2) = 1$ for $|\lambda| \leq \lambda_0/2$ and $\Phi(\lambda^2) = 0$ for $|\lambda| \geq \lambda_0$ for a suitable constant $0 < \lambda_0 \ll 1$. This allows us to write $W = W_< + W_>$, with $W_<$ the 'low energy' portion of the wave operator and $W_>$ the 'high energy' portion. The high energy term $W_>$ is controlled in [16], and this argument remains valid when zero energy is a resonance or eigenvalue, whose effects are limited to only an arbitrary small neighborhood of zero energy. Our technical analysis proceeds much in the same vein as [12]. We isolate the leading order terms of $W_<$ caused by the singularity of $R_V^\pm(\lambda^2)$ near $\lambda = 0$ and determine their $L^p(\mathbb{R}^4)$ operator bounds through a careful, pointwise analysis of their integral kernels.

The next section introduces some ideas for controlling the size of the leading order expression in $W_<$ when there is a zero energy eigenvalue, and describes how certain pointwise bounds on the integral kernel correspond to operator estimates in $L^p(\mathbb{R}^4)$. In Section 3 we calculate the leading order expression in detail and complete the proof of Theorem 1.1, modulo a number of integral estimates that are stated in Appendix A. The discussion concludes in Section 4 with some remarks about the case where a zero energy resonance is present.

2. PRELIMINARY STEPS

In four dimensions it is well-known (see [16, 5, 13]) that if there is a zero energy eigenvalue but no zero energy resonance, then the perturbed resolvent $R_V^+(\lambda^2)$ in (6) has a pole of order two whose residue is the finite-rank projection P_e onto the eigenspace. Furthermore, each zero energy eigenfunction ψ has the cancellation property

$$
\int_{\mathbb{R}^4} V(x)\psi(x)\, dx = 0,
$$

(7)

which we express in the shorthand $P_e V 1 = 0$. This fact is crucial in obtaining the full range of p as it permits improved estimates for the most singular terms in the expansion of $W_<$ that dictate the allowable range of p. The extra cancellation condition in part ii) of Theorem 1.1, namely that $\int_{\mathbb{R}^4} x_j V \psi(x) \, dx = 0$ for each $j \in [1, n]$, will be called $P_e V x = 0$.

Using the low energy expansion for $W_<$ in [16], the leading term is given by the operator

$$(8) \qquad W_s = \frac{1}{\pi i} \int_0^\infty R_0^+(\lambda^2) V P_e V (R_0^+(\lambda^2) - R_0^-(\lambda^2)) \tilde{\Phi}(\lambda) \lambda^{-1} \, d\lambda.$$

In the absence of a zero energy resonance, the next operator we need to control is

$$(9) \qquad W_{log} = \frac{1}{\pi i} \int_0^\infty R_0^+(\lambda^2) L_1 (R_0^+(\lambda^2) - R_0^-(\lambda^2)) \tilde{\Phi}(\lambda) \lambda \log(\lambda) \, d\lambda,$$

where L_1 is a finite rank operator with kernel

$$L_1(x, y) = \sum_{j,k=1}^d a_{jk} V \psi_j(x) V \psi_k(y), \qquad a_{jk} \in \mathbb{R},$$

and $\{\psi_j\}_{j=1}^d$ form an orthonormal basis for the zero energy eigenspace.

One can show that the remaining terms in the expansion of $W_<$ are better behaved. Thus the estimates on W_s, and to a lesser extent W_{log}, dictate the mapping properties of $W_<$ itself. The presence or absence of threshold eigenvalues has little effect on properties of the resolvent outside a small neighborhood of $\lambda = 0$, so the estimates for $W_>$ are unchanged. Therefore our primary effort will be to control the mapping properties of the operator W_s.

Proposition 2.1. *Assume that $|V(z)| \lesssim \langle z \rangle^{-\delta}$ for some $\delta > 0$.*

   i) If $\delta > 4$, then W_s is bounded on $L^p(\mathbb{R}^4)$ for $1 \leq p < 4$.

   ii) If $\delta > 6$, and the zero energy eigenspace satisfies the cancellation condition $P_e V x = 0$, then W_s is bounded on $L^p(\mathbb{R}^4)$ for $1 \leq p < \infty$.

The proof of this proposition is based on pointwise bounds for the integral kernel $K(x, y)$ of the operator W_s. To get started, the kernel of W_s is a sum of integrals of the form

$$(10) \qquad K^{jk}(x, y) = \int_0^\infty \iint_{\mathbb{R}^8} R_0^+(\lambda^2)(x, z) V(z) \psi_j(z) V(w) \psi_k(w)$$

$$(R_0^+ - R_0^-)(\lambda^2)(w, y) \frac{\tilde{\Phi}(\lambda)}{\lambda} \, dw dz \, d\lambda,$$

where the functions $\{\psi_j\}_{j=1}^N$ form an orthonormal basis for the zero energy eigenspace, and $\tilde{\Phi}(\lambda) \in C_c^\infty(\mathbb{R})$ is such that $\tilde{\Phi}(\lambda) \Phi(\lambda^2) = \Phi(\lambda^2)$.

For the remainder of the paper, we omit the subscripts on the eigenfunctions as our calculations will be satisfied for any such ψ. Our estimates are stated for an operator kernel

$K(x, y)$ with the understanding that each $K^{jk}(x, y)$ obeys the same bounds. To analyze the kernel, we split into three regimes based on the relative size of $|x|$ and $|y|$: $|x| > 2|y|$, $|y| > 2|x|$, and $|x| \approx |y|$. The operator estimates resulting from a typical pointwise bound in each regime are summarized in the following three lemmas.

Lemma 2.2. *Suppose* $K(x, y)$ *is an integral kernel supported in the region of* $\mathbb{R}^8$ *where* $|x| > 2|y|$, *and* $|K(x, y)| \lesssim \langle x \rangle^{-4-\alpha}$.

i) *If* $\alpha > 0$, *then* K *defines a bounded operator on* $L^p(\mathbb{R}^4)$ *for all* $1 \leq p \leq \infty$.

ii) *If* $\alpha = 0$, *then* K *defines a bounded operator on* $L^p(\mathbb{R}^4)$ *for all* $1 < p \leq \infty$ *but it may not be bounded on* $L^1(\mathbb{R}^4)$.

Lemma 2.3. *Suppose* $K(x, y)$ *is an integral kernel supported in the region of* $\mathbb{R}^8$ *where* $|y| > 2|x|$, *and* $|K(x, y)| \lesssim \langle x \rangle^{-\gamma} \langle y \rangle^{-\beta}$ *for some* $0 < \beta \leq 4$ *and* $\gamma \geq 4 - \beta$. *Then* K *defines a bounded operator on* $L^p(\mathbb{R}^4)$ *for all* $1 \leq p < \frac{4}{4-\beta}$.

