STRICHARTZ AND SMOOTHING ESTIMATES FOR SCHRÖDINGER OPERATORS WITH ALMOST CRITICAL MAGNETIC POTENTIALS IN THREE AND HIGHER DIMENSIONS.

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Abstract. In this paper we consider Schrödinger operators

$$H = -\Delta + i(A\cdot\nabla + \nabla\cdot A) + V = -\Delta + L$$

in $\mathbb{R}^n$, $n \geq 3$. Under almost optimal conditions on A and V both in terms of decay and regularity we prove smoothing and Strichartz estimates, as well as a limiting absorption principle. For large gradient perturbations the latter is not an immediate corollary of the free case as $T(\lambda) := L(-\Delta-(\lambda^2+i0))^{-1}$ is not small in operator norm on weighted L^2 spaces as $\lambda\to\infty$. We instead deduce the existence of inverses $(I+T(\lambda))^{-1}$ by showing that the spectral radius of $T(\lambda)$ decreases to zero. In particular, there is an integer m such that $\limsup_{\lambda\to\infty}\|T(\lambda)^m\| < \frac{1}{2}$. This is based on an angular decomposition of the free resolvent for which we establish the limiting absorption bound

$$\|D^\alpha \mathcal{R}_{d,\delta}(\lambda^2)f\|_{B^*} \leq C_n\lambda^{-1+|\alpha|}\|f\|_B \tag{0.1}$$

where $0\leq|\alpha|\leq 2$, B is the Agmon-Hörmander space, and $\mathcal{R}_{d,\delta}(\lambda^2)$ is the free resolvent operator at energy λ^2 whose kernel is restricted in angle to a cone of size δ and by d away from the diagonal $x = y$. The main point is that C_n only depends on the dimension, but not on the various cut-offs. The proof of (0.1) avoids the Fourier transform and instead uses Hörmander's variable coefficient Plancherel theorem for oscillatory integrals.

1. Introduction

In this paper we prove Strichartz and smoothing bounds for the magnetic Schrödinger operator on $L^2(\mathbb{R}^n)$

$$H = -\Delta + i(A\cdot\nabla + \nabla\cdot A) + V = -\Delta + L \tag{1.1}$$

under almost optimal assumptions on the large perturbations A and V. As usual we will assume that zero energy is neither an eigenvalue nor a resonance. This means that the perturbed resolvent $(H-z)^{-1}$ remains bounded on the weighted spaces $L^{2,1+}\to L^{2,1-}$ as $z\to 0$, $\Im z > 0$. This condition is equivalent (assuming sufficient decay on A, V) to the absence of nonzero solutions f of $Hf = 0$ with $f\in L^{2,\frac{n-4}{2}-}$. When $n\geq 5$ any such

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solution belongs to $L^2(\mathbb{R}^n)$ itself, so it suffices to check that zero energy is not an eigenvalue.

Theorem 1.1. *Let A and V be real-valued such that for all $x \in \mathbb{R}^n$, $n \geq 3$, and some fixed but arbitrary $\varepsilon > 0$ and all sufficiently small $0 < \varepsilon' < \varepsilon$,*

$$|A(x)| + \langle x\rangle |V(x)| \lesssim \langle x\rangle^{-1-\varepsilon} \tag{1.2}$$

$$\langle x\rangle^{1+\varepsilon'} A(x) \in \dot{W}^{\frac{1}{2},2n}(\mathbb{R}^n) \tag{1.3}$$

$$A \in C^0(\mathbb{R}^n) \tag{1.4}$$

Furthermore, assume that zero energy is neither an eigenvalue nor a resonance of H. Then, with P_c being the projection onto the continuous spectrum,

$$\|e^{itH}P_c f\|_{L^q_t(L^p_x)} \lesssim \|f\|_{L^2(\mathbb{R}^n)} \tag{1.5}$$

provided $\frac{2}{q} + \frac{n}{p} = \frac{n}{2}$ and $2 \leq p < \frac{2n}{n-2}$. Moreover, the Kato smoothing estimates

$$\begin{aligned} &\int_0^\infty \big\|\langle x\rangle^{-\sigma} |\nabla|^{\frac{1}{2}} e^{itH} P_c f\big\|_2^2\, dt \leq C\|f\|_2^2 \\ &\int_0^\infty \big\|\langle x\rangle^{-2\sigma} \langle\nabla\rangle^{\frac{1}{2}} e^{itH} P_c f\big\|_2^2\, dt \leq C\|f\|_2^2 \end{aligned} \tag{1.6}$$

hold with $\sigma > \frac{1}{2}$.

The secondary condition (1.3) deals with the regularity of A but does not impose any extra decay beyond what is assumed in (1.2). Note that $\langle x\rangle^{1+\varepsilon'}A(x)$ must decay like $\langle x\rangle^{-(\varepsilon-\varepsilon')}$ which already belongs to $\dot{W}^{\frac{1}{2},2n}(\mathbb{R}^n)$. A stronger, but more easily verifiable hypothesis would be to require A to be Lipschitz continuous with $|\nabla A(x)| \lesssim \langle x\rangle^{-2-\varepsilon}$. However stated, this condition permits the commutation of A with $|\nabla|^{\frac{1}{2}}$, which is essential to our factorization of L into pairs of pseudo-differential operators each having order $\frac{1}{2}$. The continuity assumption (1.4) is required for our treatment of large energies. To relax it, one needs to carry out some of our large energy analysis on spaces other than the L^2 based spaces B, B^* which we use here. See [13] for such work on Stein-Tomas type spaces.

Since $L^1 \to L^\infty$ dispersive bounds are currently unknown for any $A \neq 0$, we cannot follow the usual interpolation method. Instead, we adopt an argument introduced in [21], where the validity of Strichartz inequalities is instead derived from Kato's theory of smooth perturbations. This paper is related to our three-dimensional paper [8], where a result similar to Theorem 1.1 was proved but under much stronger conditions on A, V, both in terms of decay as well as regularity. In [24] and [10] Strichartz and smoothing estimates were obtained for *small* A and V, and in [7] smallness is not required for the positive part of $|A|^2 + V$. They are also known when the free Laplacian is taken with respect to a nontrapping asymptotically flat manifold, as in [3], [22]. Finally, [18] states a Strichartz estimate for time-dependent first order perturbations of such a Schrödinger operator, subject to (the existence of and) a pointwise bound on $\operatorname{div} A$ and an *a priori* local smoothing condition on the solution. In our setting the latter is enforced as a consequence of projection onto the continuous spectrum and the low-energy arguments in Section 5.

For more background and many references on magnetic operators see Erdös's survey [9].

The approach of this work is perturbative around the free case despite the fact that we make no smallness assumption. The main novel ingredient in this paper is a limiting absorption estimate for large energies on almost optimal weighted spaces. To see the difficulty with large energies, recall that in [2] and [13] it is proved that for H as in (1.1) under suitable decay conditions on A and V and with $\sigma > \frac{1}{2}$,

$$\sup_{\lambda\in[\delta,\delta^{-1}]} \|\langle x\rangle^{-\sigma}\langle\nabla\rangle(H-(\lambda^2+i0))^{-1}\langle\nabla\rangle\langle x\rangle^{-\sigma}\|_{2\to 2} \leq C(\delta) < \infty \tag{1.7}$$

provided there are no imbedded eigenvalues in the continuous spectrum (which is known due to recent work by Koch and Tataru [17]). It is well–known that this *limiting absorption principle* is of fundamental importance for proving dispersive estimates, at least for the case of large potentials. However, one needs to consider all real λ instead of restricting to a compact interval in the positive halfline. To extend (1.7) toward zero energies is similar to the case $A = 0$. This step requires the assumption on zero energy.

Note that (1.7) as stated cannot be extended to a semi-infinite interval since it would fail even for the free resolvent. Indeed, with $\sigma > \frac{1}{2}$

$$\|\langle x\rangle^{-\sigma}\langle\nabla\rangle^{\alpha}(H_0-(\lambda^2+i0))^{-1}\langle\nabla\rangle^{\alpha}\langle x\rangle^{-\sigma}\|_{2\to 2} \sim \lambda^{2\alpha-1} \tag{1.8}$$

for any $\alpha \in [0,1]$ and all $\lambda > 1$. This shows that no more than one derivative in total can be gained here while still preserving a uniform upper bound. Furthermore, in the borderline case $\alpha = \frac{1}{2}$ there is no decay of the operator norm in the limit $\lambda \to \infty$. This is the main difficulty we face when A and λ are large.

We will adopt the shorthand notation

$$R_0(z) := (H_0 - z)^{-1}$$

for the resolvent of the Laplacian. The resolvent of a general operator H will be indicated by $R_H(z)$, or else $R_L(z)$ in the case where H is specifically of the form $H_0 + L$. Formally, the relationship between R_L and R_0 is captured in the identity

$$R_L(z) = (I + R_0(z)L)^{-1}R_0(z).$$

In this paper we extend (1.8) to $H = H_0 + L$ for the class of first-order perturbations described in Theorem 1.1. A unified statement of the mapping properties of the resolvent of H over the entire spectrum $\lambda > 0$ is as follows.

Theorem 1.2. *Suppose H is a magnetic Schrödinger operator whose potentials satisfy the conditions $\langle x\rangle^{1+\varepsilon}(|A|+|V|)\in L^\infty(\mathbb{R}^n)$ with A being continuous. Then for $\sigma>\frac{1}{2}$ and $\alpha\in[0,\frac{1}{2}]$,*

$$\sup_{\lambda>1}\lambda^{1-2\alpha}\|\langle x\rangle^{-\sigma}\langle\nabla\rangle^{\alpha}(H-(\lambda^2+i0))^{-1}\langle\nabla\rangle^{\alpha}\langle x\rangle^{-\sigma}\|_{2\to2}\lesssim 1. \tag{1.9}$$

If one further assumes that A, V satisfy the full conditions of Theorem 1.1, and zero is not an eigenvalue or resonance of H, then this bound can be extended to $\lambda\in[0,\infty]$ in either of two ways:

$$\begin{aligned}\sup_{\lambda\geq0}\langle\lambda\rangle^{1-2\alpha}\|\langle x\rangle^{-\sigma}|\nabla|^{\alpha}(H-(\lambda^2+i0))^{-1}|\nabla|^{\alpha}\langle x\rangle^{-\sigma}\|_{2\to2}&\lesssim 1\\ \sup_{\lambda\geq0}\langle\lambda\rangle^{1-2\alpha}\|\langle x\rangle^{-2\sigma}\langle\nabla\rangle^{\alpha}(H-(\lambda^2+i0))^{-1}\langle\nabla\rangle^{\alpha}\langle x\rangle^{-2\sigma}\|_{2\to2}&\lesssim 1\end{aligned} \tag{1.10}$$

As a consequence, the spectrum of H is purely absolutely continuous over the entire interval $[0,\infty)$, and both the operators $\langle x\rangle^{-\sigma}|\nabla|^{\frac{1}{2}}$ and $\langle x\rangle^{-2\sigma}\langle\nabla\rangle^{\frac{1}{2}}$ are H-smooth on this interval.

Remark 1.3. In fact, (1.9) is valid for any $\alpha\in[0,1]$, and with a somewhat wider class of potentials than is described here. This is made evident in the statement and proof of Corollary 4.4. Furthermore, it is only necessary to verify the first assertion in (1.10). The second line follows immediately because the operator $\langle x\rangle^{-2\sigma}\langle\nabla\rangle^{\frac{1}{2}}|\nabla|^{-\frac{1}{2}}\langle x\rangle^{\sigma}$ and its transpose are bounded on L^2, see Lemma 5.1.

Remark 1.4. A result of type (1.9), in the case $\alpha=0$, is proved in [20] using the method of Mourre commutators and micro-local analysis. In that work the potentials require only very slight polynomial decay, however they are also assumed to be infinitely differentiable, with the derivatives satisfying a symbol-like decay condition.

This paper is organized as follows: In Section 2 we present the reduction of the Strichartz estimates to the Kato smoothing property [15]. More precisely, we are reduced to proving that $Z_0:=\langle x\rangle^{-\sigma}|\nabla|^{\frac{1}{2}}$ is smoothing relative to H for $\sigma>\frac{1}{2}$ (it is a classical result that Z_0 is smoothing relative to H_0). In Section 3 we establish our main technical ingredient, i.e., the limiting absorption principle for the angularly truncated free resolvent kernel. It is essential here that the bound does not deteriorate as the size of the truncation decreases to zero.

In Section 4 we use this bound to prove a limiting absorption principle for the perturbed resolvent via the "power method", i.e., by showing that $(LR_0)^m$ has small norm for large energies and large m. The idea is to write this power as a sum of products involving conically restricted free resolvents and to obtain a gain for both the "directed" (where all the factors have almost aligned cones) and the "undirected" summands. In the former case this takes the form of a Volterra-type gain, whereas in the latter one exploits a gain coming from angular separation (for this one needs Schwartz potentials and general A that are approximated by Schwartz functions; it is here that $A \in C(\mathbb{R}^n)$ is needed).

Finally, Section 5 presents the low energy case. Although this is similar to the case of $A = 0$ in that we use Fredholm's alternative and a Neumann series, it does have some challenges of its own mainly in form of commutator estimates. Finally, in the appendix we collect some tools from harmonic analysis.

2. The basic setup

The Strichartz estimates stated in Theorem 1.1 will be proved using Proposition 2.1 below, which was proved in [21], see Theorem 4.1 in that paper. It is based on Kato's notion of smoothing operators, see [15]. We recall that for a self-adjoint operator H, an operator Γ is called H-smooth in Kato's sense if for any $f \in \mathcal{D}(H_0)$

$$\|\Gamma e^{itH} f\|_{L^2_t L^2_x} \leq C_\Gamma(H) \|f\|_{L^2_x} \tag{2.1}$$

or equivalently, for any $f \in L^2_x$

$$\sup_{\varepsilon>0} \|\Gamma R_H(\lambda \pm i\varepsilon) f\|_{L^2_\lambda L^2_x} \leq C_\Gamma(H) \|f\|_{L^2_x}. \tag{2.2}$$

We shall call $C_\Gamma(H)$ the smoothing bound of Γ relative to H. Let $\Omega \subset \mathbb{R}$ and let P_Ω be a spectral projection of H associated with a set Ω. We say that Γ is H-smooth on Ω if ΓP_Ω is H-smooth. We denote the corresponding smoothing bound by $C_\Gamma(H, \Omega)$. It is not difficult to show (see e.g. [19]) that, equivalently, Γ is H-smooth on Ω if

$$\sup_{\beta>0} \|\chi_\Omega(\lambda) \Gamma R_H(\lambda \pm i\beta) f\|_{L^2_\lambda L^2_x} \leq C_\Gamma(H, \Omega) \|f\|_{L^2_x}. \tag{2.3}$$

The estimate (1.5) of Theorem 1.1 is obtained by means of the following result. The remainder of the paper is devoted to verifying the conditions needed in Proposition 2.1. Furthermore, this verification will establish the smoothing estimate (1.6).

Proposition 2.1. *Let $H_0 = -\Delta$ and $H = H_0 + L$ with $L = \sum_{j=1}^J Y_j^* Z_j$. We assume that each Y_j is H_0-smooth with a smoothing bound $C_B(H_0)$ and that for some $\Omega \subset \mathbb{R}$ the operators Z_j are H-smooth on Ω with the smoothing bound $C_A(H, \Omega)$. Assume also that the unitary semigroup e^{itH_0} satisfies the estimate*

$$\|e^{itH_0} \psi_0\|_{L^q_t L^r_x} \leq C_{H_0} \|\psi_0\|_{L^2_x} \tag{2.4}$$

for some $q \in (2, \infty]$ and $r \in [1, \infty]$. Then the semigroup e^{itH} associated with $H = H_0 + L$, restricted to the spectral set Ω, also verifies the estimate (2.4)*, i.e.,*

$$\|e^{itH} P_\Omega \psi_0\|_{L^q_t L^r_x} \leq J C_{H_0} C_B(H_0) C_A(H, \Omega) \|\psi_0\|_{L^2_x} \tag{2.5}$$

We refer the reader to [21] for the proof.