Lemma 2.4. *Suppose* $K(x, y)$ *is an integral kernel supported in the region of* $\mathbb{R}^8$ *where* $\frac{1}{2}|x| \leq |y| \leq 2|x|$, *and* $|K(x, y)| \lesssim \langle x \rangle^{-3-\alpha} \langle |x| - |y| \rangle^{-1-\beta}$ *for some* $\alpha, \beta \geq 0$.

i) *If* $\alpha + \beta > 0$, *then* K *defines a bounded operator on* $L^p(\mathbb{R}^n)$ *for all* $1 \leq p \leq \infty$.

ii) *If* $\alpha = \beta = 0$, *it is not guaranteed that the operator is bounded on* $L^p(\mathbb{R}^4)$ *for any* p.

The proofs of Lemmas 2.2-2.4 are technical and are provided in Appendix A. There are two main factors which control the integrability and size of (10). The integral in λ is highly oscillatory due to the presence of Bessel functions in the formula for the free resolvents $R_0^\pm(\lambda^2)$. In the w and z variables, decay of the potential V and the eigenfunctions ψ effectively localize most integrals to a neighborhood of the origin. A representative example of each kind of estimate are as follows.

Lemma 2.5 ([12], Lemma 2.2). *Let* $R_0^\pm(\lambda^2, A)$ *denote the convolution kernel of* $R_0^\pm(\lambda^2)$ *evaluated at a point with* $|x - y| = A$. *For each* $j \geq 0$,

$$(11) \qquad \int_0^\infty R_0^+(\lambda^2, A) \partial_B^j (R_0^+ - R_0^-)(\lambda^2, B) \lambda^{-1} \tilde{\Phi}(\lambda) \, d\lambda \lesssim \frac{1}{A^2 \langle A+B \rangle \langle A-B \rangle^{1+j}}.$$

Two variations of this bound, with different powers of λ in the integrand and different placement of the partial derivatives, will also be required. These are stated as Lemmas A.3 and A.5 in the Appendix. The first one is proved in [12] along with Lemma 2.5.

Lemma 2.6 ([12], Lemma 4.3). *Let $\beta \geq 1$ and $0 \leq \alpha < n - 1$. If $N \geq n + \beta$, then for each fixed constant $R \geq 0$, we have the bound*

$$\int_{\mathbb{R}^n} \frac{\langle z \rangle^{-N}}{|x - z|^\alpha \langle |x - z| + R \rangle \langle |x - z| - R \rangle^\beta} \, dz \lesssim \frac{1}{\langle x \rangle^\alpha \langle |x| + R \rangle \langle R - |x| \rangle^\beta}.$$

The $\langle z \rangle^{-N}$ decay in the numerator of Lemma 2.6 is achieved by the combined decay of the potential $V(z)$ and eigenfunctions $\psi(z)$. Eigenfunctions at zero energy have a characteristic rate of decay which comes from the Green's function of the Laplacian and the additional cancellation property $P_e V 1 = 0$.

Lemma 2.7. *If $|V(x)| \lesssim \langle x \rangle^{-3-}$, and ψ is a zero energy eigenfunction, then $|\psi(x)| \lesssim \langle x \rangle^{-3}$.*

Proof. By definition, every eigenfunction $\psi(x)$ satisfies $\Delta \psi = V \psi$. After applying the Green's function of the Laplacian to both sides, $\psi(x) = C(|x|^{-2} * V\psi)$. Using $P_e V 1 = 0$ allows us to write

$$\psi(x) = C \int_{\mathbb{R}^4} V\psi(y) \Big(\frac{1}{|x - y|^2} - \frac{1}{|x|^2} \Big) \, dy.$$

When $|y| < \frac{1}{2}|x|$, we have the bound $\big| |x - y|^{-2} - |x|^{-2} \big| \lesssim |y| \, |x|^{-3}$, hence

$$|\psi(x)| \lesssim \int_{|y| < \frac{1}{2}|x|} \frac{|y| \, |V\psi(y)|}{|x|^3} \, dy + \int_{|x - y| < \frac{1}{2}|x|} \frac{|V\psi(y)|}{|x - y|^2} \, dy + \int_{|y| > \frac{1}{2}|x|} \frac{|V\psi(y)|}{|x|^2} \, dy \lesssim \frac{1}{|x|^3},$$

provided $|V\psi(y)| \lesssim \langle y \rangle^{-5-}$. Since we have the *a priori* estimate that $|\psi(x)| \lesssim \langle x \rangle^{-2}$, see Lemma 2.1 in [12], it suffices to assume that $|V(x)| \lesssim \langle x \rangle^{-3-}$. $\qquad\square$

3. Main Estimates for $K(x, y)$

The technical tools required to prove Proposition 2.1 are somewhat different depending on the relative size of $|x|$ and $|y|$ in the integral kernel $K(x, y)$. In order to employ Lemmas 2.2-2.4, we proceed by making separate estimates where $|y|$ is greater than, approximately equal to, or smaller than $|x|$. The region where $|y| > 2|x|$ plays a key role in distinguishing the two cases of Proposition 2.1. In the following subsections we provide the necessary bounds on the integral kernel of W_s to verify both claims in Proposition 2.1 and formulate a similar proposition about the operator W_{log} defined in (9). Once these facts are in hand, Theorem 1.1 is an immediate consequence.

3.1. **Estimates when $|y| > 2|x|$.** We begin the analysis of $K(x, y)$ in the region where y is large compared to x. As suggested by Lemma 2.3, the decay as $|y| \to \infty$ dictates the upper range of exponents for which W_s is bounded on $L^p(\mathbb{R}^4)$. Our use of the cancellation

condition $P_e V 1 = 0$ (which always holds, see (7)), and (when it is assumed) $P_e V x = 0$ mirrors its treatment in [12].

We first rewrite the $K(x, y)$ integral in the following manner.

$$(12) \quad K(x, y) = \iint_{\mathbb{R}^8} \int_0^\infty V\psi(z) V\psi(w) R_0^+(\lambda^2, |x - z|)$$

$$\left((R_0^+ - R_0^-)(\lambda^2, |y - w|) - (R_0^+ - R_0^-)(\lambda^2, |y|) \right) \frac{\tilde{\Phi}(\lambda)}{\lambda} \, d\lambda \, dz \, dw.$$

Subtracting $(R_0^+ - R_0^-)(\lambda^2, |y|)$, which is independent of w, from the integrand does not affect the final value due to (7).

The strong decay of $V\psi(z)$ and $V\psi(w)$, together with compact support of $\tilde{\Phi}(\lambda)$ and boundedness as $\lambda \to 0$ (due to (34)) allows the order of integration to be changed freely.

For any function $F(\lambda, |y|)$ one can express

$$(13) \qquad F(\lambda, |y - w|) - F(\lambda, |y|) = \int_0^1 \partial_r F(\lambda, |y - sw|) \frac{(-w) \cdot (y - sw)}{|y - sw|} \, ds.$$

Here we are interested in $F(\lambda, |y|) = (R_0^+ - R_0^-)(\lambda^2, |y|)$, whose radial derivatives are considered in the statement of Lemma 2.5. Identity (13) is most useful in the region of (12) where $|w| < \frac{1}{2}|y|$. In this region we use the righthand side of (13) to express the contribution to $K(x, y)$ as

$$(14) \quad \int_{|w| < \frac{|y|}{2}} \int_{\mathbb{R}^4} \int_0^1 \int_0^\infty V\psi(z) V\psi(w) R_0^+(\lambda^2, |x - z|) \partial_r \left((R_0^+ - R_0^-)(\lambda^2, |y - sw|) \right)$$

$$\frac{(-w) \cdot (y - sw)}{|y - sw|} \frac{\tilde{\Phi}(\lambda)}{\lambda} \, ds \, d\lambda \, dz \, dw,$$

where ∂_r indicates the partial derivative with respect to the radial variable of $R_0^\pm(\lambda^2, r)$. Applying Fubini's Theorem and then Lemma 2.5 with $j = 1$ we obtain the upper bound

$$\int_0^1 \int_{|w| < \frac{|y|}{2}} \int_{\mathbb{R}^4} \frac{|V\psi(z)| \, |wV\psi(w)|}{|x - z|^2 \langle |x - z| + |y - sw| \rangle \langle |x - z| - |y - sw| \rangle^2} \, dz \, dw \, ds.$$

By Lemma 2.7 and our assumption that $|V(z)| \lesssim \langle z \rangle^{-4-}$, we can control the decay of the numerator with $|V\psi(z)| \lesssim \langle z \rangle^{-6-}$ and $|wV\psi(w)| \lesssim \langle w \rangle^{-6-}$. These are sufficient to apply Lemma 2.6 in the z variable, then Lemma A.1 in the w variable to obtain

$$|(14)| \lesssim \int_0^1 \int_{|w| < \frac{|y|}{2}} \frac{|wV\psi(w)|}{\langle x \rangle^2 \langle |x| + |y - sw| \rangle \langle |x| - |y - sw| \rangle^2} \, dw \, ds$$

$$(15) \qquad \lesssim \int_0^1 \frac{1}{\langle x \rangle^2 \langle |x| + |y| \rangle \langle |x| - |y| \rangle^2} \, ds \lesssim \frac{1}{\langle x \rangle^2 \langle |x| + |y| \rangle \langle |x| - |y| \rangle^2}.$$

In the region where $|y| > 2|x|$, this yields $|(14)| \lesssim \langle x \rangle^{-2} \langle y \rangle^{-3}$.

For the portion of (12) where $|w| > \frac{1}{2}|y|$, we do not seek out cancellation between $(R_0^+ - R_0^-)(\lambda^2, |y - w|)$ and $(R_0^+ - R_0^-)(\lambda^2, |y|)$ and instead treat the two terms separately. For the term with $(R_0^+ - R_0^-)(\lambda^2, |y - w|)$, we have

$$\int_{\mathbb{R}^4} \int_{|w| > \frac{|y|}{2}} \int_0^\infty R_0^+(\lambda^2, |x - z|) V\psi(z) V\psi(w)(R_0^+ - R_0^-)(\lambda^2, |y - w|) \frac{\tilde{\Phi}(\lambda)}{\lambda} \, d\lambda \, dw \, dz$$

$$\lesssim \int_{\mathbb{R}^4} \int_{|w| > \frac{|y|}{2}} \frac{|V\psi(z)| \, |V\psi(w)|}{|x - z|^2 \langle |x - z| + |y - w| \rangle \langle |x - z| - |y - w| \rangle} \, dw \, dz$$

$$(16) \qquad\qquad\qquad\qquad \lesssim \frac{1}{\langle x \rangle^2 \langle |x| + |y| \rangle \langle |x| - |y| \rangle \langle y \rangle}.$$

The first inequality is Lemma 2.5 with $j = 0$. The second is a combination of Lemma A.2 with $k = 1$ and $\alpha = 0$ for the w integral and Lemma 2.6 for the z integral. When $|y| > 2|x|$ this is also bounded by $\langle x \rangle^{-2} \langle y \rangle^{-3}$.

The estimate for the term with $(R_0^+ - R_0^-)(\lambda^2, |y|)$ is more straightforward, we obtain the same bound as before,

$$\int_{\mathbb{R}^4} \int_{|w| > \frac{|y|}{2}} \int_0^\infty R_0^+(\lambda^2, |x - z|) V\psi(z) V\psi(w)(R_0^+ - R_0^-)(\lambda^2, |y|) \frac{\tilde{\Phi}(\lambda)}{\lambda} \, d\lambda \, dw \, dz$$

$$\lesssim \int_{\mathbb{R}^n} \int_{|w| > \frac{|y|}{2}} \frac{|V\psi(z)| \, |V\psi(w)|}{|x - z|^2 \langle |x - z| + |y| \rangle \langle |x - z| - |y| \rangle} \, dw \, dz$$

$$(17) \qquad\qquad\qquad\qquad \lesssim \frac{1}{\langle x \rangle^2 \langle |x| + |y| \rangle \langle |x| - |y| \rangle \langle y \rangle}.$$

We have used Lemma 2.6 in z, and the estimate $\int_{|w| > |y|/2} \langle w \rangle^{-N} \, dw \lesssim \langle y \rangle^{4-N}$ (for $N > 4$) in lieu of Lemma A.2. Put together, the integral bounds (15)-(17) show that where $|y| > 2|x|$

$$(18) \qquad\qquad\qquad |K(x, y)| \lesssim \frac{1}{\langle x \rangle^2 \langle y \rangle^3},$$

so its contribution to W_s is bounded on $L^p(\mathbb{R}^4)$ for $1 \leq p < 4$ by Lemma 2.3 with $\beta = 3$ and $\gamma = 2$.