To apply this proposition we write, with a decreasing weight $w(x) = \langle x\rangle^{-\tau}$ chosen from the range $\tau \in (\frac{1}{2}, \frac{1}{2} + \varepsilon')$,

$$i(A \cdot \nabla + \nabla \cdot A) = \sum_{j=1}^{2} Y_j^* Z_j, \quad V = Y_3^* Z_3 \tag{2.6}$$

where

$$\begin{aligned} &Y_1^* := iAw^{-1} \cdot \nabla|\nabla|^{-\frac{1}{2}}, \quad Z_1 := |\nabla|^{\frac{1}{2}} w \\ &Y_2 := Z_1, \quad Z_2 := Y_1, \quad Y_3 = |V|^{\frac{1}{2}} \mathrm{sign}\, V, \quad Z_3 = |V|^{\frac{1}{2}} \end{aligned} \tag{2.7}$$

Note that the cross-term produced by $Y_1^* Z_1$ is point-wise multiplication by the purely imaginary function $i(Aw^{-1} \cdot (\nabla w))$. It is canceled by the corresponding cross-term in $Y_2^* Z_2$.

We now reduce the smoothing properties of Y_j and Z_j, $1 \le j \le 3$, relative to H_0 and H, respectively, to the smoothing properties of

$$Z_0 := \langle x\rangle^{-\sigma} |\nabla|^{\frac{1}{2}}, \tag{2.8}$$

where σ is chosen so that $\frac{1}{2} < \sigma < \tau$. It is standard that Z_0 is smoothing relative to H_0. Theorem 1.2, once proven, demonstrates that Z_0 is also smoothing relative to H. We first state a technical lemma which explains the role of our regularity assumption (1.3).

Lemma 2.2. *Let A, ε' be as in Theorem 1.1. Then the operator*

$$|\nabla|^{\frac{1}{2}} A \langle x\rangle^{1+\varepsilon'} |\nabla|^{-\frac{1}{2}}$$

is bounded on L^2.

Proof. This is a straightforward application of the fractional Leibniz rule in Lemma 6.3.

$$\begin{aligned} \| \, |\nabla|^{\frac{1}{2}} \widetilde{A} f \|_2 &\lesssim \|\widetilde{A}\|_\infty \| \, |\nabla|^{\frac{1}{2}} f \|_2 + \| \, |\nabla|^{\frac{1}{2}} \widetilde{A} \|_{2n} \|f\|_{\frac{2n}{n-1}} \\ &\lesssim (\|\widetilde{A}\|_\infty + \| \, |\nabla|^{\frac{1}{2}} \widetilde{A}\|_{2n}) \| \, |\nabla|^{\frac{1}{2}} f \|_2 \lesssim \| |\nabla|^{\frac{1}{2}} f\|_2 \end{aligned}$$

which is equivalent to $|\nabla|^{\frac{1}{2}} \widetilde{A} |\nabla|^{-\frac{1}{2}} : L^2 \to L^2$. $\square$

Returning to our discussion of the decomposition of L, observe that

$$\begin{aligned} Z_1 &= \big(|\nabla|^{\frac{1}{2}} w |\nabla|^{-\frac{1}{2}} \langle x\rangle^\sigma\big) \langle x\rangle^{-\sigma} |\nabla|^{\frac{1}{2}} =: S_1 Z_0 \\ Z_2 &= i\big(\nabla|\nabla|^{-\frac{1}{2}} A w^{-1} |\nabla|^{-\frac{1}{2}} \langle x\rangle^\sigma\big) \langle x\rangle^{-\sigma} |\nabla|^{\frac{1}{2}} =: S_2 Z_0 \\ Z_3 &= \big(|V|^{\frac{1}{2}} |\nabla|^{-\frac{1}{2}} \langle x\rangle^\sigma\big) \langle x\rangle^{-\sigma} |\nabla|^{\frac{1}{2}} =: S_3 Z_0 \end{aligned}$$

with S_1 being L^2 bounded by Lemma 6.2 in the appendix. Similarly, S_3 can be expanded as

$$S_3 = \big(|V|^{\frac{1}{2}} w^{-1} |\nabla|^{-\frac{1}{2}}\big) S_1$$

and the operator in parentheses is bounded on L^2 by fractional integration. For S_2, we need to invoke the local regularity of A:

$$-iS_2 = \nabla|\nabla|^{-\frac{1}{2}} A w^{-1} \langle x\rangle^{(1+\varepsilon')-\tau} |\nabla|^{-\frac{1}{2}} \big(|\nabla|^{\frac{1}{2}} \langle x\rangle^{\tau-(1+\varepsilon')} |\nabla|^{-\frac{1}{2}} \langle x\rangle^{\sigma} \big),$$

and the operator in parentheses is again L^2 bounded by Lemma 6.2, whereas, by (1.3) we can rewrite the remaining expression on the right-hand side as

$$\nabla|\nabla|^{-\frac{1}{2}} A \langle x\rangle^{1+\varepsilon'} |\nabla|^{-\frac{1}{2}} = \sum_{j=1}^{n} \partial_j |\nabla|^{-1} |\nabla|^{\frac{1}{2}} A \langle x\rangle^{1+\varepsilon'} |\nabla|^{-\frac{1}{2}}.$$

The sum here is L^2 bounded; indeed, obviously the Riesz transforms $\partial_j|\nabla|^{-1}$ are L^2 bounded and now apply Lemma 2.2. In conclusion it will suffice to prove that Z_0 is H-smooth.

Let us first consider intermediate energies λ^2, i.e., $\lambda \in [\lambda_0, \lambda_1] = \mathcal{J}_0$ with λ_0 small and λ_1 large. Then it was shown in [13], see also [2], that the resolvent of H satisfies the following bound

$$\sup_{\lambda\in\mathcal{J}_0} \|\langle x\rangle^{-\sigma} \langle\nabla\rangle R_L(\lambda^2 + i0) f\|_2 \le C(\lambda_0, \lambda_1)\, \|\langle x\rangle^{\sigma} \langle\nabla\rangle^{-1} f\|_2$$

(in fact, a stronger bound was proved in [13]). More precisely, this bound follows provided there are no eigenvalues of H in the interval $\mathcal{J}_0$. The latter property (absence of imbedded eigenvalues) is shown in [17] to hold for the entire family of potentials under consideration. It is not difficult to replace the derivative $\langle\nabla\rangle$ with $|\nabla|$, since the operator $|\nabla|\langle\nabla\rangle^{-1}$ is bounded on a wide range of weighted L^2 spaces, see Lemma 5.1. Thus,

$$\sup_{\lambda\in\mathcal{J}_0} \|Z_0 R_L(\lambda^2 + i0) Z_0^*\|_{2\to 2} \le C(\lambda_0, \lambda_1)$$

Finally, by Kato's smoothing theory, see [19] Theorem XIII.30, we conclude that Z_0 is H-smooth on $\Omega = \mathcal{J}_0$ as desired. In the following two sections we treat the case of large energies, which takes up the most work. The small energy case is then treated in Section 5. Finally, in the appendix we collect some bounds from harmonic analysis. Although they can all be found in the literature in some form, the specific version required here appears to be somewhat different.

3. The Directed Resolvent Estimate

This section, which can be read independently of the other sections, presents a limiting absorption estimate for the truncated free resolvent kernel. The crucial point is that the constants in our estimate do not depend on the truncation. Our main tool is Hörmander's variable coefficient Plancherel theorem from the appendix.

The kernel of the free resolvent $R_0^+(\lambda^2)$ in $\mathbb{R}^n$ is given by[1]

$$R_0^+(\lambda^2)(x,y) = C_n \frac{\lambda^{\frac{n-2}{2}}}{|x-y|^{\frac{n-2}{2}}} H^+_{\frac{n-2}{2}}(\lambda|x-y|)$$

where H^+_ν is a Hankel function. There is the scaling relation

$$R_0^+(\lambda^2)(x,y) = \lambda^{n-2} R_0^+(1)(\lambda x, \lambda y) \quad \forall\, \lambda > 0 \tag{3.1}$$

and the representation, see the asymptotics of H^+_ν in [1],

$$R_0^+(1)(x,y) = \frac{e^{i|x-y|}}{|x-y|^{\frac{n-1}{2}}} a(|x-y|) + \frac{b(|x-y|)}{|x-y|^{n-2}} \tag{3.2}$$

provided $n \geq 3$ where

$$|a^{(k)}(r)| \lesssim r^{-k} \quad \forall\, k \geq 0, \quad a(r) = 0 \quad \forall\, 0 < r < 1 \tag{3.3}$$

and $b(r) = 0$ for all $r > 2$, with

$$|b^{(k)}(r)| \lesssim 1 \quad \forall\, k \geq 0, \quad n \text{ odd} \tag{3.4}$$

$$\left.\begin{array}{lll} |b^{(k)}(r)| & \lesssim 1 & \forall\, 0 \leq k < n-2 \\ |b^{(k)}(r)| & \lesssim r^{n-k-2}|\log r| & \forall\, k \geq n-2 \end{array}\right\} \; n \geq 4 \text{ even} \tag{3.5}$$

for all $r > 0$. As in Chapter XIV of [11] define

$$\|f\|_B := \sum_{j=0}^\infty 2^{\frac{j}{2}} \|f\|_{L^2(D_j)}, \quad \|f\|_{B^*} := \sup_{j\geq 0} 2^{-\frac{j}{2}} \|f\|_{L^2(D_j)}$$

where $D_j = \{x \,:\, |x| \sim 2^j\}$ for $j \geq 1$ and $D_0 = \{|x| \leq 1\}$.

Lemma 3.1. *For any $\lambda \geq 1$,*

$$\|f(\lambda^{-1}\cdot)\|_B \lesssim \lambda^{\frac{n+1}{2}} \|f\|_B, \quad \|g(\lambda\cdot)\|_{B^*} \lesssim \lambda^{-\frac{n-1}{2}} \|g\|_{B^*}$$

provided the right-hand sides are finite.

Proof. By duality, it suffices to prove the first estimate. Assume without loss of generality that $\lambda = 2^N$ for some $N \geq 0$. Then

$$\|f(\lambda^{-1}\cdot)\|_B \lesssim \sum_{j=N}^\infty 2^{\frac{j}{2}} 2^{\frac{nN}{2}} \|f\|_{L^2(D_{j-N})} + 2^{\frac{N}{2}} 2^{\frac{nN}{2}} \|f\|_{L^2(D_0)} \lesssim 2^{\frac{N(n+1)}{2}} \|f\|_B$$

as claimed. $\square$

This lemma and the scaling relation (3.1) immediately imply the following statement. In what follows, R_0 stands for either of $R_0^\pm$.

Corollary 3.2. *If $R_0(1) \,:\, B \to B^*$, then*

$$\|R_0(\lambda^2)\|_{B\to B^*} \lesssim \lambda^{-1} \|R_0(1)\|_{B\to B^*}$$

for all $\lambda \geq 1$.

[1] Constants C_n are allowed to change from line to line.

Proof. First, from (3.1)

$$(R_0^+(\lambda^2)f)(x) = \lambda^{-2}[R_0^+(1)f(\cdot\lambda^{-1})](\lambda x)$$

Hence, by the previous lemma,

$$\|R_0^+(\lambda^2)f\|_{B^*} \lesssim \lambda^{-2}\lambda^{-\frac{n-1}{2}}\|R_0^+(1)f(\cdot\lambda^{-1})\|_{B^*} \lesssim \lambda^{-1}\|R_0^+(1)f\|_{B^*}$$

as claimed. □

For any $\delta \in (0,1)$, let Φ_δ be a smooth cut-off function to a δ-neighborhood of the north pole in S^{n-1}. Also, for any $d \in (0,\infty)$, $\eta_d(x) = \eta(|x|/d)$ denotes a smooth cut-off to the set $|x| > d$. In what follows, we shall use the notation

$$\mathcal{R}_{d,\delta}(\lambda^2)(x,y) = [R_0(\lambda^2)\eta_d\Phi_\delta](x,y) = R_0(\lambda^2)(x,y)\eta_d(|x-y|)\Phi_\delta\Big(\frac{x-y}{|x-y|}\Big)$$

Note that this operator obeys the same scaling as R_0, see (3.1). More precisely,

$$\mathcal{R}_{d,\delta}(\lambda^2)(x,y) = \lambda^{n-2}\mathcal{R}_{d\lambda,\delta}(1)(\lambda x, \lambda y)$$

Thus, Corollary 3.2 applies to $\mathcal{R}_{d,\delta}(\lambda^2)$ in the form

$$\|\mathcal{R}_{d,\delta}(\lambda^2)\|_{B\to B^*} \lesssim \lambda^{-1}\|\mathcal{R}_{d\lambda,\delta}(1)\|_{B\to B^*} \tag{3.6}$$

for all $\lambda \geq 1$ or, more generally,

$$\|D^\alpha\mathcal{R}_{d,\delta}(\lambda^2)\|_{B\to B^*} \lesssim \lambda^{-1+|\alpha|}\|D^\alpha\mathcal{R}_{d\lambda,\delta}(1)\|_{B\to B^*} \tag{3.7}$$

for all multi-indices α and $\lambda \geq 1$.

The main goal of this section is to prove a limiting absorption bound for $\mathcal{R}_{d,\delta}$ and its derivatives of order at most two uniformly in the parameters $d, \delta \in (0,1)$, see Proposition 3.5 below. This will be based on the oscillatory integral estimate in Lemma 3.4. We first state a simple technical fact which will be used repeatedly.

Lemma 3.3. *Let $K(x,y)$ be the kernel of the L^2 bounded operator $T : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^m)$ with*

$$(Tf)(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\,dy$$

Let $L_1 : \mathbb{R}^n \to \mathbb{R}^n$ and $L_2 : \mathbb{R}^m \to \mathbb{R}^m$ be invertible linear transformations and define

$$(\widetilde{T}f)(x) = \int_{\mathbb{R}^n} K(L_1x, L_2y)f(y)\,dy$$

Then

$$\sqrt{|\det L_1||\det L_2|}\ \|\widetilde{T}\|_{2\to 2} = \|T\|_{2\to 2}$$

The following lemma is the main technical tool of this section.

Lemma 3.4. *Let χ denote a smooth cut-off function to the region $1 < |x| < 2$. With $a(r)$ as in (3.3), define*

(3.8)
$$(T_{\delta,p,R_1,R_2}f)(x) = \int \chi\big(\frac{x}{R_1}\big)\frac{e^{i|x-y|}}{|x-y|^p}a(|x-y|)\Phi_\delta\Big(\frac{x-y}{|x-y|}\Big)\chi\big(\frac{y}{R_2}\big)f(y)\,dy$$

Then, for any $n \geq 3$, and $\frac{n-1}{2} \leq p \leq \frac{n+3}{2}$,

$$\|T_{\delta,p,R_1,R_2}f\|_2 \leq C_n\,\delta^{p-\frac{n-1}{2}}\sqrt{R_1R_2}\,\|f\|_2 \tag{3.9}$$

for all $R_1, R_2 \geq 1$, $\delta \in (0,1)$. The constant C_n only depends on $n \geq 3$.