3.2. **Improvement when $P_e V x = 0$.** If we assume that $P_e V x = 0$, this permits us to introduce a linear approximation of $(R_0^+ - R_0^-)(\lambda^2, |y - w|)$ in the w variable without changing the value of the integral (12). That is, we have the equality

$$(19) \qquad \int_0^\infty R_0^+(\lambda^2, |x - z|) V(z)\psi(z) V(w)\psi(w)(R_0^+ - R_0^-)(\lambda^2, |w - y|) \frac{\tilde{\Phi}(\lambda)}{\lambda} \, d\lambda$$

$$= \int_0^\infty R_0^+(\lambda^2, |x - z|) V\psi(z) V\psi(w) \left[(R_0^+ - R_0^-)(\lambda^2, |w - y|) - F(\lambda, y) - G(\lambda, y) \frac{w \cdot y}{|y|} \right] \frac{\tilde{\Phi}(\lambda)}{\lambda} \, d\lambda$$

for any functions $F(\lambda, y)$ and $G(\lambda, y)$. In place of (13), we utilize the second level of cancellation to write

$$(20) \quad K(\lambda, |y - w|) - K(\lambda, |y|) + \partial_r K(\lambda, |y|) \frac{w \cdot y}{|y|}$$

$$= \int_0^1 (1 - s) \left[\partial_r^2 K(\lambda, |y - sw|) \frac{(w \cdot (y - sw))^2}{|y - sw|^2} \right.$$

$$\left. + \partial_r K(\lambda, |y - sw|) \left(\frac{|w|^2}{|y - sw|} - \frac{(w \cdot (y - sw))^2}{|y - sw|^3} \right) \right] ds.$$

The formula above suggests that we choose $F(\lambda, y) = (R_0^+ - R_0^-)(\lambda^2, |y|)$ and $G(\lambda, y) = \partial_r (R_0^+ - R_0^-)(\lambda^2, |y|)$ in (19) respectively.

As in the arguments of the previous section, we use the left side of (20) when $|w| > |y|/2$ because there is no significant cancellation of these three terms. One can imitate (16) more or less exactly to show that they contribute no more than $\langle x \rangle^{-2} \langle y \rangle^{-4}$ to the size of $K(x, y)$, provided $V\psi(w)$ decays rapidly enough to apply Lemma A.2 with $k = 2$ instead of $k = 1$.

The portion of $K(x, y)$ originating from the region $|w| < |y|/2$, consists of new terms of the form

$$\int_{|w| < \frac{|y|}{2}} \int_{\mathbb{R}^4} \int_0^\infty \int_0^1 V\psi(z) V\psi(w) R_0^+(\lambda^2, |x - z|) \partial_r^j \big((R_0^+ - R_0^-)(\lambda^2, |y - sw|) \big)$$

$$(1 - s) \Gamma_j(s, w, y) \frac{\tilde{\Phi}(\lambda)}{\lambda} \, ds \, d\lambda \, dz \, dw$$

with $j = 1, 2$ and $\Gamma_j(s, w, y)$ denoting

$$\Gamma_1 = \left(\frac{|w|^2}{|y - sw|} - \frac{(w \cdot (y - sw))^2}{|y - sw|^3} \right), \qquad \text{and} \qquad \Gamma_2 = \frac{(w \cdot (y - sw))^2}{|y - sw|^2}.$$

When $|w| < |y|/2$ and $0 \leq s \leq 1$, these factors obey the bounds $|\Gamma_1(s, w, y)| \lesssim |y|^{-1} |w|^2$ and $|\Gamma_2(s, w, y)| \leq |w|^2$. The calculation proceeds in the same manner as the estimate for (14), first using Lemma 2.5 with $j = 1, 2$, then Lemma 2.6 in the z integral and Lemma A.1 (with $\alpha = 2 - j$) in the w integral. For both terms we have the bound $\langle x \rangle^{-2} \langle y \rangle^{-4}$ when $|y| > 2|x|$.

In this case the use of Lemma A.1 with $\beta = 1 + j$ requires that $|w|^2 |V\psi(w)| \lesssim \langle w \rangle^{-7}$, from which Lemma 2.7 shows that $|V(w)| \lesssim \langle w \rangle^{-6-}$ is needed.

Put together with the previous claim, this implies that if $P_e V x = 0$, then

$$(21) \qquad\qquad |K(x, y)| \lesssim \frac{1}{\langle x \rangle^2 \langle y \rangle^4} \quad \text{when } |y| > 2|x|.$$

The operator with kernel $K(x, y)$ in this region is therefore bounded on $L^p(\mathbb{R}^4)$ for all $p \in [1, \infty)$, by Lemma 2.3.

3.3. **Estimates when $|x| \approx |y|$.** No new work is required to control $K(x,y)$ adequately when x and y are of similar size. We need only use the fact that $P_e V 1 = 0$ as before, then combining (15) and (16) when $\frac{1}{2}|x| \le |y| \le 2|x|$ leads to the bound

$$(22) \qquad |K(x,y)| \lesssim \frac{1}{\langle x \rangle^3 \langle |x| - |y| \rangle^2} + \frac{1}{\langle x \rangle^4 \langle |x| - |y| \rangle} \lesssim \frac{1}{\langle x \rangle^3 \langle |x| - |y| \rangle^2}.$$

This is bounded on $L^p(\mathbb{R}^4)$ for all $1 \le p \le \infty$ by Lemma 2.4.

Remark 3.1. *In dimensions $n \ge 5$ it is possible to prove adequate bounds for $K(x,y)$ in the region where $|x| \approx |y|$ without assuming that $P_e V 1 = 0$. Here the cancellation is essential, as a straightforward estimation of (10) gives the bound $|K(x,y)| \lesssim \langle x \rangle^{-3} \langle |x| - |y| \rangle^{-1}$ in this region, which does not lead to operator bounds on $L^p(\mathbb{R}^4)$ for any exponent p.*

3.4. **Estimates when $|x| > 2|y|$.** In the region where $|x| > 2|y|$ the combined bounds (15) and (16) imply that

$$(23) \qquad |K(x,y)| \lesssim \frac{1}{\langle x \rangle^5} + \frac{1}{\langle x \rangle^4 \langle y \rangle} \lesssim \frac{1}{\langle x \rangle^4},$$

which according to Lemma 2.2 gives rise to a bounded operator on $L^p(\mathbb{R}^4)$ for all $p > 1$. It is certainly bounded on $L^1(\mathbb{R}^4)$ as well if we restrict to the compact region where $2|y| < |x| < 1$.

In order to show that the entire kernel of W_s is bounded on $L^1(\mathbb{R}^4)$, we must improve the decay as $|x| \to \infty$ enough to make it integrable. This is accomplished by applying the cancellation condition $P_e V 1 = 0$ in the z variable in (10). Then we may write

$$(24) \quad K(x,y) = \iint_{\mathbb{R}^8} \int_0^\infty V\psi(z) V\psi(w) \big(R_0^+(\lambda^2, |x-z|) - R_0^+(\lambda^2, |x|) \big)$$

$$(R_0^+ - R_0^-)(\lambda^2, |y-w|) \frac{\tilde{\Phi}(\lambda)}{\lambda} \, d\lambda \, dz \, dw.$$

We split this integral into two regions: where $|z| > \frac{1}{2}|x|$, and where $|z| < \frac{1}{2}|x|$.

The first region is evaluated in the same spirit as (16).

$$\int_{\mathbb{R}^4} \int_{|z| > \frac{|x|}{2}} \int_0^\infty R_0^+(\lambda^2, |x-z|) V\psi(z) V\psi(w) (R_0^+ - R_0^-)(\lambda^2, |y-w|) \frac{\tilde{\Phi}(\lambda)}{\lambda} \, d\lambda \, dz \, dw$$

$$\lesssim \int_{\mathbb{R}^4} \int_{|z| > \frac{|x|}{2}} \frac{|V\psi(z)| \, |V\psi(w)|}{|x-z|^2 \langle |x-z| + |y-w| \rangle \langle |x-z| - |y-w| \rangle} \, dz \, dw$$

$$\lesssim \frac{1}{\langle x \rangle^3 \langle |x| + |y| \rangle \langle |x| - |y| \rangle} \lesssim \frac{1}{\langle x \rangle^5}.$$

The first inequality is Lemma 2.5, and the second follows from Lemma A.2 with $\alpha = 2$, $k = 1$, and Lemma 2.6.

The integral with $R_0^+(\lambda^2, |x|)$ is similar. Assuming $|x| > \max(1, 2|y|)$ we have

$$\int_{\mathbb{R}^4} \int_{|z| > \frac{|x|}{2}} \int_0^\infty R_0^+(\lambda^2, |x|) V\psi(z) V\psi(w) (R_0^+ - R_0^-)(\lambda^2, |y-w|) \frac{\tilde{\Phi}(\lambda)}{\lambda} \, d\lambda \, dz \, dw$$

$$\lesssim \int_{\mathbb{R}^4} \int_{|z| > \frac{|x|}{2}} \frac{|V\psi(z)| \, |V\psi(w)|}{|x|^2 \langle |x| + |y-w| \rangle \langle |x| - |y-w| \rangle} \, dz \, dw$$

$$\lesssim \frac{1}{\langle x \rangle^3 \langle |x| + |y| \rangle \langle |x| - |y| \rangle} \lesssim \frac{1}{\langle x \rangle^5}.$$

These again follow by Lemmas 2.5, A.2, and 2.6.

Finally the part of (24) that comes from integrating where $|z| < \frac{1}{2}|x|$ is evaluated in the same manner as (14). Namely,

$$\int_{|z| < \frac{|x|}{2}} \int_{\mathbb{R}^4} \int_0^1 \int_0^\infty V\psi(z) V\psi(w) \partial_r \big(R_0^+(\lambda^2, |x - sz|) \big) (R_0^+ - R_0^-)(\lambda^2, |y-w|)$$

$$\times \frac{(-z) \cdot (x - sz)}{|x - sz|} \frac{\tilde{\Phi}(\lambda)}{\lambda} \, ds \, d\lambda \, dw \, dz$$

$$\lesssim \int_0^1 \int_{|z| < \frac{|x|}{2}} \int_{\mathbb{R}^4} \frac{|z V\psi(z)| \, |V\psi(w)|}{|x - sz|^3 \langle |x - sz| - |y-w| \rangle^2} \, dw \, dz \, ds$$

$$\lesssim \int_0^1 \frac{1}{\langle x \rangle^3 \langle |x| - |y| \rangle^2} \, ds \lesssim \frac{1}{\langle x \rangle^5}.$$

Put together, we conclude that in the entire region where $|x| > 2|y|$,

(25) $$|K(x, y)| \lesssim \frac{1}{\langle x \rangle^5},$$

and this describes a bounded operator on all $L^p(\mathbb{R}^4)$, $1 \le p \le \infty$ by Lemma 2.2.

Proof of Proposition 2.1. In the preceding subsections, we have assembled the following bounds (18), (22), and (25) for the integral kernel of W_s:

$$|K(x, y)| \lesssim \begin{cases} \frac{1}{\langle x \rangle^5} & \text{if } |x| > 2|y|, \\ \frac{1}{\langle x \rangle^3 \langle |x| - |y| \rangle^2} & \text{if } |x| \approx |y|, \\ \frac{1}{\langle x \rangle^2 \langle y \rangle^3} & \text{if } |y| > 2|x| \end{cases} \quad .$$

The pointwise estimates in the first two regions are sufficient to satisfy the Schur test, so they describe bounded operators on $L^p(\mathbb{R}^4)$ for all $1 \le p \le \infty$. When $|y|$ is large, the integral kernel describes a bounded operator only if $1 \le p < 4$ by Lemma 2.3.

With the further assumption that $P_e V x = 0$ and some additional decay of $|V(z)|$, we are able to replace (18) with the stronger pointwise bound (21), which asserts that $|K(x, y)| \lesssim$

$\langle x \rangle^{-2} \langle y \rangle^{-4}$ when $|y| > 2|x|$. This now describes a bounded operator on $L^p(\mathbb{R}^4)$ in the range $1 \leq p < \infty$. $\qquad \square$

3.5. **Estimates for W_{log}.** Pointwise estimates for the integral kernel of W_{log} follow from the same arguments as with W_s above, only with Lemma A.3 used in place of Lemma 2.6.

Proposition 3.2. *If $|V(z)| \lesssim \langle z \rangle^{-4-}$, then the operator W_{log} is bounded on $L^p(\mathbb{R}^4)$ for $1 \leq p < \infty$.*

Remark 3.3. *In fact W_{log} is bounded on $L^\infty(\mathbb{R}^4)$ as well. We omit this case as a matter of convenience, in order to present a shorter proof with no reliance on cancellation properties.*

Proof. The kernel of W_{log} is given by

$$W_{log}(x, y) = \frac{1}{\pi i} \int_{\mathbb{R}^8} \int_0^\infty R_0^+(\lambda^2, |x - z|) \Bigg[\sum_{j,k=1}^d a_{jk} V\psi_j(z) V\psi_k(w) \Bigg]$$
$$[R_0^+ - R_0^-](\lambda^2, |y - w|) \tilde\Phi(\lambda) \lambda \log(\lambda) \, d\lambda \, dz \, dw.$$

Applying Lemma A.3 with $j = 2$ and $\ell = 1$, we have

$$|W_{log}(x, y)| \lesssim \int_{\mathbb{R}^8} \frac{\langle \log\langle |x - z| - |y - w| \rangle \rangle |V\psi_j(z)| \, |V\psi_k(w)|}{|x - z|^2 \langle |x - z| + |y - w| \rangle \langle |x - z| - |y - w| \rangle^3} \, dz \, dw$$
$$\lesssim \int_{\mathbb{R}^8} \frac{\langle z \rangle^{-7-} \langle w \rangle^{-7-}}{|x - z|^2 \langle |x - z| + |y - w| \rangle \langle |x - z| - |y - w| \rangle^{3-\varepsilon}} \, dz \, dw,$$

for any sufficiently small $\varepsilon > 0$. The last inequality follows from the assumption on the decay of V and Lemma 2.7. Now, applying Lemma 2.6 in both the z and w integrals, we have the upper bound

$$|W_{log}(x, y)| \lesssim \frac{1}{\langle x \rangle^2 \langle |x| + |y| \rangle \langle |x| - |y| \rangle^{3-\varepsilon}} \lesssim \begin{cases} \frac{1}{\langle x \rangle^{6-\varepsilon}} & \text{if } |x| > 2|y| \\ \frac{1}{\langle x \rangle^3 \langle |x| - |y| \rangle^{3-\varepsilon}} & \text{if } |x| \approx |y| \\ \frac{1}{\langle x \rangle^2 \langle y \rangle^{4-\varepsilon}} & \text{if } |y| > 2|x| \end{cases}.$$