Proof. We first consider the cases where $R_2 > 4R_1$ or $R_1 > 4R_2$. By duality it suffices to treat the first case. We then distinguish two further cases, depending on whether $\delta R_2 > R_1$ or not.

Case 1: $\delta R_2 > R_1$
On the support of the integrand in (3.8), we have

$$|y'| \lesssim \delta R_2,\ y_n \sim R_2,\ |x| \lesssim R_1$$

where $y = (y', y_n)$. By the change of variables $y = (\delta R_2 v', R_2 v_n)$ and $x = R_1 u$ and Lemma 3.3,

(3.10)
$$\begin{aligned}&\|T_{\delta,p,R_1,R_2}\|_{2\to2}\\ &= \Big\|e^{iR_2\left|\frac{R_1}{R_2}u-(\delta v',v_n)\right|}\frac{\chi(u)\chi(\delta v',v_n)(a\Phi_\delta)(u,v)}{\left|\frac{R_1}{R_2}u-(\delta v',v_n)\right|^p}\Big\|_{L^2_v\to L^2_u}R_1^{\frac{n}{2}}R_2^{\frac{n}{2}-p}\delta^{\frac{n-1}{2}}\end{aligned}$$

where

$$(a\Phi_\delta)(u,v) = a(|R_1u-(R_2\delta v', R_2v_n)|)\Phi_\delta\Big(\frac{R_1u-(R_2\delta v',R_2v_n)}{|R_1u-(R_2\delta v',R_2v_n)|}\Big).$$

We will apply Proposition 6.1 to the operator in (3.10). First note that the derivatives of

$$\frac{\chi(u)\chi(\delta v',v_n)(a\Phi_\delta)(u,v)}{\left|\frac{R_1}{R_2}u-(\delta v',v_n)\right|^p}$$

in u' are bounded using the property that $\frac{R_1}{\delta R_2} \lesssim 1$, the symbol-like decay of a, the bounds $|D^\alpha\Phi_\delta| \lesssim \delta^{-|\alpha|}$ and the bound

$$\Big|v_n - \frac{R_1}{R_2}u_n\Big| \sim 1. \tag{3.11}$$

Second, the phase $\Psi(u,v) = \left|\frac{R_1}{R_2}u - (\delta v', v_n)\right|$ satisfies the hypothesis of Proposition 6.1. Indeed,

$$\nabla_{u'}\Psi(u',u_n,v',v_n) = \frac{R_1}{R_2}\frac{(\frac{R_1}{R_2}u'-\delta v',0)}{\left|\frac{R_1}{R_2}u-(\delta v',v_n)\right|} = \frac{R_1}{R_2}\frac{(\frac{R_1}{R_2}u'-\delta v',0)}{\left|(\frac{R_1}{R_2}u'-\delta v', \frac{R_1}{R_2}u_n - v_n)\right|}$$

so that

$$
\begin{aligned}
&\nabla_{u'}\Psi(u',u_n,v',v_n)-\nabla_{u'}\Psi(u',u_n,w',w_n)\\
&=\frac{R_1}{R_2}\frac{(\frac{R_1}{R_2}u'-\delta v',0)}{|(\frac{R_1}{R_2}u'-\delta v',\frac{R_1}{R_2}u_n-v_n)|}-\frac{R_1}{R_2}\frac{(\frac{R_1}{R_2}u'-\delta w',0)}{|(\frac{R_1}{R_2}u'-\delta w',\frac{R_1}{R_2}u_n-w_n)|}
\end{aligned}
$$

Now observe the following: if $x,y\in\mathbb{R}^k$, satisfy $|x|,|y|\ll 1$, then

$$
\begin{aligned}
&\Big|\frac{x}{\sqrt{1+|x|^2}}-\frac{y}{\sqrt{1+|y|^2}}\Big|=\frac{|x-y|}{\sqrt{1+|x|^2}}+|y|\,O\Big(\frac{1}{\sqrt{1+|x|^2}}-\frac{1}{\sqrt{1+|y|^2}}\Big)\\
&\sim|x-y|
\end{aligned}
$$

Thus, in view of (3.11),

$$
|\nabla_{u'}\Psi(u',u_n,v',v_n)-\nabla_{u'}\Psi(u',u_n,w',w_n)|\sim\frac{R_1}{R_2}\delta|v'-w'|
$$

as desired. Moreover, the higher derivatives satisfy

$$
\Big|D^\beta_{u'}\Big[\nabla_{u'}\Psi(u',u_n,v',v_n)-\nabla_{u'}\Psi(u',u_n,w',w_n)\Big]\Big|\lesssim\frac{R_1}{R_2}\delta|v'-w'|
$$

for any β. In fact, we gain factors of $\frac{R_1}{R_2}$ for the higher derivatives, but this is of no use to us. Thus, we apply Proposition 6.1 with $\lambda=R_2$, $\mu=\frac{R_1}{R_2}\delta$, $n_1=n-1$ to obtain

$$
\|T_{\delta,p,R_1,R_2}\|_{2\to2}\lesssim R_1^{\frac{n}{2}}R_2^{\frac{n}{2}-p}\delta^{\frac{n-1}{2}}(\delta R_1)^{-\frac{n-1}{2}}\lesssim\sqrt{R_1R_2}R_2^{\frac{n-1}{2}-p}
$$

which implies the stated bound since $R_2^{-1}\le\delta/R_1\le\delta$.

Case 2: $\delta R_2\le R_1$

Let η be a smooth bump function supported in a neighborhood of the origin such that it defines a partition of unity of $\mathbb{R}^{n-1}$ via

$$
\sum_{k'\in\mathbb{Z}^{n-1}}\eta(x'-k')=1\quad\forall\,x'\in\mathbb{R}^{n-1}
$$

so that also

$$
\sum_{k'\in\mathbb{Z}^{n-1}}\eta\Big(\frac{x'-\delta R_2k'}{\delta R_2}\Big)=1
$$

This latter partition of unity induces a partition of the x and y supports in (3.8) into cylinders of dimensions $\delta R_2\times\ldots\times\delta R_2\times R_1$, and $\delta R_2\times\ldots\times\delta R_2\times R_2$, respectively. If x belongs to a fixed cylinder, then $\Phi_\delta(x,y)\ne0$ implies that y belongs to a finite number of adjacent cylinders, and this number is uniformly controlled. By almost orthogonality, it suffices to prove the desired bound for the kernel localized to such cylinders. After a translation we can assume that the cylinders are

$$
|x'|\lesssim R_2\delta,\quad|x_n|\lesssim R_1,\qquad|y'|\lesssim R_2\delta,\quad y_n\sim R_2
$$

Let

$$
(x',x_n)=(R_2\delta u',R_1u_n),\quad(y',y_n)=(R_2\delta v',R_2v_n)
$$

By Lemma 3.3,

$$\|T_{\delta,p,R_1,R_2}\|_{2\to 2}$$
$$\lesssim \delta^{n-1} R_1^{\frac{1}{2}} R_2^{n-\frac{1}{2}-p} \Big\| e^{iR_2|(\delta u', \frac{R_1}{R_2}u_n)-(\delta v', v_n)|} \frac{\chi(u)\chi(v)(a\Phi_\delta)(u,v)}{\big|(\delta u', \frac{R_1}{R_2}u_n) - (\delta v', v_n)\big|^p} \Big\|_{L^2_v \to L^2_u}$$

where

$$(a\Phi_\delta)(u,v) = a(|(R_2\delta u', R_1 u_n) - (R_2\delta v', R_2 v_n)|)\times \tag{3.12}$$
$$\times\, \Phi_\delta\Big(\frac{(R_2\delta u', R_1 u_n) - (R_2\delta v', R_2 v_n)}{|(R_2\delta u', R_1 u_n) - (R_2\delta v', R_2 v_n)|}\Big)$$

On the support of the integrand, $|u|, |v| \lesssim 1$, and $v_n \sim 1$. Here the kernel is bounded in absolute value by $(\frac{4}{3})^p \chi(u)\chi(v)$ since $a\Phi_\delta$ is bounded, $v_n \sim 1$, and $\frac{R_1}{R_2} < \frac{1}{4}$. Schur's test gives the immediate bound

$$\|T_{\delta,p,R_1,R_2}\|_{2\to 2} \lesssim R_1^{\frac{1}{2}} R_2^{n-\frac{1}{2}-p} \delta^{n-1}$$

If $R_2\delta^2 \le 1$, this estimate is sufficient because $R_2^{n-1-p}\delta^{n-1} \le \delta^{p-\frac{n-1}{2}}$. The last inequality is verified in two ways: if $\frac{n-1}{2} \le p \le n-1$, one can write

$$R_2^{n-1-p}\delta^{n-1} \le \delta^{-2(n-1-p)}\delta^{n-1} \le \delta^{2p-(n-1)} \le \delta^{p-\frac{n-1}{2}}$$

On the other hand, if $p > n-1$, then

$$R_2^{n-1-p}\delta^{n-1} \le \delta^{n-1} \le \delta^{p-\frac{n-1}{2}}$$

since $p \le \frac{n+3}{2} \le 3\frac{n-1}{2}$ when $n \ge 3$.

When $R_2\delta^2 \ge 1$, an improved operator estimate can be obtained via Proposition 6.1. Observe that the u'-derivatives of (3.12) are uniformly bounded. Furthermore, the same analysis as in the previous case applies to the phase

$$\Psi(u,v) = |(\delta u', \frac{R_1}{R_2}u_n) - (\delta v', v_n)|$$

with $\mu = \delta^2$, $\lambda = R_2$, since we still have $|u| \lesssim 1$, $|v| \lesssim 1$, as well as $v_n \sim 1$. Proposition 6.1 now provides the desired estimate

$$\begin{aligned}\|T_{\delta,p,R_1,R_2}\|_{2\to 2} &\lesssim \delta^{n-1} R_1^{\frac{1}{2}} R_2^{n-\frac{1}{2}-p} (R_2\delta^2)^{-\frac{n-1}{2}} \\ &= R_2^{\frac{n-1}{2}-p}\sqrt{R_1R_2} \lesssim \delta^{p-\frac{n-1}{2}}\sqrt{R_1R_2}\end{aligned}$$

where we have used the condition $p \ge \frac{n-1}{2}$ twice in the last line.

Finally, we need to consider the case $R_1 \sim R_2 \sim R$ where $R \ge 1$. Let $\|f\|_2 \le 1$. Then

$$\|T_{\delta,p,R_1,R_2} f\|_2$$
$$\le \sum_{1\le 2^j \le R} \Big\| \int e^{iR|x-y|} \frac{\chi_j(x,y)(a\Phi_\delta)(Rx,Ry)}{|x-y|^p} f(Ry)\, dy \Big\|_{L^2_x} R^{\frac{3n}{2}-p} \tag{3.13}$$

where $\chi_j(x,y)$ is a smooth cut-off function on the set $\{(x,y) : |x|,|y| < 1, |x-y| \sim 2^{-j}\}$. Performing a Whitney decomposition of the integrand away from the diagonal $x=y$, we can estimate (3.13) by

$$R^{n-p} \sum_{1\le 2^j \le R} \max_{Q_1^{(j)} \sim Q_2^{(j)}} \Big\| \chi_{Q_1^{(j)}}(x) \frac{e^{iR|x-y|}}{|x-y|^p} \chi_{Q_2^{(j)}}(y)(a\Phi_\delta)(Rx,Ry) \Big\|_{L^2_y \to L^2_x} \tag{3.14}$$

Thus, $Q_1^{(j)}, Q_2^{(j)}$ are cubes of side length 2^{-j} and $Q_1^{(j)} \sim Q_2^{(j)}$ denotes that they are "related", i.e., $\mathrm{dist}(Q_1^{(j)}, Q_2^{(j)}) \sim 2^{-j}$. Now fix j and cubes $Q = Q_1^{(j)}, Q' = Q_2^{(j)}$. We break Q and Q' into cylinders of size $2^{-j}\delta \times \ldots \times 2^{-j}\delta \times 2^{-j}$. Because of the directional cut-off Φ_δ, each Q cylinder interacts with at most finitely many Q' cylinders. For one such pair of cylinders, we can assume (after translation) that

$$x = (2^{-j}\delta u', 2^{-j}u_n), \qquad y = (2^{-j}\delta v', 2^{-j}v_n)$$

where $|u|,|v| \lesssim 1$, $v_n - u_n \sim 1$. By Lemma 3.3

$$\begin{aligned}
&\Big\| \chi_Q(x) \frac{e^{iR|x-y|}}{|x-y|^p}\, \chi_{Q'}(y)(a\Phi_\delta)(Rx,Ry) \Big\|_{L^2_y \to L^2_x} \\
&\lesssim 2^{j(p-n)}\delta^{n-1} \Big\| e^{iR2^{-j}|(\delta u',u_n)-(\delta v',v_n)|}\, \frac{\chi(u)\chi(v)(a\Phi_\delta)(u,v)}{|(\delta u',u_n)-(\delta v',v_n)|^p} \Big\|_{L^2_v \to L^2_u} \\
&\lesssim 2^{j(p-n)}\delta^{n-1} \min\big(1, (R\delta^2 2^{-j})^{-\frac{n-1}{2}}\big) \qquad (3.15)
\end{aligned}$$

where

$$(a\Phi_\delta)(u,v) = a(R2^{-j}|(\delta u',u_n)-(\delta v',v_n)|)\Phi_\delta\Big(\frac{(\delta u',u_n)-(\delta v',v_n)}{|(\delta u',u_n)-(\delta v',v_n)|}\Big)$$

(3.15) follows from Schur's test and Proposition 6.1. For the latter note that the u' derivatives of $(a\Phi_\delta)(u,v)$ are uniformly bounded on the support of the integrand. Furthermore, the phase is $\Psi(u,v) = |(\delta u',u_n)-(\delta v',v_n)|$ and we have $|u|,|v| \lesssim 1$, $v_n - u_n \sim 1$. Thus, as in the previous cases, the proposition applies with $\mu = \delta^2$, $\lambda = R2^{-j}$.

Combining (3.13), (3.14), and (3.15), yields

$$\|T_{\delta,p,R_1,R_2}\|_{2\to 2} \lesssim \sum_{1\le 2^j\le R} R^{n-p} 2^{j(p-n)} \delta^{n-1} \min\big(1, (R\delta^2 2^{-j})^{-\frac{n-1}{2}}\big) \tag{3.16}$$

Note that $p < n$ unless $n = 3 = p$. In that case the right-hand side of (3.16) is $\lesssim \delta^2 \log R \lesssim R\delta^2$. For the remainder of the proof, therefore, we may assume $p < n$. First consider the case $R\delta^2 \le 1$ where we have

$$(3.16) \lesssim R^{n-p}\delta^{n-1} = R\delta^{p-\frac{n-1}{2}} R^{n-1-p}\delta^{3\frac{n-1}{2}-p} \lesssim R\delta^{p-\frac{n-1}{2}}$$

To prove the final inequality distinguish the cases $p \ge n-1$ and $p < n-1$ and note that $p \le \frac{n+3}{2} \le 3\frac{n-1}{2}$. Henceforth $R\delta^2 > 1$ and we distinguish between $R\delta^2 \leq 2^j \leq R$ and $1 \leq 2^j \leq R\delta^2$. The contribution to the sum in (3.16) by the former is

$$R^{n-p}\delta^{n-1}(R\delta^2)^{p-n} = \delta^{2p-(n+1)} = R(R\delta^2)^{-1}\delta^{2(p-(n-1)/2)} \lesssim R\delta^{p-\frac{n-1}{2}}$$

since $p \geq \frac{n-1}{2}$. The contribution by $1 \leq 2^j \leq R\delta^2$ to (3.16) is

$$R^{\frac{n+1}{2}-p} \sum_{1\leq 2^j \leq \delta^2 R} 2^{-j(\frac{n+1}{2}-p)} \tag{3.17}$$

If $\frac{n-1}{2} \leq p < \frac{n+1}{2}$, then

$$(3.17) \lesssim R^{\frac{n+1}{2}-p} \lesssim R\delta^{p-\frac{n-1}{2}}$$

since $R\delta \geq R\delta^2 \geq 1$. If $p = \frac{n+1}{2}$, then

$$(3.17) \lesssim \log(R\delta^2) \lesssim R\delta^2 \lesssim R\delta$$

since again $R\delta^2 \geq 1$. Finally, if $p > \frac{n+1}{2}$, then

$$(3.17) \lesssim \delta^{2p-n-1} \lesssim R\delta^{p-\frac{n-1}{2}}$$

which concludes the proof. □

Proposition 3.5. *Let $n \geq 3$. Then for any $d \in (0,\infty)$, $\delta \in (0,1)$, and $\lambda \geq 1$ there is the bound*

$$\|D^\alpha \mathcal{R}_{d,\delta}(\lambda^2)f\|_{B^*} \leq C_n \lambda^{-1+|\alpha|}\|f\|_B \tag{3.18}$$

for any $0 \leq |\alpha| \leq 2$. The constant C_n depends only on the dimension $n \geq 3$.

Proof. In view of (3.6) and (3.7) it suffices to prove these estimates for $\lambda = 1$. We need to prove that for any $0 \leq |\alpha| \leq 2$

$$\|\chi(\cdot/R_1)D^\alpha \mathcal{R}_{d,\delta}(1)\,\chi(\cdot/R_2)f\|_2 \leq C_n\sqrt{R_1R_2}\,\|f\|_2 \tag{3.19}$$

where $R_1, R_2 \geq 1$ are arbitrary. We write

$$\mathcal{R}_{d,\delta}(1) = R_0^+(1)\eta_d\Phi_\delta = T_0 + T_1 \tag{3.20}$$

where the kernels of T_0, T_1 are

$$\begin{aligned} T_0(x,y) &= \frac{b(|x-y|)}{|x-y|^{n-2}}\eta_d(|x-y|)\Phi_\delta(x,y) \\ T_1(x,y) &= \frac{e^{i|x-y|}}{|x-y|^{\frac{n-1}{2}}}\eta_d(|x-y|)a(|x-y|)\Phi_\delta(x,y), \end{aligned} \tag{3.21}$$

respectively, see (3.2). The modified function $\eta_d(r)a(r)$ satisfies all decay estimates in (3.3) with constants independent of the choice of d.

We begin by showing that $\widehat{T_0f} = m_0\hat{f}$ where $|m_0(\xi)| \lesssim \langle\xi\rangle^{-2}$. This will imply (3.19) for T_0. By definition

$$m_0(\xi) = \int_0^\infty \int_{S^{n-1}} rb(r)\eta_d(r)e^{-ir\omega\cdot\xi}\Phi_\delta(\omega)\,\sigma(d\omega)\,dr$$

Since $b(r) = 0$ if $r > 2$, $|m_0(\xi)| \lesssim 1$. Hence we may assume that $|\xi| \geq 1$. If $|\xi_n| \geq |\xi|/10$, then $|\omega \cdot \xi| \gtrsim |\xi|$ and

$$|m_0(\xi)| \lesssim \int_{S^{n-1}} \Phi_\delta(\omega)\langle \omega \cdot \xi\rangle^{-2}\,\sigma(d\omega) \lesssim \delta^{n-1}|\xi|^{-2}$$

where we have used that

$$\Big|\int_0^\infty e^{-ir\rho} r b(r)\eta_d(r)\chi(r)\,dr\Big| \lesssim \langle\rho\rangle^{-2}$$

This follows from (3.4) and (3.5) after two integrations by parts. Now suppose that $|\xi_n| \leq |\xi|/10$. Set $\xi = |\xi|\hat{\xi}$ and change integration variables as follows:

$$\begin{aligned}
&\int_{S^{n-1}} \int_0^\infty r b(r)\eta_d(r)\chi(r) e^{-ir|\xi|\omega\cdot\hat{\xi}}\,dr\,\Phi_\delta(\omega)\,\sigma(d\omega)\\
&= \int_{\mathbb{R}^{n-1}} \int_0^\infty r b(r)\eta_d(r)\chi(r) e^{-ir|\xi|u_1}\,dr\,\widetilde{\Phi}_\delta(u_1,\ldots,u_{n-1})\,du_1 du_2\ldots du_{n-1}\\
&= \delta^{n-2}\int_0^\infty \int_{\mathbb{R}} r b(r)\eta_d(r)\chi(r) e^{-ir|\xi|u_1}\Psi_\delta(u_1)\,du_1 dr,
\end{aligned}$$

where $(u_1,\ldots,u_{n-1})$ is a parametrization of the support of Φ_δ, aligning u_1 with $\hat{\xi}$. The function Ψ_δ is a smooth cut-off supported on an interval of length $\sim \delta$ resulting from the integration of $\widetilde{\Phi}_\delta$. Thus,

$$|m_0(\xi)| \lesssim \delta^{n-2}\int_0^1 r|\widehat{\Psi_\delta}(r|\xi|)|\,dr \lesssim \delta^{n-2}|\xi|^{-2}\|u\widehat{\Psi_\delta}(u)\|_{L^1_u} \lesssim \delta^{n-3}|\xi|^{-2}.$$

In conclusion, $|m_0(\xi)| \lesssim \langle\xi\rangle^{-2}$ as claimed.

Next, consider T_1. By the Leibniz rule,

$$\begin{aligned}
D_x^\alpha T_1(x,y) &= \sum_{\beta\leq\alpha} c_{\alpha,\beta}\, D_x^{\alpha-\beta}\Big[\frac{e^{i|x-y|}}{|x-y|^{\frac{n-1}{2}}}\eta_d(|x-y|)a(|x-y|)\Big] D_x^\beta \Phi_\delta(x,y)\\
&= \sum_{\beta\leq\alpha} \delta^{-|\beta|}\, c_{\alpha,\beta}\,\frac{e^{i|x-y|}}{|x-y|^{\frac{n-1}{2}+|\beta|}} a_{\alpha,\beta,d}(|x-y|)\Phi_{\delta,\beta}(x,y) \qquad (3.22)
\end{aligned}$$

where $\Phi_{\delta,\beta} = \delta^{|\beta|}D^\beta\Phi_\delta$ is a modified angular cut-off and $a_{\alpha,\beta,d}$ satisfies the same bounds as a, see (3.3), with constants that do not depend on d. The estimate (3.19) for T_1 follows from Lemma 3.4 with $p = \frac{n-1}{2} + |\beta|$. □

For any $\lambda \geq 1$ define

$$\begin{aligned}
X_\lambda^* &:= \{f \in B^* \,:\, \langle\nabla\rangle f \in B^*\}\\
\|f\|_{X_\lambda^*} &:= \|f\|_{B^*} + \lambda^{-1}\|\langle\nabla\rangle f\|_{B^*}
\end{aligned}$$

The dual norm is

$$\|f\|_{X_\lambda} := \inf_{f=f_1+f_2}\Big(\|f_1\|_B + \lambda\|\langle\nabla\rangle^{-1}f_2\|_B\Big)$$

Corollary 3.6. *Let $\mathcal{R}_{d,\delta}$ be as above. Then for all $\lambda \geq 1$*

$$\|\mathcal{R}_{d,\delta}(\lambda^2)f\|_{X_\lambda^*} \leq C_n\lambda^{-1}\|f\|_{X_\lambda} \tag{3.23}$$

uniformly in $d \in (0,\infty)$, $\delta \in [0,1]$.

Proof. This follows from Proposition 3.5 provided the estimate

$$\|\langle\nabla\rangle f\|_{B^*} \lesssim \|f\|_{B^*} + \|\nabla f\|_{B^*}$$

holds. This in turn will follow if we can show that $\|(m\hat{f})^\vee\|_{B^*} \lesssim \|f\|_{B^*}$ for any symbol m with bounded derivatives. However, this is guaranteed by Corollary 14.1.5 in [11]. □

4. The high energies limiting absorption principle

The main result of this section is a limiting absorption principle for the perturbed resolvent

$$R_L^+(\lambda^2) = (I + R_0^+(\lambda^2)L)^{-1}R_0^+(\lambda^2) \tag{4.1}$$

where $L = i(\nabla\cdot A + A\cdot\nabla) + V$, see Proposition 4.3 below. As before, we shall mostly drop the superscript $+$ on the resolvent. We shall assume throughout this section that $A, V \in Y$ where

$$Y := \Big\{f \in L^\infty \;:\; \sum_{j=0}^\infty 2^j\|f\|_{L^\infty(D_j)} < \infty\Big\}$$

This is the space of functions that take $B^* \to B$ by multiplication.

Lemma 4.1. *For any $\lambda \geq 1$*

$$\|Lf\|_{X_\lambda} \leq C_n(\lambda\|A\|_Y + \|V\|_Y)\|f\|_{X_\lambda^*} \tag{4.2}$$

Proof. Multiplication by V is bounded $B^* \to B$. Also,

$$\|\langle\nabla\rangle^{-1}\nabla Af\|_B \lesssim \|Af\|_B \leq \|A\|_Y\|f\|_{B^*}$$

where the first inequality follows from Corollary 14.1.5 in [11]. Hence $\nabla\cdot A : X_\lambda^* \to X_\lambda$ with norm $\lesssim \lambda$. By duality the same holds for $A\cdot\nabla$. □

From this and Corollary 3.6 it follows that

$$\|(I + R_0(\lambda^2)L)^{-1}f\|_{X_\lambda^*} \leq 2\|f\|_{X_\lambda^*}$$

for all $\lambda \geq 1$ provided A is small in Y.

The main goal of this section is to show that even when A is not small the Neumann series

$$(I + R_0(\lambda^2)L)^{-1} = \sum_{\ell=0}^\infty (-1)^\ell (R_0(\lambda^2)L)^\ell \tag{4.3}$$

converges for large λ. This cannot be deduced from the size of $R_0(\lambda^2)L$ alone, but is instead a consequence of the following crucial lemma.

Lemma 4.2. *Assume that $A, V \in Y$, with A also being continuous. Given any constant $c > 0$, there exist sufficiently large $m = m(c, A, V)$ and $\lambda_1 = \lambda_1(c, A, V)$ such that*

$$\sup_{\lambda > \lambda_1} \|(R_0(\lambda^2)L)^m\|_{X_\lambda^* \to X_\lambda^*} \leq c \tag{4.4}$$

More generally, given any $r > 0$, there exist sufficiently large $m = m(r, A, V)$ and $\lambda_1(r, A, V)$ such that

$$\sup_{\lambda > \lambda_1} \|(R_0(\lambda^2)L)^m\|_{X_\lambda^* \to X_\lambda^*} \leq (2r)^m$$

By choosing $c = \frac{1}{2}$, the series in (4.3) becomes absolutely convergent. In view of (4.1), we thus conclude the following limiting absorption principle for large energies:

Proposition 4.3. *Under the conditions of the previous lemma, there exists $\lambda_1 = \lambda_1(A, V)$ so that for all $\lambda \geq \lambda_1$ one has $R_L(\lambda^2) : X_\lambda \to X_\lambda^*$ with norm estimate*

$$\|R_L(\lambda^2)f\|_{X_\lambda^*} \leq C_n \, \lambda^{-1} \|f\|_{X_\lambda}$$

for all $\lambda \geq \lambda_1$.

As a corollary, we obtain the desired L^2 bounds on $Z_0 R_L(\lambda^2) Z_0^*$ as required in Section 2.

Corollary 4.4. *With λ_1 as above, there are the bounds*

$$\sup_{\lambda \geq \lambda_1} \left\| \langle x \rangle^{-\sigma} \langle \nabla \rangle^{\frac{1}{2}} R_L(\lambda^2 + i0) \langle \nabla \rangle^{\frac{1}{2}} \langle x \rangle^{-\sigma} \right\|_{2 \to 2} \leq C_n$$

$$\sup_{\lambda \geq \lambda_1} \left\| \langle x \rangle^{-\sigma} |\nabla|^{\frac{1}{2}} R_L(\lambda^2 + i0) |\nabla|^{\frac{1}{2}} \langle x \rangle^{-\sigma} \right\|_{2 \to 2} \leq C_n$$

for any $\sigma > \frac{1}{2}$. In particular, $Z_0 R_L(\lambda^2) Z_0^*$ is uniformly bounded in L^2 for $\lambda \geq \lambda_1$.

Proof. Let $Z = \langle x \rangle^{-\sigma} \langle \nabla \rangle^{\frac{1}{2}}$. In view of Proposition 4.3, in order for $Z R_L(\lambda^2) Z^*$ to be uniformly bounded in L^2, we need to prove that $Z : X_\lambda^* \to L^2$ with norm $\lesssim \sqrt{\lambda}$, and, equivalently that $Z^* : L^2 \to X_\lambda$ with the same norm. These estimates follow rather directly from the definition of the space X_λ^*.

If $\|f\|_{X_\lambda^*} = 1$, then $f \in B^*$ and $\langle x \rangle^{-\sigma} f \in L^2$, each with bounded norm. At the same time, $\|\langle x \rangle^{-\sigma} \langle \nabla \rangle f\|_2 \lesssim \lambda$. By the commutator bound in the appendix, it is possible to interchange the weight and the derivative. Therefore by Parseval's identity,

$$\|\langle \nabla \rangle^{\frac{1}{2}} \langle x \rangle^{-\sigma} f\|_2^2 \;\lesssim\; \|\langle \nabla \rangle \langle x \rangle^{-\sigma} f\|_2 \|\langle x \rangle^{-\sigma} f\|_2 \;\lesssim\; \lambda \|f\|_{X_\lambda^*}^2$$

Once again the weight and fractional derivative can be interchanged to prove the bound for $Z R_L(\lambda^2) Z^*$. The bound for $Z_0 R_L(\lambda^2) Z_0^*$ follows immediately because $\langle x \rangle^{-\sigma} |\nabla|^{\frac{1}{2}} \langle \nabla \rangle^{-\frac{1}{2}} \langle x \rangle^{\sigma}$ is a bounded operator on L^2, see Lemma 5.1. $\square$

The remainder of this section is devoted to the proof of Lemma 4.2. Due to the estimate

$$\|R_0(\lambda^2)Vf\|_{X^*_\lambda} \lesssim \lambda^{-1}\|V\|_Y\|f\|_{X^*_\lambda}.$$

we can henceforth assume that $L = i(\nabla \cdot A + A \cdot \nabla)$, with $V \equiv 0$. A partition of unity $\{\Phi_i\}$ over S^{n-1} induces a directional decomposition of the free resolvent, namely

$$R_0(\lambda^2) = \sum_i \mathcal{R}_i(\lambda^2) + R_d(\lambda^2) \tag{4.5}$$

where $\mathcal{R}_i(\lambda^2) := \mathcal{R}_{d,\delta}(\lambda^2)$ with Φ_i playing the role of Φ_δ from the previous section. Moreover, $R_d(\lambda^2)(x) = (1-\eta_d(|x|))R_0(\lambda^2)(|x|)$ is the "short range piece". Heuristically speaking, $R_d(\lambda^2)$ behaves like $R_0((\lambda+i\frac{d}{\lambda})^2)$ and should therefore be bounded on L^2 with operator norm $\lesssim \frac{d}{\lambda}$. The following lemma makes this precise.