Applying Lemmas 2.2-2.4 concludes the proof, with Lemma 2.3 specifically showing that W_{log} is bounded on $L^p(\mathbb{R}^4)$ for $1 \leq p < \frac{4}{\varepsilon}$. $\qquad \square$

Proof of Theorem 1.1. Section 4 of [16] provides the decomposition $W = W_< + W_>$, with $W_< = \Phi(H)(1 - (W_s + W_{log} + W_0 + W_{-1} + W_r))\Phi(H_0)$, where

$$(8) \qquad W_s = \frac{1}{\pi i} \int_0^\infty R_0^+(\lambda^2) V P_e V (R_0^+(\lambda^2) - R_0^-(\lambda^2)) \tilde\Phi(\lambda) \lambda^{-1} \, d\lambda.$$

$$(9) \qquad W_{log} = \frac{1}{\pi i} \int_0^\infty R_0^+(\lambda^2) L_1 (R_0^+(\lambda^2) - R_0^-(\lambda^2)) \tilde\Phi(\lambda) \lambda \log(\lambda) \, d\lambda.$$

The operator W_{log} as defined here is called W_1 in [16]. The remaining terms W_0, W_{-1}, and W_r are shown to be bounded on L^p for $1 \leq p \leq \infty$ in Lemma 4.1 of [16].

Under the assumptions on the decay of V and (5), it was shown in [16] that $W_>$ is also bounded on $L^p(\mathbb{R}^4)$ for $1 \leq p \leq \infty$. Proposition 3.2 asserts that W_{log} is bounded on $L^p(\mathbb{R}^4)$ in the range $1 \leq p < \infty$.

The kernels of $\Phi(H)$ and $\Phi(H_0)$ are bounded by $C_N \langle x - y \rangle^{-N}$ for each $N = 1, 2, \ldots$, see Lemma 2.2 of [23]. Following (8), (10), W_s is bounded on $L^p(\mathbb{R}^4)$ exactly when the operators K^{jk} are. The range of exponents p for which this occurs is determined by Proposition 2.1. $\square$

Remark 3.4. *We note that the endpoint $p = \infty$ is not covered in Theorem 1.1. With the additional assumptions $P_e V x^2 = 0$ and $|V(z)| \lesssim \langle z \rangle^{-8-}$, one can use the techniques above to show that the wave operators are bounded on $L^p(\mathbb{R}^4)$ for the full range of $1 \leq p \leq \infty$. Here the assumption $P_e V x^2 = 0$ means that $\int_{\mathbb{R}^n} P_2(x) V(x) \psi(x) \, dx = 0$ for any quadratic monomial P_2. As with the L^∞ bound for W_{log}, we leave the details to the reader.*

4. Remarks On Threshold Resonances

Finally, we discuss some aspects of how threshold resonances affect the L^p boundedness of the wave operators. There is strong evidence that they are not bounded on $L^p(\mathbb{R}^4)$ for any $p > 2$ in this case. By the intertwining identity (3), L^p boundedness of wave operators with $p > 2$ would imply a power-law decay of the linear evolution $e^{itH} P_{ac}(H)$ at the rate $|t|^{(4/p)-2}$ as in (4). Instead it is known that, as an operator from $L^1(\mathbb{R}^4) \to L^\infty(\mathbb{R}^4)$,

$$e^{itH} P_{ac}(H) = \phi(t) P + O(t^{-1}),$$

with $\phi(t) \sim (\log t)^{-1}$ and P a finite rank operator, [5, 13]. The exact form of P depends on whether or not there is a zero energy eigenvalue as well. We suspect that the wave operators are bounded in $L^p(\mathbb{R}^4)$ precisely in the range $1 < p \leq 2$.

The low energy resolvent expansion of $R_V^{\pm}(\lambda^2)$ is considerably more complicated when a zero energy resonance is present. See, for example [15, 5, 13]. For the sake of (relative) simplicity we consider the case of a resonance and no eigenvalue at zero. Then the most singular λ term is of the form

$$\frac{P_r(x,y)}{\lambda^2(a \log \lambda + z)}, \qquad a \in \mathbb{R} \setminus \{0\}, \quad z \in \mathbb{C} \setminus \mathbb{R},$$

where $P_r(x,y)$ is a Riesz projection onto the canonical zero energy resonance. One crucial feature is that $P_r V 1 \neq 0$ for resonances, so most cancellation arguments do not apply.

A heuristic argument suggests that the leading order term of $W_<$ would take the form

$$\int_{\mathbb{R}^8} \int_0^\infty R_0^+(\lambda^2, |x-z|) \frac{V\varphi(z)V\varphi(w)}{\lambda(a\log\lambda + z)} [R_0^+ - R_0^-](\lambda^2, |y-w|) \, d\lambda \, dz \, dw,$$

with φ the canonical resonance at zero. This should have the same order of magnitude as what we calculated for W_s, perhaps even better by a logarithmic factor, except that no cancellation may be applied. If the estimate (23) can only be improved by a factor of $\log\langle x\rangle$ for large $|x|$, then the operator may not be bounded in $L^1(\mathbb{R}^4)$.

In the region where $|x| \approx |y|$, the lack of cancellation suggests that the best pointwise bounds for $K(x,y)$ will fall into the poorly behaved $\alpha = \beta = 0$ case of Lemma 2.4. Establishing boundedness on $L^p(\mathbb{R}^4)$ for such an operator would require more precise information about the sign and smoothness of $K(x,y)$. This seems closer in spirit to the approach taken in [26], where wave operator bounds are linked to weighted L^p estimates for the Hilbert transform rather than operators with a positive kernel.

APPENDIX A. INTEGRAL ESTIMATES

We begin by proving the assertions in Lemmas 2.2-2.4 regarding the $L^p(\mathbb{R}^4)$ behavior of integral operators with certain pointwise bounds. We then proceed to prove technical lemmas on integral bounds required to complete our arguments.