Lemma 4.5. *With $R_d^+(\lambda^2)$ defined as above, the mapping estimate*

$$\|D^\alpha R_d(\lambda^2)f\|_2 \le C_n\, \lambda^{-2+|\alpha|}\langle d\lambda\rangle\|f\|_2 \tag{4.6}$$

holds uniformly for every choice of $d \in (0,\infty)$, $0 \le |\alpha| \le 2$, and $\lambda \ge 1$.

Proof. By the scaling relation (3.1), for any α,

$$\|D^\alpha R_0(\lambda^2)\chi_{[|x|<d]}\|_{2\to 2} = \lambda^{-2+|\alpha|}\|D^\alpha R_0(1)\chi_{[|x|<\lambda d]}\|_{2\to 2}$$

where $\chi_{[|x|<\rho]} = \chi(|x|/\rho)$ is a smooth cut-off to the set $|x| < \rho$ with $\rho > 0$ arbitrary. The notation is somewhat ambiguous here; we are seeking an estimate for the convolution operator with kernel $D^\alpha R_0(1)\chi_{[|x|<\lambda d]}$. The lemma is proved by showing that the Fourier transform of $R_0(1)\chi_{[|x|<\rho]}$ is bounded point-wise by $\langle\rho\rangle\langle\xi\rangle^{-2}$.

Consider first the case $\rho \le 1$. The decomposition (3.2) implies that $\int_{\mathbb{R}^n} |R_0(1)(x)\chi(|x|/\rho)|\,dx \lesssim \rho^2$. Furthermore, since $(\Delta+1)R_0(1)$ is a point mass at the origin, the distribution $\Delta[R_0(1)\chi_{[|x|<\rho]}]$ consists of a point mass plus a function of bounded L^1 norm. The desired Fourier transform estimates follow immediately.

When $\rho > 1$, it is more convenient to estimate

$$\rho^n\Big|\int \big[\mathrm{P.V.}\frac{1}{|\eta|^2-1} + i\sigma_{S^{n-1}}(d\eta)\big]\,\hat{\chi}((\xi-\eta)\rho)\,d\eta\Big|$$

A standard calculation shows this to be less than $\rho\langle\rho(|\xi|^2-1)\rangle^{-1} < \rho\langle\xi\rangle^{-2}$. ☐

We shall use Lemma 4.5 in the following somewhat less precise form:

Lemma 4.6. *For any $0 < d < 1$*

$$\|R_d(\lambda^2)f\|_{X^*_\lambda} \le C_n\,\lambda^{-1}d\|f\|_{X_\lambda}$$

uniformly in $\lambda \ge d^{-1}$.

Proof. In view of the definition of the spaces X_λ, X_λ^* this follows from Lemma 4.5 via the imbedding $B \to L^2 \to B^*$ and the identity $\lambda^{-1}\langle d\lambda\rangle \sim d$ for $\lambda > d^{-1}$. □

Decomposing each free resolvent in the m-fold product $(R_0(\lambda^2)L)^m$ as in (4.5) yields the identity

$$(R_0(\lambda^2)L)^m = \sum_{i_1\ldots i_m} \prod_{k=1}^m \big(R_{i_k}(\lambda^2)L\big). \tag{4.7}$$

The indices i_k may take numerical values corresponding to the partition of unity $\{\Phi_i\}$, or else the letter d to indicate a short-range resolvent. There are two main types of products represented here, namely:

- *Directed Products*, where the support of functions Φ_{i_k} and $\Phi_{i_{k+1}}$ are separated by less than 10δ for each k. A product is also considered to be directed if it has this property once all instances of $i_k = d$ are removed. The term $(R_d(\lambda^2)L)^m$ is a vacuous example of a directed product.
- All other terms not meeting the above criteria are *Undirected Products*. An undirected product must contain two adjacent numerical indices (i.e., after discarding all instances where $i_k = d$) for which the corresponding functions Φ_i have disjoint support with distance at least 10δ between them.

Lemma 4.7. *For any $\delta > 0$, there exists a partition of unity $\{\Phi_i\}$ with approximately δ^{1-n} elements, having* $\operatorname{diam}\operatorname{supp}(\Phi_i) < \delta$ *for each i and admitting no more than $\delta^{1-n}(C_n)^m$ directed products of length m in* (4.7)*.*

Proof. The first claim is a standard fact from differential geometry. For the second claim note that there are $\lesssim \delta^{1-n}$ choices for the first element in a directed product, but only C_n choices at each subsequent step. □

If $\delta < \frac{1}{20m}$, then a directed product is truly "directed" in the sense that all the participating functions Φ_{i_k} have support well within a single hemisphere. The convolution operators $R_{i_k}(\lambda^2)$ are therefore biased consistently to one side. In the one-dimensional setting this is reminiscent of a product of Volterra operators, where a norm improvement of $m!$ is typical.

Lemma 4.8. *Suppose $L = i(\nabla \cdot A + A \cdot \nabla)$, with $A(x) \in Y$. Given any $r > 0$, there exists a distance $d = d(r) > 0$ such that each directed product in* (4.7) *satisfies the estimate*

$$\Big\| \prod_{k=1}^m \big(R_{i_k}(\lambda^2)L\big) f \Big\|_{X_\lambda^*} \le C_{n,A,r}\, r^m \|f\|_{X_\lambda^*} \tag{4.8}$$

uniformly over all $\lambda > d^{-1}$ and all choices of m and δ satisfying $\delta \le \frac{1}{20m}$.

Consequently, given any $c > 0$, there exists a number $m = m(c, A)$ and a partition of unity governed by $\delta = \frac{1}{20m}$ so that the sum over all directed *products achieves the bound*

$$\sum_{\substack{i_1 \dots i_m \\ directed}} \Big\| \prod_{k=1}^{m} \big(R_{i_k}(\lambda^2)L\big) \Big\|_{X_\lambda^* \to X_\lambda^*} \le \tfrac{c}{2}$$

uniformly in $\lambda > d^{-1}$.

Proof. In this proof, we will keep track of the superscripts $\pm$ on the resolvents. Also, we will write $\|A\|_Y = C_A$. There is no loss of generality if we assume that $r < C_n C_A$, where C_n is the product of the constants in (3.23) and (4.2).

After a rotation, we may assume that every function Φ_{i_k} which appears in the product has support within a half-radian neighborhood of the north pole, where $x_n > \frac{2}{3}$. If $f \in X_\lambda$ is supported on the half plane $\{x_n > a\}$, then the support of $R_{i_k}^+(\lambda^2)f$ must be translated upward to $\{x_n > a + \frac{2}{3}d\}$. The short-range resolvent $R_d^+(\lambda^2)$ does not have a preferred direction; however if $f \in X_\lambda$ is supported on $\{x_n > a\}$ then $\operatorname{supp} R_d^+(\lambda^2)f \subset \{x_n > a - 2d\}$.

The purpose of keeping track of supports is that if $f \in X_\lambda^*$ is supported away from the origin, in the set $\{|x| > a\}$, then the estimate in Lemma 4.1 can be improved to

$$\|Lf\|_{X_\lambda} \lesssim \lambda \|A\chi_{[|x|>a]}\|_Y \|f\|_{X_\lambda^*}, \tag{4.9}$$

since we are assuming that $V \equiv 0$. For $a > 0$, the half-plane $\{x_n > a\}$ is sufficiently far from the origin for this improved estimate to hold. Note that the compactly supported functions are dense in Y. Given any $A \in Y$ and any $r > 0$, we can choose $R < \infty$ so that

$$\|A\chi_{[x_n > R]}\|_Y < \frac{r^2}{C_n^2 C_A}$$

Let χ be a smooth function supported on the interval $[-1, \infty)$ such that $\chi(x_n) + \chi(-x_n) = 1$. We will initially estimate the operator norm of $\big(\prod_k (R_{i_k}^+(\lambda^2)L)\big)\chi(x_n)$. Multiplication by $\chi(x_n)$ is bounded operator of approximately unit norm in all spaces X_λ^* and X_λ.

The support of $\chi(x_n)f$ lies in the half-space $\{x_n > -1\}$. Suppose every one of the indices i_k is numerical. Then each application of an operator $R_{i_k}^+(\lambda^2)L$ translates the support upward by $\frac{2}{3}d$. For the first $\frac{3R}{2d}$ steps the operator norm of $R_{i_k}^+(\lambda^2)L$ is bounded by (3.23) and (4.2). Thereafter it is possible to use the stronger bound of (4.9) in place of (4.2) because the support will have moved into the half-space $\{x_n > R\}$. The combined estimate is

$$\begin{aligned} \Big\| \prod_{k=1}^{m} (R_{i_k}^+(\lambda^2)L)\,\chi(x_n) f \Big\|_{X_\lambda^*} &\le (C_n C_A)^m \Big(\frac{r^2}{(C_n C_A)^2}\Big)^{m - \frac{3R}{2d}} \|f\|_{X_\lambda^*} \\ &= (C_n C_A)^{-m} (r^{-1} C_n C_A)^{\frac{3R}{d}} r^{2m} \|f\|_{X_\lambda^*} \end{aligned} \tag{4.10}$$

This is valid for small m by our assumption that $r < C_n C_A$.

If each directed resolvent $R^+_{\Phi_i}(\lambda^2)$ is seen as taking one step forward, then the short-range resolvent $R^+_d(\lambda^2)$ may take as many as three steps back. Suppose a directed product includes exactly one index $i_k = d$. This will have the most pronounced effect if it occurs near the beginning of the product, delaying the upward progression of supports by a total of 4 steps. In this case one combines (4.9), (3.23), and Lemma 4.6 to obtain

$$\Big\| \prod_{k=1}^{m} (R^+_{i_k}(\lambda^2)L)\,\chi(x_n) f \Big\|_{X^*_\lambda} \leq (C_n C_A)^m d \Big(\frac{r^2}{(C_nC_A)^2}\Big)^{m-(\frac{3R}{2d}+4)} \|f\|_{X^*_\lambda}$$

Notice that this estimate agrees with the one in (4.10) up to a factor of $d(r^{-1}C_nC_A)^8$. By setting $d = d(r) = \big(\frac{r}{C_nC_A}\big)^8$, the bound in (4.10) is strictly larger. Similar arguments yield the same result for any directed product with one or more instances of the short-range resolvent $R^+_d(\lambda^2)$.

To remove the spatial cutoff, write

$$\prod_{k=1}^{m} R^+_{i_k}(\lambda^2)L \;=\; \Big(\prod_{k=1}^{m/2}(R^+_{i_k}(\lambda^2)L)\Big)(\chi(x_n)+\chi(-x_n))\Big(\prod_{k=\frac{m}{2}+1}^{m}(R^+_{i_k}(\lambda^2)L)\Big)$$

Consider the $\chi(x_n)$ term. By (4.10), the first half of the product carries an operator norm bound of $(C_n\,C_A)^{-\frac{m}{2}}(r^{-1}C_nC_A)^{\frac{3R}{d(r)}}r^m$. The second half contributes at most $(C_nC_A)^{m/2}$, based on (3.23) and Lemma 4.1. Put together, this product has an operator norm less than $C_{n,A,r}\,r^m$, where $C_{n,A,r} = (r^{-1}C_nC_A)^{\frac{3R}{d(r)}}$.

The $\chi(-x_n)$ term has nearly identical estimates, by duality. The adjoint of any directed resolvent $R^+_\Phi(\lambda^2)$ is precisely $R^-_{\widetilde{\Phi}}(\lambda^2)$, with $\widetilde{\Phi}$ being the antipodal image of Φ. Because the order of multiplication is reversed, one applies the geometric argument above (modulo the antipodal map) to a product of the form

$$\Big(\prod_{k=1}^{m/2}(LR^-_{i_k}(\lambda^2))\Big)\chi(-x_n),$$

which is an operator on X_λ. The estimates (4.9), (3.23), and (4.6) are used in the same manner as in deriving the main bound (4.10).

According to Lemma 4.7 there are at most $\delta^{1-n}(C_n)^m$ directed products of length m. To prove (4.8), it therefore suffices to let $r = \frac{1}{2C_n}$, and $\delta = \frac{1}{20m}$ so that the sum of the operator norms of all directed products is bounded by $20^{n-1}C_{n,A}m^{n-1}2^{-m}$. This can be made smaller than $\frac{c}{2}$ by choosing m sufficiently large. □

As for the undirected products, recall that their defining feature is the presence of adjacent resolvents $R^+_i(\lambda^2)$ oriented in distinct directions. The resulting oscillatory integral has no region of stationary phase, and therefore exhibits improved bounds at high energy provided the potential $A(x)$ is smooth.

Lemma 4.9. *Let Φ_1 and Φ_2 be chosen from a partition of unity of S^{n-1} so that their supports are separated by a distance greater than 10δ. Suppose $A \in C^\infty(\mathbb{R}^n)$ with compact support. Then for each $j \geq 0$, and any $N \geq 1$,*

$$\big\| R^+_{d,\Phi_2}(\lambda^2)(LR^+_d(\lambda^2))^j LR^+_{d,\Phi_1}(\lambda^2)\big\|_{X_\lambda \to X^*_\lambda} = \mathcal{O}(\lambda^{-N}) \tag{4.11}$$

as $\lambda \to \infty$ and similarly for $R^-(\lambda^2)$.

Proof. In view of (3.1) and (3.2) we can write

$$\begin{aligned}
&R_0^+(\lambda^2)(|x|)\eta_d(|x|)\Phi(x/|x|)\\
&= \lambda^{n-2} R_0(1)(\lambda|x|)\eta_d(|x|)\Phi(x/|x|)\\
&= \lambda^{\frac{n-3}{2}} \frac{e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}}\Big[a(\lambda|x|) + \frac{e^{-i\lambda|x|}b(\lambda|x|)}{|\lambda x|^{\frac{n-3}{2}}}\Big]\eta_{\lambda d}(\lambda|x|)\Phi(x/|x|)\\
&= \lambda^{\frac{n-3}{2}} \frac{e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}} a_{\lambda d}(\lambda|x|)\Phi(x/|x|)
\end{aligned}$$

where for arbitrary $\widetilde{d} = \lambda d > 0$

$$a_{\widetilde{d}}(r) = \Big[a(r) + \frac{e^{-ir}b(r)}{r^{\frac{n-3}{2}}}\Big]\eta_{\widetilde{d}}(r)$$

is supported on $\{r \geq d\}$ and satisfies the bounds, for all $\ell \geq 0$ and $r > d$,

$$|\partial_r^\ell a_{\widetilde{d}}(r)| \leq C_\ell\, \widetilde{d}^{\,-\frac{n-3}{2}}\, r^{-\ell} \leq C_\ell\, d^{-\frac{n-3}{2}}\, r^{-\ell}$$

uniformly in $\lambda \geq 1$. The kernel of the operator of (4.11) with $j = 0$ equals

$$\begin{aligned}
K_{d,\lambda}(x,y) := \lambda^{n-3} \int_{\mathbb{R}^n} &\frac{e^{i\lambda|x-u|}}{|x-u|^{\frac{n-1}{2}}}\Phi_1\Big(\frac{x-u}{|x-u|}\Big) a_{\lambda d}(\lambda|x-u|)(\nabla_u A(u) +\\
&+ A(u)\nabla_u)\frac{e^{i\lambda|u-y|}}{|u-y|^{\frac{n-1}{2}}}\, a_{\lambda d}(\lambda|u-y|)\Phi_2\Big(\frac{u-y}{|u-y|}\Big)\, du
\end{aligned} \tag{4.12}$$

By our assumption on A we can integrate by parts any number of times in the u variable since

$$\big|\partial_u[|x-u| + |u-y|]\big| = \Big|\frac{x-u}{|x-u|} - \frac{u-y}{|u-y|}\Big| > \delta$$

by the angular separation hypothesis between $\operatorname{supp}\Phi_1$ and $\operatorname{supp}\Phi_2$. In conclusion, for arbitrary N,

$$|K_{d,\lambda}(x,y)| \leq C_N(A,d,\delta,n)\lambda^{-N}\langle y\rangle^{-\frac{n-1}{2}}\langle x\rangle^{-\frac{n-1}{2}}$$

Here we also used the compact support assumption on A which restricts the size of u in (4.12). This kernel takes $B \to B^*$ with norm $\lesssim \lambda^{-N}$. In the same way one bounds the kernel $D^\alpha_{x,y}K_{d,\lambda}(x,y)$ for any α which concludes the argument for $j = 0$.

If $j \geq 1$, then write

$$\begin{aligned}
&R_0^+(\lambda^2)(|x|)[1-\eta_d(|x|)] \\
&= \lambda^{n-2} R_0(1)(\lambda|x|)[1-\eta_d(|x|)] \\
&= \lambda^{\frac{n-3}{2}} \frac{e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}} \Big[a(\lambda|x|) + \frac{e^{-i\lambda|x|} b(\lambda|x|)}{|\lambda x|^{\frac{n-3}{2}}}\Big][1-\eta_{\lambda d}(\lambda|x|)] \\
&= \lambda^{\frac{n-3}{2}} \frac{e^{i\lambda|x|}}{|x|^{\frac{n-1}{2}}} b_{\lambda d}(\lambda|x|)
\end{aligned}$$

where for arbitrary $\widetilde{d} = \lambda d > 0$

$$b_{\widetilde{d}}(r) = \Big[a(r) + \frac{e^{-ir} b(r)}{r^{\frac{n-3}{2}}}\Big][1-\eta_{\widetilde{d}}(r)]$$

satisfies the bounds $|\partial_r^\ell b_{\widetilde{d}}(r)| \leq C_\ell\, \widetilde{d}^{\frac{n-3}{2}}\, r^{-\frac{n-3}{2}-\ell}$ for all $\ell \geq 0$ and $r > 0$. In particular,

$$\big|D_x^\ell[b_{\lambda d}(\lambda|x|)]\big| \leq C_\ell d^{\frac{n-3}{2}} |x|^{\frac{n-3}{2}-\ell}$$

uniformly in $\lambda \geq 1$. The kernel of the operator of (4.11) with $j > 0$ now equals

$$\begin{aligned}
K_{j,d,\lambda}(x,y) &:= \lambda^{(n-3)(j+2)/2} \int_{\mathbb{R}^{(j+1)n}} \frac{e^{i\lambda|x-u_0|}}{|x-u_0|^{\frac{n-1}{2}}} \Phi_1\Big(\frac{x-u_0}{|x-u_0|}\Big) a_{\lambda d}(\lambda|x-u_0|) \\
&\prod_{i=0}^{j-1} (\nabla_{u_i} A(u_i) + A(u_i)\nabla_{u_i}) \frac{e^{i\lambda|u_i-u_{i+1}|}}{|u_i-u_{i+1}|^{\frac{n-1}{2}}} b_{\lambda d}(\lambda|u_i-u_{i+1}|) \\
&(\nabla_{u_j} A(u_j) + A(u_j)\nabla_{u_j}) \frac{e^{i\lambda|u_j-y|}}{|u_j-y|^{\frac{n-1}{2}}}\, a_{\lambda d}(\lambda|u_j-y|) \Phi_2\Big(\frac{u_j-y}{|u_j-y|}\Big)\, du
\end{aligned}$$

We change variables

$$w_i = \frac{1}{2}(u_i - u_{i+1}), \quad 0 \leq i \leq j-1, \quad w_j = \frac{1}{2}(u_0 + u_j)$$

so that $u_0 = \sum_{i=0}^j w_i$, $u_j = w_j - \sum_{i=0}^{j-1} w_i$, and $u_1 = u_0 - 2w_0$, $u_2 = u_1 - 2w_1$ etc. After this substitution we obtain

$$\begin{aligned}
K_{j,d,\lambda}(x,y) &:= c\lambda^{(n-3)(j+2)/2} \int_{\mathbb{R}^{(j+1)n}} \frac{e^{i\lambda|x-u_0|}}{|x-u_0|^{\frac{n-1}{2}}} \Phi_1\Big(\frac{x-u_0}{|x-u_0|}\Big) a_{\lambda d}(\lambda|x-u_0|) \\
&\prod_{i=0}^{j-1} (\nabla_{u_i} A(u_i) + A(u_i)\nabla_{u_i}) \frac{e^{2i\lambda|w_i|}}{|w_i|^{\frac{n-1}{2}}} b_{\lambda d}(2\lambda|w_i|) \\
&(\nabla_{u_j} A(u_j) + A(u_j)\nabla_{u_j}) \frac{e^{i\lambda|u_j-y|}}{|u_j-y|^{\frac{n-1}{2}}}\, a_{\lambda d}(\lambda|u_j-y|) \Phi_2\Big(\frac{u_j-y}{|u_j-y|}\Big)\, dw
\end{aligned}$$

where it is understood that $u_0 = u_0(w)$ and $u_j = u_j(w)$. Of particular interest, the phase functions involving x, y contain the variable w_j, viz.

$$\lambda|x - u_0| = \lambda\big|x - w_j - \sum_{i=0}^{j-1} w_i\big|, \qquad \lambda|y - u_j| = \lambda\big|y - w_j + \sum_{i=0}^{j-1} w_i\big|$$

whereas none of the short range free resolvent kernels contains w_j. Thus, since

$$\Big|\frac{x - u_0}{|x - u_0|} - \frac{u_j - y}{|u_j - y|}\Big| > \delta$$

integration by parts in w_j yields as before

$$|D^\alpha_{x,y} K_{j,d,\lambda}(x, y)| \le C_{N,\alpha}(A, d, \delta, n, j)\lambda^{-N} \langle y\rangle^{-\frac{n-1}{2}} \langle x\rangle^{-\frac{n-1}{2}}$$

for any α, N and we are done. $\square$

Remark 4.10. One should be careful that the integrand above is locally integrable. Each short range resolvent contains a singularity on the order of $|w_i|^{2-n}$ which becomes more severe with repeated differentiation.

Fortunately, in the full change of coordinates $\nabla_{u_i} = \frac{1}{2}(\nabla_{w_i} - \nabla_{w_{i-1}})$ for each $i = 1, 2, \ldots, j$, while $\nabla_{u_0} = \frac{1}{2}(\nabla_{w_0} + \nabla_{w_j})$. Therefore the dangerous piece $\frac{b_{\lambda d}(2\lambda|w_i|)}{|w_i|^{(n-1)/2}}$ experiences the ∇_{u_i} immediately preceding it, but no other derivatives, creating local singularities no worse than $|w_i|^{1-n}$.

Note that under the conditions of Lemma 4.9 each undirected product in (4.7) satisfies the bound

$$\Big\| \prod_{k=1}^{m} \big(R_{i_k}(\lambda^2)L\big)\Big\|_{X^*_\lambda \to X^*_\lambda} = \mathcal{O}(\lambda^{-N}) \tag{4.13}$$

for any $N \ge 1$. We now show by approximation that vanishing still holds for merely continuous A, but without any control over the rate.

Lemma 4.11. *Let Φ_1 and Φ_2 be chosen as in Lemma 4.9. Suppose A is a continuous function with $A \in Y$. Then each undirected product in* (4.7) *satisfies the limiting bound*

$$\lim_{\lambda\to\infty} \Big\| \prod_{k=1}^{m} \big(R_{i_k}(\lambda^2)L\big)\Big\|_{X^*_\lambda \to X^*_\lambda} = 0. \tag{4.14}$$

for any $\lambda \ge 1$.

Proof. For any small $\gamma > 0$, there exists a smooth approximation $A_\gamma \in C^\infty(\mathbb{R}^n)$ of compact support so that $\|A - A_\gamma\|_Y < \gamma$ and $\|A_\gamma\|_Y < 2\|A\|_Y$. Define the operator L_γ accordingly. By Lemma 4.1 and Corollary 3.6

$$\Big\| \prod_{k=1}^{m} \big(R_{i_k}(\lambda^2)L\big) - \prod_{k=1}^{m} \big(R_{i_k}(\lambda^2)L_\gamma\big)\Big\|_{X^*_\lambda \to X^*_\lambda} \lesssim \gamma(2\|A\|_Y)^{m-1}$$

uniformly in $\lambda \geq 1$. Thus, by (4.13),

$$\limsup_{\lambda\to\infty} \Big\| \prod_{k=1}^{m} \big(R_{i_k}(\lambda^2)L\big) \Big\|_{X^*_\lambda \to X^*_\lambda} \lesssim \gamma(2\|A\|_Y)^{m-1}$$

Sending $\gamma \to 0$ finishes the proof. □

Proof of Lemma 4.2. Lemma 4.8 provides a recipe for selecting a value of m, together with a partition of unity $\{\Phi_i\}$ and a short-range threshold d, so that the sum over all directed products in (4.7) will be an operator of norm less than $\frac{c}{2}$, or $C_r m^{n-1} r^m$. We may choose m so that $2^m > C_r m^{n-1}$. This fixes the number of undirected products as approximately $\delta^{m(1-n)} = (20m)^{m(n-1)}$. For each of these, Lemma 4.11 asserts that its operator norm tends to zero as $\lambda \to \infty$. The same is true for the finite sum over all undirected products of length m. In particular it is less than the directed product estimate provided $\lambda > \lambda_1(m)$ is sufficiently large. □

5. Small energies

The remaining task is to verify that for sufficiently small λ_0 (and following our convention regarding $\lambda^2 \pm i0$ from before)

$$\sup_{0<\lambda<\lambda_0} \|Z_0 R_L(\lambda^2) Z_0^*\|_{2\to 2} < \infty, \tag{5.1}$$

where $Z_0 = \langle x\rangle^{-\sigma}|\nabla|^{\frac{1}{2}}$ for some $\sigma > \frac{1}{2}$. As in the high energy case (and implicitly for intermediate energies), we need to impose an invertibility condition which allows the resolvent $R_L(\lambda^2)$ to be bounded between suitable spaces. More precisely, by the resolvent identity,

$$R_L(\lambda^2 + i0) = (1 + R_0(\lambda^2 + i0)L)^{-1} R_0(\lambda^2 + i0)$$

provided the inverse on the right-hand side exists. We have

$$\begin{aligned} &\|Z_0 R_L(\lambda^2) Z_0^*\|_{2\to 2} \\ &= \|Z_0(1 + R_0(\lambda^2)L)^{-1} Z_0^{-1} Z_0 R_0(\lambda^2) Z_0^*\|_{2\to 2} \\ &\leq \|Z_0(1 + R_0(\lambda^2)L)^{-1} Z_0^{-1}\|_{2\to 2}\ \|Z_0 R_0(\lambda^2) Z_0^*\|_{2\to 2} \end{aligned}$$

By the smoothing properties of Z_0 relative to H_0,

$$\sup_{\lambda} \|Z_0 R_0(\lambda^2) Z_0^*\|_{2\to 2} < \infty$$

Thus, it will suffice to verify that

$$\sup_{|\lambda|<\lambda_0} \|Z_0(1 + R_0(\lambda^2)L)^{-1} Z_0^{-1}\|_{2\to 2} < \infty \tag{5.2}$$

Let $G = R_0(0)$, and $B_\lambda = R_0(\lambda^2) - G$. We will prove that under suitable conditions

$$Z_0(I + GL)^{-1} Z_0^{-1} = (I + Z_0 G L Z_0^{-1})^{-1} : L^2 \to L^2 \tag{5.3}$$

$$\|Z_0 B_\lambda L Z_0^{-1}\|_{2\to 2} \to 0 \qquad \text{as } \lambda \to 0 \tag{5.4}$$

This implies (5.2) by summing the Neumann series directly. The proof of (5.3) is a standard Fredholm alternative argument, while (5.4) will follow from properties of the kernel of B_λ.

Lemma 5.1. *Let $n > \beta > 0$, $\beta \geq \alpha$. Then*

$$\langle \nabla \rangle^{\alpha} |\nabla|^{-\beta} \,:\, L^{2,\sigma_1} \to L^{2,-\sigma_2}$$

for all pairs $\sigma_1, \sigma_2 > \beta - \frac{n}{2}$ satisfying $\sigma_1 + \sigma_2 > \beta$, and is a compact operator whenever the strict inequality $\beta > \alpha$ holds.

The same result holds when $\beta \leq 0$ and $\beta \geq \alpha$, under the conditions $\sigma_1, \sigma_2 > \beta - \frac{n}{2}$ and $\sigma_1 + \sigma_2 \geq 0$. Compactness in this case requires strict inequalities for both $\beta > \alpha$ and $\sigma_1 + \sigma_2 > 0$.

Proof. Note that $\langle x \rangle^{-\varepsilon} \langle \nabla \rangle^{-\delta}$ is compact on L^2 for any choice of $\varepsilon, \delta > 0$. Therefore, it suffices to establish the boundedness of $\langle x \rangle^{-\sigma_2'} \langle \nabla \rangle^{\beta} |\nabla|^{-\beta} \langle x \rangle^{-\sigma_1}$ on L^2 for any value $\sigma_2' < \sigma_2$. Strict inequalities are necessary to ensure that σ_2' can also be chosen to satisfy the hypotheses of the lemma. Let

$$K(x,y) = \langle x \rangle^{-\sigma_2'} [\langle \xi \rangle^{\beta} |\xi|^{-\beta}]^{\vee}(x-y) \langle y \rangle^{-\sigma_1}$$

Then

$$|I - K(x,y)| \lesssim \begin{cases} \langle x \rangle^{-\sigma_2} |x-y|^{2-n} \langle y \rangle^{-\sigma_1} & |x-y| \leq 1 \\ \langle x \rangle^{-\sigma_2} |x-y|^{\beta-n} \langle y \rangle^{-\sigma_1} & |x-y| > 1 \end{cases}$$

based on the fact that $1 - \langle \xi \rangle^{\beta} |\xi|^{-\beta} \sim |\xi|^{-2}$ for all $|\xi| \geq 1$.

Define $K_{ij}(x,y) = [I - K(x,y)] \chi_i(x) \chi_j(y)$ where $\chi_i(x) = 1_{[|x| \leq 1]}$ if $i = 0$ and $\chi_i(x) = 1_{[2^{i-1} < |x| \leq 2^i]}$ if $i \geq 1$. Then

$$|K_{ij}(x,y)| \lesssim 2^{-i\sigma_2' - j\sigma_1} 2^{-\max(i,j)(n-\beta)}$$

provided $|i - j| > 1$. Hence, $K_1 := \sum_{|i-j|>1} K_{ij}$ defines a bounded operator (in fact, compact operator) on L^2 since its Hilbert-Schmid norm is controlled by

$$\|K_1\|_{HS}^2 \lesssim \sum_{|i-j|>1} 2^{-2(i\sigma_2' + j\sigma_1)} 2^{-2\max(i,j)(n-\beta)} 2^{(i+j)n} < \infty$$

For fixed i, the sum over $j = i+2, i+3, \ldots$ is only finite if $\sigma_1 > \beta - \frac{n}{2}$, and its value is then comparable to $2^{-2i(\sigma_1 + \sigma_2' - \beta)}$. Finite summation over i then requires that $\sigma_1 + \sigma_2' > \beta$. Similar conditions are noted, with the roles of σ_1 and σ_2' reversed, when considering the summation over all $j > i + 1$.