Proof of Lemma 2.2. Recall the Schur test, if

$$\sup_x \int_{\mathbb{R}^4} |K(x,y)| \, dy + \sup_y \int_{\mathbb{R}^4} |K(x,y)| \, dx < \infty,$$

then the integral operator with kernel $K(x,y)$ is bounded on $L^p(\mathbb{R}^4)$ for all $1 \le p \le \infty$. If $|K(x,y)| \lesssim \langle x\rangle^{-4-\alpha}$ for $\alpha > 0$, the full range of p is attained as

$$\sup_x \int_{\mathbb{R}^4} |K(x,y)| \, dy \lesssim \sup_x \langle x\rangle^{-4-\alpha} \int_{|y|<\frac{1}{2}|x|} dy \lesssim \sup_x \langle x\rangle^{-\alpha} \lesssim 1.$$

$$\sup_y \int_{\mathbb{R}^4} |K(x,y)| \, dx \lesssim \sup_y \int_{|x|>2|y|} \langle x\rangle^{-4-\alpha} \, dx \lesssim \int_{\mathbb{R}^4} \langle x\rangle^{-4-\alpha} \, dx \lesssim 1.$$

If $\alpha = 0$, then $|K(x,y)| \lesssim \langle x\rangle^{-4}$. Note that for each $x \in \mathbb{R}^4$,

$$\int_{\mathbb{R}^4} K(x,y)f(y) \, dy \lesssim |x|^{-4} \int_{|y|<|x|} |f(y)| \, dy \lesssim \mathcal{M}f(x),$$

where $\mathcal{M}f$ is the Hardy-Littlewod maximal function. The claimed range of p follows from the well-known bounds of the Hardy-Littlewood maximal function, see [20] for example. $\square$

Proof of Lemma 2.3. It suffices to prove this for $\gamma = 4 - \beta$, as $\langle x \rangle^{-\gamma} \leq \langle x \rangle^{\beta - 4}$ in all other cases. Note that K is bounded on $L^1(\mathbb{R}^4)$ since, if $\beta > 0$,

$$\sup_y \int_{\mathbb{R}^4} |K(x,y)|\, dx \lesssim \sup_y \langle y \rangle^{-\beta} \int_{|x| < \frac{1}{2}|y|} \langle x \rangle^{\beta - 4}\, dx \lesssim 1.$$

Now consider $0 < \beta < 4$. For each $x \in \mathbb{R}^4$, the weak-$L^{4/\beta}$ norm of $K(x, \cdot)$ is uniformly bounded by $\langle x \rangle^{\beta - 4}$. Hence K defines a bounded operator between the Lorentz spaces $L^{\frac{4}{4-\beta}, 1}(\mathbb{R}^4)$ and $L^{\frac{4}{4-\beta}, \infty}(\mathbb{R}^4)$. Boundedness on $L^p(\mathbb{R}^4)$, $1 \leq p < \frac{4}{4-\beta}$ follows by interpolation, [14]. The case $\beta = 4$ is the dual formulation of Lemma 2.2, hence K defines a bounded operator on $L^p(\mathbb{R}^4)$ for all $1 \leq p < \infty$. $\qquad\square$

Proof of Lemma 2.4. Once again we can use the Schur test. This time we integrate in spherical coordinates for y:

$$\sup_x \int_{\mathbb{R}^4} |K(x,y)|\, dy \lesssim \sup_x \langle x \rangle^{-3-\alpha} \int_{|y| \approx |x|} \langle |x| - |y| \rangle^{-1-\beta}\, dy$$

$$\lesssim \sup_x \langle x \rangle^{-\alpha} \int_{\frac{1}{2}|x|}^{2|x|} \langle |x| - r \rangle^{-1-\beta}\, dr \lesssim \begin{cases} \sup_x \langle x \rangle^{-\alpha} & \text{if } \beta > 0 \\ \sup_x \langle x \rangle^{-\alpha}(1 + \log|x|) & \text{if } \beta = 0 \end{cases}.$$

These are bounded by 1 unless α and β are both zero. By symmetry the result is the same if x and y are reversed.

If $\alpha = \beta = 0$, let f be a characteristic function of the annulus $R < |y| < 2R$ with radius $R > 1$. Then for every point x in the same annulus, $\int_{\mathbb{R}^4} |K(x,y)| f(y)\, dy \gtrsim \log \langle R \rangle$. $\qquad\square$

The following integral estimates, along with Lemma 2.5 and Lemma 2.6, are direct quotes or minor adaptations of statements in [12]. Proofs are presented where they differ from the previous work.

Lemma A.1. *Suppose $0 < s \leq 1$, $0 \leq \alpha \leq n$, $\beta \geq 1$, and $\gamma \in \mathbb{R}$. If $N \geq n + \beta$, then for each fixed constant $R \geq 0$, we have the bound*

$$(26) \quad \int_{|w| < \frac{|y|}{2}} \frac{\langle w \rangle^{-N}}{|y - sw|^\alpha \langle |y - sw| + R \rangle^\gamma \langle |y - sw| - R \rangle^\beta}\, dw \lesssim \frac{1}{\langle y \rangle^\alpha \langle |y| + R \rangle^\gamma \langle R - |y| \rangle^\beta}.$$

Proof. The claim for large $|y| > 10$ essentially follows from Lemma 4.4 in [12], which considers only the case $\gamma = 1$. However one of the first steps is to observe that $\langle |y - sw| + R \rangle \approx \langle |y| + R \rangle$ for all $|w| < \frac{1}{2}|y|$ and $|s| \leq 1$, so the calculations proceed independently of the choice of γ. To complete the proof when $|y| \leq 10$, we simply note that the region of

integration has volume comparable to $|y|^n$ and the integrand is nearly constant. Thus

$$\int_{|w|<|y|/2} \frac{\langle w \rangle^{-N}}{|y-sw|^\alpha \langle |y-sw|+R \rangle^\gamma \langle |y-sw|-R \rangle^\beta} \, dw \lesssim \frac{|y|^{n-\alpha}}{\langle |y|+R \rangle^\gamma \langle R-|y| \rangle^\beta}.$$

$\square$

Lemma A.2. *Let $k \geq 0$ and $0 \leq \alpha < n-1$. If $N \geq n+1+k$ and $R \geq 0$ is fixed, then*

$$(27) \qquad \int_{|w|>\frac{|y|}{2}} \frac{\langle w \rangle^{-N}}{\langle R+|y-w| \rangle \langle R-|y-w| \rangle |y-w|^\alpha} \, dw \lesssim \frac{1}{\langle R+|y| \rangle \langle R-|y| \rangle \langle y \rangle^{\alpha+k}}.$$

Proof. The $\alpha = 0$ case is proved as Lemma 3.2 in [12]. If $\alpha > 0$, split the domain of integration into two pieces according to whether $|y-w| > \frac{1}{2}|y|$ or $|y-w| < \frac{1}{2}|y|$. In the former region, $|w|$, and $|y-w|$ are of comparable size, so one may reduce to the $\alpha = 0$ case

$$\int_{|w|>\frac{|y|}{2}} \frac{\langle w \rangle^{-N}|w|^{-\alpha}}{\langle R+|w| \rangle \langle R-|w| \rangle} \, dw \lesssim |y|^{-\alpha} \int_{|w|>\frac{|y|}{2}} \frac{\langle w \rangle^{-N}}{\langle R+|w| \rangle \langle R-|w| \rangle} \, dw,$$

which suffices if $|y| > 1$. If $|y| < 1$, we bound the integral by

$$\int_{\mathbb{R}^4} \frac{\langle w \rangle^{-N}|w|^{-\alpha}}{\langle R+|w| \rangle \langle R-|w| \rangle} \, dw \lesssim \int_{|w|<1} \frac{|w|^{-\alpha}}{\langle R \rangle^2} + \int_{|w|>1} \frac{\langle w \rangle^{-N}}{\langle R+|w| \rangle \langle R-|w| \rangle} \lesssim \frac{1}{\langle R \rangle^2}.$$