By almost orthogonality, $K_0 = \sum_{|i-j| \leq 1} K_{ij}$ satisfies

$$\|K_0\|_{2 \to 2} \lesssim \max_{|i-j| \leq 1} \|K_{ij}\|_{2 \to 2}$$

By Schur's test,

$$\|K_{ij}\|_{2 \to 2} \lesssim 2^{-i(\sigma_1 + \sigma_2')} 2^{i \max(\beta, 0)}$$

when $|i - j| \leq 1$, which is uniformly bounded provided $\sigma_1 + \sigma_2' \geq \max(\beta, 0)$. We are done evaluating the three components of the decomposition $K = I - K_0 - K_1$. $\square$

Remark 5.2. To be precise, the above proof did not capture the points $\beta = 0$, $\sigma_1 + \sigma_2 = 0$, but this can be shown trivially as a special case.

Next, we apply this result to prove compactness of the zero energy operators.

Lemma 5.3. *Assume that L is as in* (1.2), (1.3), *and* $Z_0 = \langle x\rangle^{-\frac{1}{2}-}|\nabla|^{\frac{1}{2}}$. *Then $Z_0 G L Z_0^{-1}$ is a compact operator on L^2.*

Proof. We shall use the decomposition

$$L = Y_1^* Z_1 + Y_2^* Z_2 + V$$

where

$$Y_1 := i\nabla|\nabla|^{-\frac{1}{2}} \cdot A w^{-1}, \quad Z_1 := |\nabla|^{\frac{1}{2}} w, \quad Y_2 := Z_1, \quad Z_2 := Y_1$$

As before, $w = \langle x\rangle^{-\tau}$ for some $\tau \in (\frac{1}{2}, \frac{1}{2} + \varepsilon')$. For convenience we will take $\tau = \frac{1}{2}(1+\varepsilon')$ in the calculations below. Our goal is to prove that the operators

$$\begin{aligned} O_1 &= \langle x\rangle^{-\sigma}|\nabla|^{\frac{1}{2}} G V |\nabla|^{-\frac{1}{2}}\langle x\rangle^{\sigma} \\ O_2 &= \langle x\rangle^{-\sigma}|\nabla|^{\frac{1}{2}} G Y_1^* Z_1 |\nabla|^{-\frac{1}{2}}\langle x\rangle^{\sigma} \\ O_3 &= \langle x\rangle^{-\sigma}|\nabla|^{\frac{1}{2}} G Y_2^* Z_2 |\nabla|^{-\frac{1}{2}}\langle x\rangle^{\sigma} \end{aligned}$$

are compact in L^2 for some $\sigma > \frac{1}{2}$. In what follows, we will use the commutator bounds of the appendix without further mention. The same applies to the fact that $|\nabla|^{\frac{1}{2}} A \langle x\rangle^{1+}|\nabla|^{-\frac{1}{2}}$ (and therefore its adjoint) are bounded on L^2, see Lemma 2.2 above.

For O_1, it suffices to observe that

$$\langle x\rangle^{-\sigma}|\nabla|^{\frac{1}{2}} G V |\nabla|^{-\frac{1}{2}}\langle x\rangle^{\sigma} = \big(\langle x\rangle^{-\sigma}|\nabla|^{-\frac{3}{2}}\langle x\rangle^{-1}\big)\big(\langle x\rangle V |\nabla|^{-\frac{1}{2}}\langle x\rangle^{\sigma}\big)$$

is compact by Lemma 5.1 provided $\sigma \in (\frac{1}{2}, \frac{1}{2} + \varepsilon)$.

Denote the bounded and compact operators on L^2 by $\mathcal{B}$ and $\mathcal{C}$, respectively. Then,

$$\begin{aligned} O_2 &= \langle x\rangle^{-\sigma}|\nabla|^{-\frac{3}{2}} A w^{-1}\nabla|\nabla|^{-\frac{1}{2}}\big(|\nabla|^{\frac{1}{2}} w |\nabla|^{-\frac{1}{2}}\langle x\rangle^{\sigma}\big) \\ &\in \langle x\rangle^{-\sigma}|\nabla|^{-\frac{3}{2}} w |\nabla|^{\frac{1}{2}}\big(|\nabla|^{-\frac{1}{2}} A w^{-2}|\nabla|^{\frac{1}{2}}\big)\nabla|\nabla|^{-1}\mathcal{B} \\ &\subset \big(\langle x\rangle^{-\sigma}|\nabla|^{-1}\langle x\rangle^{-\frac{1}{2}}\big)\big(\langle x\rangle^{\frac{1}{2}}|\nabla|^{-\frac{1}{2}} w |\nabla|^{\frac{1}{2}}\big)\mathcal{B} \subset \mathcal{C}. \end{aligned}$$

requires $\sigma \in (\frac{1}{2}, \tau)$. Finally, under the same conditions,

$$\begin{aligned} O_3 &= \langle x\rangle^{-\sigma}|\nabla|^{-\frac{3}{2}} w |\nabla|^{\frac{1}{2}}(\nabla|\nabla|^{-1})\big(|\nabla|^{\frac{1}{2}} A w^{-2}|\nabla|^{-\frac{1}{2}}\big)\big(|\nabla|^{\frac{1}{2}} w |\nabla|^{-\frac{1}{2}}\langle x\rangle^{\sigma}\big) \\ &\in \langle x\rangle^{-\sigma}|\nabla|^{-\frac{3}{2}} w |\nabla|^{\frac{1}{2}}\mathcal{B} \\ &\subset \big(\langle x\rangle^{-\sigma}|\nabla|^{-1}\langle x\rangle^{-\frac{1}{2}}\big)\big(\langle x\rangle^{\frac{1}{2}}|\nabla|^{-\frac{1}{2}} w |\nabla|^{\frac{1}{2}}\big)\mathcal{B} \subset \mathcal{C} \end{aligned}$$

and we are done. □

As an immediate consequence we arrive at the following.

Corollary 5.4. *Let* $Z_0 = \langle x\rangle^{-\sigma}|\nabla|^{\frac12}$ *with* $\sigma \in (\frac12, \frac12+\varepsilon')$. *Assume that* $\ker(I + Z_0GLZ_0^{-1}) = \{0\}$ *as an operator on* $L^2(\mathbb{R}^n)$. *Then* $I + Z_0GLZ_0^{-1}$ *is invertible on* L^2.

Proof. The statement follows from Fredholm's alternative. Note that

$$(I + Z_0GLZ_0^{-1})^{-1} = Z_0(I+GL)^{-1}Z_0^{-1}$$

where GL on the right-hand side is an operator on $Z_0^{-1}(L^2(\mathbb{R}^n))$. □

Now, we verify the vanishing norm condition (5.4).

Lemma 5.5. *For any* $L = i(A\cdot\nabla + \nabla\cdot A) + V$ *satisfying conditions* (1.2), (1.3), *one has*

$$\lim_{\lambda\to 0^+} \|\langle x\rangle^{-\sigma}|\nabla|^{\frac12}B_\lambda L|\nabla|^{-\frac12}\langle x\rangle^{\sigma}\|_{2\to 2} = 0$$

Proof. By the commutator identities, the assumptions on A and V, and fractional integration, the claim above is a consequence of the bound

$$\lim_{\lambda\to 0^+} \|\langle x\rangle^{-\sigma}\nabla B_\lambda \langle x\rangle^{-\sigma}\|_{2\to 2} = 0 \tag{5.5}$$

To be precise, the reduction proceeds as follows. Recall that B_λ is defined as a function of $-\Delta$, and therefore commutes with all derivatives. First,

$$\begin{aligned}\langle x\rangle^{-\sigma}|\nabla|^{\frac12}B_\lambda V|\nabla|^{-\frac12}\langle x\rangle^{\sigma} &= \sum_{i=1}^n \big(\langle x\rangle^{-\sigma}\partial_i B_\lambda\langle x\rangle^{-\sigma}\big)\big(\langle x\rangle^{\sigma}\partial_i|\nabla|^{-\frac32}V|\nabla|^{-\frac12}\langle x\rangle^{\sigma}\big)\\ &= \sum_i \big(\langle x\rangle^{-\sigma}\partial_i B_\lambda\langle x\rangle^{-\sigma}\big)S_1^i\end{aligned}$$

where each S_1^i is bounded on L^2 by the fractional integration estimates in Lemma 5.1. For the gradient term $\nabla\cdot A$ we have

$$\langle x\rangle^{-\sigma}|\nabla|^{-\frac12}B_\lambda\nabla\cdot A|\nabla|^{-\frac12}\langle x\rangle^{\sigma} = \big(\langle x\rangle^{-\sigma}\nabla B_\lambda\langle x\rangle^{-\sigma}\big)\cdot S_2$$

where the operator

$$S_2 = \big(\langle x\rangle^{\sigma}|\nabla|^{\frac12}w|\nabla|^{-\frac12}\big)\big(|\nabla|^{\frac12}Aw^{-2}|\nabla|^{-\frac12}\big)\big(|\nabla|^{\frac12}w|\nabla|^{-\frac12}\langle x\rangle^{\sigma}\big)$$

is bounded on L^2 by Lemma 2.2 and the commutator estimates in Lemma 6.2. The second gradient term, $A\cdot\nabla$, requires a slightly more intricate decomposition.

$$\langle x\rangle^{-\sigma}|\nabla|^{-\frac12}B_\lambda A\cdot\nabla|\nabla|^{-\frac12}\langle x\rangle^{\sigma} = \sum_{i,j=1}^n \big(\langle x\rangle^{-\sigma}\partial_i B_\lambda\langle x\rangle^{-\sigma}\big)S_3^{i,j}$$

where each $S_3^{i,j}$ has the structure

$$\begin{aligned}S_{i,j} = {}&\big(\langle x\rangle^{\sigma}\partial_i|\nabla|^{-1}\langle x\rangle^{-\sigma}\big)\big(\langle x\rangle^{\sigma}|\nabla|^{-\frac12}w|\nabla|^{\frac12}\big)\\ &\times\big(|\nabla|^{-\frac12}A_jw^{-2}|\nabla|^{\frac12}\big)\big(|\nabla|^{-\frac12}w|\nabla|^{\frac12}\langle x\rangle^{\sigma}\big)\big(\langle x\rangle^{-\sigma}\partial_j|\nabla|^{-1}\langle x\rangle^{\sigma}\big)\end{aligned}$$

The central term is bounded on L^2 by Lemma 2.2; it is flanked by a pair of commutators as in Lemma 6.2. The boundedness of the outer operators simply reflects the boundedness of the Riesz transforms on the weighted space $\langle x\rangle^{\pm\sigma}L^2$.

Now it remains to verify (5.5). With the notation of Section 3, we have

$$B_\lambda(x,y) = \frac{b(\lambda|x-y|)}{|x-y|^{n-2}} - \frac{b(0)}{|x-y|^{n-2}} + \lambda^{\frac{n-3}{2}} e^{i\lambda|x-y|} \frac{a(\lambda|x-y|)}{|x-y|^{\frac{n-1}{2}}}$$

We write $\nabla B_\lambda = T_{\lambda,0} + T_{\lambda,1}$, where

$$|T_{\lambda,0}(x,y)| \lesssim \frac{\lambda}{|x-y|^{n-2}}$$

$$T_{\lambda,1}(x,y) = \lambda^{\frac{n-3}{2}}\Big(\lambda + \frac{1}{|x-y|}\Big) e^{i\lambda|x-y|} \frac{\widetilde{a}(\lambda|x-y|)}{|x-y|^{\frac{n-1}{2}}}$$

where $\widetilde{a}$ is a modified symbol with the same properties as a. The support of $T_{\lambda,0}$ is restricted to the set $\{\lambda|x-y| \lesssim 1\}$, so the point-wise estimate

$$|T_{\lambda,0}(x,y)| \lesssim \frac{\lambda^{\sigma-\frac{1}{2}}}{|x-y|^{n-\frac{1}{2}-\sigma}}$$

is also valid. Thanks to the positive power of λ in the numerator, $T_{\lambda,0}$ satisfies (5.5) by fractional integration. $T_{\lambda,1}$ requires more care. Let χ be a smooth cut-off for the region $\{x : |x| \sim 1\}$. It suffices to prove that for $R_2 \gtrsim R_1 > 1$ and $R_2 \gtrsim 1/\lambda$ (since $\widetilde{a}(\lambda|x-y|) = 0$ for $|x-y| < 1/\lambda$),

$$\|\chi(x/R_1)T_{\lambda,1}(x,y)\chi(y/R_2)\|_{2\to2} \lesssim \lambda^\varepsilon R_1^{\frac{1}{2}} R_2^{\frac{1}{2}+\varepsilon}$$

This, however, is an almost immediate corollary of Lemma 3.4. We can chop $T_{\lambda,1}$ into finitely many conical pieces with $\delta \sim 1$. A properly scaled version of (3.9) states that

$$\|\chi(x/R_1)T_{\lambda,1}(x,y)\chi(y/R_2)\|_{2\to2} \lesssim \sqrt{R_1R_2} \le \lambda^\varepsilon R_1^{\frac{1}{2}} R_2^{\frac{1}{2}+\varepsilon}$$

for each piece, because $\lambda R_2 > 1$. □

We now relate the condition in Corollary 5.4 to the notion of resonance and/or eigenvalue at zero.

Lemma 5.6. *Suppose that zero is neither an eigenvalue nor a resonance of H. Then*

$$\ker(I + Z_0 G L Z_0^{-1}) = \{0\} \quad \text{on} \quad L^2(\mathbb{R}^n)$$

for $Z_0 = \langle x\rangle^{-\sigma}|\nabla|^{\frac{1}{2}}$, with $\sigma \in (\frac{1}{2}, \frac{1}{2}+\varepsilon')$ In particular, (5.1) *holds for sufficiently small λ_0.*

Proof. Suppose $f \in L^2(\mathbb{R}^n)$ satisfies

$$f + Z_0 G L Z_0^{-1} f = 0$$

We proved in Lemma 5.3 that $Z_0 G L Z_0^{-1} : L^2 \to L^2$. By a simple modification of the proof, we can obtain $Z_0 G L Z_0^{-1} : L^{2,\rho} \to L^{2,\rho+\varepsilon}$ for $\rho \in [0, \frac{n}{2} - 1)$ and for some fixed $\varepsilon > 0$ which depends on the decay rates of A and V.

This is done by commuting $\langle x\rangle^\rho$ through each of the expressions O_1, O_2, O_3. Two representative examples from the study of O_2 are presented below.

$$\begin{aligned}&\langle x\rangle^\rho\big(\langle x\rangle^{\frac12}|\nabla|^{-\frac12}w|\nabla|^{\frac12}\big)\langle x\rangle^{-\rho}\\&\qquad\qquad=\big(\langle x\rangle^{\rho+\frac12}|\nabla|^{-\frac12}\langle x\rangle^{-\rho+\varepsilon}w|\nabla|^{\frac12}\big)\big(|\nabla|^{-\frac12}\langle x\rangle^{-\rho-\varepsilon}|\nabla|^{\frac12}\langle x\rangle^{-\rho}\big)\end{aligned}$$

The constraints in Lemma 6.2 require that $\rho+\frac12<\frac{n-1}{2}$, hence the upper bound $\rho<\frac n2-1$. The second example involves the non-smooth function A, however it does not pose any difficulties.

$$\begin{aligned}&\langle x\rangle^\rho\big(|\nabla|^{\frac12}Aw^{-2}|\nabla|^{-\frac12}\big)\langle x\rangle^{-\rho}=\\&\big(\langle x\rangle^\rho|\nabla|^{\frac12}\langle x\rangle^{-\rho-\varepsilon}|\nabla|^{-\frac12}\big)\big(|\nabla|^{\frac12}Aw^{-2}\langle x\rangle^{2\varepsilon}|\nabla|^{-\frac12}\big)\big(|\nabla|^{\frac12}\langle x\rangle^{\rho-\varepsilon}|\nabla|^{-\frac12}\langle x\rangle^{-\rho}\big)\end{aligned}$$

These examples also demonstrate the flexibility to adjust the weights up or down by a factor of $\langle x\rangle^\varepsilon$. In this manner it is possible to accommodate a weight of $\langle x\rangle^{\rho+\varepsilon}$ on one side and $\langle x\rangle^{-\rho}$ on the other.