The region where $|y-w| < \frac{1}{2}|y|$ is best integrated in spherical coordinates centered at the point y. This leads to the expression

$$(28) \qquad \int_0^{\frac{1}{2}|y|} \frac{1}{r^\alpha \langle r+R \rangle \langle r-R \rangle} \int_{|y-w|=r} \langle w \rangle^{-N} \, dw \, dr,$$

which is precisely handled in [12, equation (31)] provided $\alpha < n-1$, with the bound

$$|(28)| \lesssim \frac{1}{\langle y \rangle^{N-n-1+\alpha} \langle |y|+R \rangle \langle R-|y| \rangle}.$$

$\square$

Lemma A.3 ([12], Lemma 5.2)**.** *Let $R_0^\pm(\lambda^2, A)$ denote the convolution kernel of $R_0^\pm(\lambda^2)$ evaluated at a point with $|x| = A$. For each $j \geq 0$,*

$$(29) \qquad \int_0^\infty R_0^+(\lambda^2, A)(R_0^+ - R_0^-)(\lambda^2, B)\lambda^{j-1}(\log \lambda)^\ell \tilde{\Phi}(\lambda) \, d\lambda \lesssim \frac{\langle \log\langle A-B \rangle \rangle^\ell}{A^{n-2}\langle A+B \rangle \langle A-B \rangle^{n-3+j}}.$$

Lemmas 2.5 and A.3 rely on the following oscillatory integral estimate.

Lemma A.4 ([12], Lemma 4.1)**.** *Suppose there exists $\beta > -1$ and $M > \beta + 1$ such that $|F^{(k)}(\lambda)| \lesssim \lambda^{\beta-k}$ for all $0 \leq k \leq M$. Then given a smooth cutoff function $\tilde{\Phi}$,*

$$(30) \qquad \left| \int_0^\infty e^{i\rho\lambda} F(\lambda)\tilde{\Phi}(\lambda) \, d\lambda \right| \lesssim \langle \rho \rangle^{-\beta-1}.$$

If F is further assumed to be smooth and supported in the annulus $L \lesssim \lambda \lesssim 1$ for some $L > \rho^{-1} > 0$, then

$$(31) \qquad \left| \int_0^\infty e^{i\rho\lambda} F(\lambda) \tilde{\Phi}(\lambda) \, d\lambda \right| \lesssim \langle\rho\rangle^{-M} L^{\beta+1-M}.$$

One additional variation is needed specifically to assist in the proof of (25) for large x.

Lemma A.5. *In four dimensions we have the bound*

$$(32) \qquad \int_0^\infty \partial_A R_0^+(\lambda^2, A)\big(R_0^+ - R_0^-\big)(\lambda^2, B)\lambda^{-1}\tilde{\Phi}(\lambda) \, d\lambda \lesssim \begin{cases} \frac{1}{A^3\langle A\rangle^2} & \text{if } A > 2B \\[2mm] \frac{1}{A^3\langle B\rangle^2} & \text{if } B > 2A \\[2mm] \frac{1}{A^3\langle A-B\rangle^2} & \text{if } A \approx B \end{cases}.$$

Proof. We recall the expansion for the free resolvent, see Section 4 of [12],

$$(33) \qquad R_0^\pm(\lambda^2, A) = \frac{1}{A^2}\Omega(\lambda A) + \frac{e^{\pm i\lambda A}}{A^2}\Psi_{\frac{1}{2}}(\lambda A),$$

where Ω is a bounded compactly supported function that is smooth everywhere except possibly at zero, and each $\Psi_{\frac{1}{2}}$ is a smooth function supported outside the unit interval that asymptotically behaves like $(\,\cdot\,)^{1/2}$ and whose k^{th} derivative behaves like $(\,\cdot\,)^{(1-2k)/2}$. This expansion follows from writing $R_0^\pm(\lambda^2, r) = \frac{-\lambda}{8\pi r}[Y_1(\lambda r) \mp iJ_1(\lambda r)]$ and using the asymptotic properties of the Bessel functions, see [1]. Additional properties include

$$(34) \qquad R_0^+(\lambda^2, B) - R_0^-(\lambda^2, B) = \lambda^2\Omega(\lambda B) + \frac{e^{i\lambda B}}{B^2}\Psi_{\frac{1}{2}}(\lambda B) + \frac{e^{-i\lambda B}}{B^2}\Psi_{\frac{1}{2}}(\lambda B),$$

$$(35) \qquad \partial_A R_0^\pm(\lambda^2, A) = \frac{e^{i\lambda A}}{A^2}\lambda\Psi_{\frac{1}{2}}(\lambda A) + \frac{1}{A^3}\widetilde{\Omega}(\lambda A).$$

Here $\widetilde{\Omega}(\lambda A)$ is a compactly supported function that satisfies $|\partial_\lambda^j \widetilde{\Omega}(\lambda A)| \lesssim \lambda^{-j}$.

Using the bounds proven in Lemma 2.2 in [12] which were considered when $j = 1$ (and $n = 4$), we have

$$\int_0^\infty \frac{e^{i\lambda A}}{A^2}\Psi_{\frac{1}{2}}(\lambda A)\left(\lambda^2\Omega(\lambda B) + \frac{e^{\pm i\lambda B}}{B^2}\Psi_{\frac{1}{2}}(\lambda B)\right)\tilde{\Phi}(\lambda) \, d\lambda \lesssim \frac{1}{A^2\langle A+B\rangle\langle A-B\rangle^2}.$$

This satisfies the desired bounds, thus we need only bound the contribution of the terms

$$\int_0^\infty \frac{1}{A^3}\widetilde{\Omega}(\lambda A)\left(\lambda^2\Omega(\lambda B) + \frac{e^{\pm i\lambda B}}{B^2}\Psi_{\frac{1}{2}}(\lambda B)\right) \, d\lambda.$$

The first term is supported on $\lambda \lesssim \min(1, A^{-1}, B^{-1})$, and hence is bounded by $A^{-3}\min(\langle A\rangle^{-2}, \langle B\rangle^{-2})$. Using Lemma A.4 with $\rho = B$ and $L = \langle B\rangle^{-1}$, the second term is bounded by $A^{-3}\langle B\rangle^{-2}$ and is zero unless $A \lesssim B$. $\qquad\square$

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