By iterating the relation $f=-Z_0GLZ_0^{-1}f$ a sufficient number of times, it follows that $f\in L^{2,(n-2)/2}$. Set $h:=Z_0^{-1}f$. Then $h=-GLh$. We have $h\in\cap_{\tau>\frac{n-4}{2}}L^{2,\tau}(\mathbb{R}^n)$ since $Z_0^{-1}:L^{2,\rho}\to L^{2,\rho-1-}$. It follows, see [14], that $Hh=0$ in the distributional sense. If $n\ge5$, we see that h is a true L^2 eigenfunction of H. In dimensions $n=3,4$ we can only conclude that h exists in polynomially weighted L^2, making it indicative of a resonance. However, by our assumption on zero energy it follows that $h=0$ and therefore $f=0$ as desired. □

6. Appendix: Hörmander's Plancherel theorem, a commutator bound, and the fractional Leibniz rule.

The following is a version of Hörmander's variable coefficient Plancherel theorem, see Theorem 1.1 in [12].

Proposition 6.1. *Let $a=a(u,v),\Psi=\Psi(u,v)\in C^\infty(\mathbb{R}^n\times\mathbb{R}^m)$ with $\mathrm{supp}\,(a)\subset B_n(0,1)\times B_m(0,1)$ and Ψ real-valued. Write $u=(u',u'')$, $v=(v',v'')$ and assume that $u',v'\in\mathbb{R}^{n_1}$ where $1\le n_1\le\min(n,m)$. Assume that on the support of a, for some finite constants $\mu>0$, and $M>1$,*

$$|\nabla_{u'}\Psi(u',u'',v',v'')-\nabla_{u'}\Psi(u',u'',w',w'')|\ge\mu|v'-w'|$$
$$\sup_{|\alpha|\le n_1+1}|D^\alpha_{u'}[\nabla_{u'}\Psi(u',u'',v',v'')-\nabla_{u'}\Psi(u',u'',w',w'')]|\le M\mu|v'-w'|$$
$$\sup_{|\alpha|\le n_1+1}\|\partial^\alpha_{u'}a\|_\infty\le M$$

Then the operator

$$(T_\lambda f)(u):=\int e^{i\lambda\Psi(u,v)}a(u,v)f(v)\,dv$$

satisfies the estimate

$$\|T_\lambda f\|_{L^2(\mathbb{R}^m)}\le C(n,m,M)\langle\lambda\mu\rangle^{-\frac{n_1}{2}}\|f\|_{L^2(\mathbb{R}^n)}$$

for all $\lambda > 0$. The constant C depends only on the dimensions n, m and M.

Proof. Define

$$T_\lambda^{(u'',v'')} f(u') = \int_{\mathbb{R}^{n_1}} e^{i\lambda\Psi(u',u'',v',v'')} a(u',u'',v',v'') f(v')\,dv'$$

where (u'', v'') are fixed parameters. We have

$$((T_\lambda^{(u'',v'')})^* T_\lambda^{(u'',v'')} f)(w') = \int K^{(u'',v'')}(w',v') f(v')\,dv'$$

$$K^{(u'',v'')}(w',v') = \int e^{i\lambda[\Psi(u,v)-\Psi(u,w)]}\,\bar{a}(u,w)a(u,v)\,du'$$

Introduce the differential operator

$$L = -i\lambda^{-1} \frac{\nabla_{u'}\Psi(u',u'',v',v'') - \nabla_{u'}\Psi(u',u'',w',w'')}{|\nabla_{u'}\Psi(u',u'',v',v'') - \nabla_{u'}\Psi(u',u'',w',w'')|^2} \cdot \nabla_{u'}$$

Note that for any $|\beta| \le n_1 + 1$,

$$Le^{i\lambda[\Psi(u',u'',v',v'')-\Psi(u',u'',w',w'')]} = e^{i\lambda[\Psi(u',u'',v',v'')-\Psi(u',u'',w',w'')]}$$

$$\Big| D_{u'}^\beta \Big[\frac{\nabla_{u'}\Psi(u',u'',v',v'') - \nabla_{u'}\Psi(u',u'',w',w'')}{|\nabla_{u'}\Psi(u',u'',v',v'') - \nabla_{u'}\Psi(u',u'',w',w'')|^2} \Big] \Big| \lesssim (\mu|v'-w'|)^{-1}$$

Hence, for any N,

$$K^{(u'',v'')}(w',v') = \int e^{i\lambda[\Psi(u,v)-\Psi(u,w)]}\,(L^*)^N \big[\bar{a}(u,w)a(u,v)\big]\,du'$$

so that by our assumptions,

$$|K^{(u'',v'')}(w',v')| \le C(n,m,M)\langle \lambda\mu|v'-w'|\rangle^{-n_1-1}$$

The lemma now follows by Schur's test. In fact there is the stronger estimate

$$\|T_\lambda f\|_{L^\infty_{u''}L^2_{u'}} \le C(n,m,M)\langle \mu\lambda\rangle^{-\frac{n_1}{2}} \|f\|_{L^1_{v''}L^2_{v'}}$$

for all $\lambda > 0$. $\square$

Next, we present three commutator bounds. The first one is from Hörmander [11], and the second two are variants which are most likely standard.

Lemma 6.2. *Suppose $\sigma, \tau \in \mathbb{R}$. Then*

$$\langle \nabla \rangle^\tau w_\sigma^{-1} \langle \nabla \rangle^{-\tau} w_\sigma$$

is L^2 bounded on $\mathbb{R}^n$. Further, let $\sigma_1 > \sigma_2$ with $\sigma_1 > -n$ and $\frac{n-1}{2} > \sigma_2$. Then

$$|\nabla|^{\frac{1}{2}} w_{\sigma_1} |\nabla|^{-\frac{1}{2}} w_{\sigma_2}^{-1}$$

is also L^2 bounded on $\mathbb{R}^n$. The reversed commutator

$$|\nabla|^{-\frac{1}{2}} w_{\sigma_1} |\nabla|^{\frac{1}{2}} w_{\sigma_2}^{-1}$$

is L^2 bounded on $\mathbb{R}^n$, $n > 1$ provided $\sigma_1 > \sigma_2$ with $\sigma_1 > -n$ and $\frac{n+1}{2} > \sigma_2$. In all these expressions, $w_\sigma(x) := \langle x \rangle^{-\sigma}$.

Proof. The first statement is from [11], see Definition 30.2.2, as well as Theorem 18.1.13. For the second we write, with $1 = \chi_{[|\xi|>1]} + \chi_{[|\xi|\le 1]}$ a smooth partition of unity,

$$
\begin{aligned}
&|\nabla|^{\frac12} w_{\sigma_1} |\nabla|^{-\frac12} w_{\sigma_2}^{-1} \\
&= |\nabla|^{\frac12}\chi_{[|\nabla|>1]} w_{\sigma_1} |\nabla|^{-\frac12}\chi_{[|\nabla|>1]} w_{\sigma_2}^{-1} + |\nabla|^{\frac12}\chi_{[|\nabla|>1]} w_{\sigma_1} |\nabla|^{-\frac12}\chi_{[|\nabla|<1]} w_{\sigma_2}^{-1} \\
&+ |\nabla|^{\frac12}\chi_{[|\nabla|<1]} w_{\sigma_1} |\nabla|^{-\frac12}\chi_{[|\nabla|>1]} w_{\sigma_2}^{-1} + |\nabla|^{\frac12}\chi_{[|\nabla|<1]} w_{\sigma_1} |\nabla|^{-\frac12}\chi_{[|\nabla|<1]} w_{\sigma_2}^{-1}
\end{aligned}
$$

We denote the terms on the right-hand side, in this order, as high-high, high-low, low-high, and low-low, respectively. By the first commutator bound it will suffice to deal with the low-low and high-low cases. We shall do this by means of standard Littlewood-Paley projections $P_j f = \phi_j * f$ where P_j denotes a projection onto frequencies 2^j. We start with the low-low case. With $w_j = w_{\sigma_j}$ it is of the form

$$
\begin{aligned}
\sum_{j,k\ge 0} 2^{-\frac{j}{2}+\frac{k}{2}} P_{-j}\big(w_1 P_{-k}(w_2^{-1} f)\big)(x) &= \sum_{j,k\ge 0} 2^{-\frac{j}{2}+\frac{k}{2}} \phi_{-j} * \big(w_1[\phi_{-k} * \frac{f}{w_2}]\big)(x) \\
&= \sum_{j,k\ge 0} 2^{-\frac{j}{2}+\frac{k}{2}} \phi_{-j} * \big(w_1[\phi_{-k} * \frac{f}{w_2}]\big)(x) \\
&= \sum_{j,k\ge 0} 2^{-\frac{j}{2}+\frac{k}{2}} \int \chi_{[|v|\sim 2^{-j}]} G_{k,f}(v) e^{-ivx}\, dv \\
\text{where}\quad G_{k,f}(v) &= \int w_1(y)\phi_{-k}(y-z) e^{ivy}\, dy \frac{f(z)}{w_2(z)}\, dz \\
&= \int P_{-k}(w_1 e^{iv\cdot})(z) \frac{f(z)}{w_2(z)}\, dz
\end{aligned}
$$

Since $|\widehat{w}_1(\xi)| \lesssim |\xi|^{-(n-\sigma_1)}$, it follows that, provided $|j-k| \gg 1$,

$$
\begin{aligned}
\sup_{|v|\sim 2^{-j}} \|P_{-k}(w_1 e^{iv\cdot})\|_{L^{2,\sigma_2}} &\lesssim 2^{k\sigma_2} 2^{-\frac{kn}{2}} (2^{-j} + 2^{-k})^{-(n-\sigma_1)} \\
\sup_{|v|\sim 2^{-j}} |G_{k,f}(v)| &\lesssim 2^{k\sigma_2} 2^{-\frac{kn}{2}} (2^{-j} + 2^{-k})^{-(n-\sigma_1)} \|f\|_2 \\
\|P_{-j}\big(w_1 P_{-k}(w_2^{-1} f)\big)\|_2 &\lesssim 2^{-j\frac{n}{2}} 2^{k\sigma_2} 2^{-\frac{kn}{2}} (2^{-j} + 2^{-k})^{-(n-\sigma_1)} \|f\|_2,
\end{aligned}
$$

whereas by Schur's lemma, for the case $|j-k| \lesssim 1$,

$$
\|P_{-j}\big(w_1 P_{-k}(w_2^{-1} f)\big)\|_2 \lesssim 2^{-j(\min(\sigma_1,n) - \max(\sigma_2,-n))}
$$

In conclusion,

$$\begin{aligned}
&\sum_{j,k\geq 0} 2^{-\frac{j}{2}+\frac{k}{2}} \|P_{-j}\big(w_1 P_{-k}(w_2^{-1} f)\big)\|_2 \\
&\qquad \lesssim \sum_{j\geq k\geq 0} 2^{-(j-k)(n+1)/2} 2^{-k(\min(\sigma_1,n)-\max(\sigma_2,-n))} \|f\|_2 \\
&\qquad\quad + \sum_{k>j\geq 0} 2^{-(k-j)((n-1)/2-\sigma_2)} 2^{-j(\min(\sigma_1,n)-\max(\sigma_2,-n))} \|f\|_2 \\
&\qquad \lesssim \|f\|_2
\end{aligned}$$

Next, consider the high-low case. It takes the form

$$\sum_{j,k\geq 0} 2^{\frac{j}{2}+\frac{k}{2}} P_j\big(w_1 P_{-k}(w_2^{-1} f)\big)(x) = \sum_{j,k\geq 0} 2^{\frac{j}{2}+\frac{k}{2}} \int \chi_{[|v|\sim 2^j]} G_{k,f}(v) e^{-ivx}\, dv$$

with $G_{k,v}$ as above. To be precise, the marginal cases $|j|+|k|\leq 1$ are already part of the previous estimate. Since $|\widehat{w}_1(\xi)| \lesssim |\xi|^{-N}$ for $|\xi|>1$,

$$\begin{aligned}
\sup_{[|v|\sim 2^j]} \|P_{-k}(w_1 e^{iv\cdot})\|_{L^{2,\sigma_2}} &\lesssim 2^{-k\frac{n}{2}} 2^{-jN} 2^{k\sigma_2} \\
\sup_{[|v|\sim 2^j]} |G_{k,f}(v)| &\lesssim 2^{-k\frac{n}{2}} 2^{-jN} 2^{k\sigma_2} \|f\|_2 \\
\|P_{-j}\big(w_1 P_{-k}(w_2^{-1} f)\big)\|_2 &\lesssim 2^{(j-k)\frac{n}{2}-jN+k\sigma_2} \|f\|_2
\end{aligned}$$

and thus, finally,

$$\begin{aligned}
&\sum_{j,k\geq 0} 2^{\frac{j}{2}+\frac{k}{2}} \|P_j\big(w_1 P_{-k}(w_2^{-1} f)\big)\|_2 \\
&\lesssim \sum_{j,k\geq 0} 2^{-j(N-(n+1)/2)} 2^{-k((n-1)/2-\sigma_2)} \|f\|_2 \lesssim \|f\|_2
\end{aligned}$$

and we are done with the second statement. The third statement is verified using the same Littlewood-Paley decomposition and many of the same estimates. The high-high term is again dominated by the corresponding piece of the first commutator bound. The low-low and high-low terms follow the analysis above since they are concerned with the same operators $P_{\pm j}\big(w_1 P_{-k}(w_2^{-1} f)\big)$. Thus we can quickly sum

$$\begin{aligned}
&\sum_{j,k\geq 0} 2^{\frac{j}{2}-\frac{k}{2}} \|P_{-j}\big(w_1 P_{-k}(w_2^{-1} f)\big)\|_2 \\
&\qquad \lesssim \sum_{j\geq k\geq 0} 2^{-(j-k)(n-1)/2} 2^{-k(\min(\sigma_1,n)-\max(\sigma_2,-n))} \|f\|_2 \\
&\qquad\quad + \sum_{k>j\geq 0} 2^{-(k-j)((n+1)/2-\sigma_2)} 2^{-j(\min(\sigma_1,n)-\max(\sigma_2,-n))} \|f\|_2 \\
&\qquad \lesssim \|f\|_2
\end{aligned}$$

for the low-low term, and

$$\begin{aligned}&\sum_{j,k\geq 0} 2^{-\frac{j}{2}-\frac{k}{2}}\|P_j\big(w_1 P_{-k}(w_2^{-1}f)\big)\|_2\\ &\lesssim \sum_{j,k\geq 0} 2^{-j(N-(n-1)/2)}2^{-k((n+1)/2-\sigma_2)}\|f\|_2 \lesssim \|f\|_2\end{aligned}$$

for the high-low term. Finally, the low-high term is also a concern. Similar to the high-low case it takes the form

$$\sum_{j,k\geq 0} 2^{\frac{j}{2}+\frac{k}{2}} P_{-j}\big(w_1 P_k(w_2^{-1}f)\big)(x) = \sum_{j,k\geq 0} 2^{\frac{j}{2}+\frac{k}{2}} \int \chi_{[|v|\sim 2^{-j}]} G_{-k,f}(v) e^{-ivx}\, dv$$

where $G_{-k,f}$ is the inverse Fourier transform of $w_1[\phi_{-k} * (w_2^{-1}f)]$ as before. Using the fact that $\widehat{w}_1(\xi)$ decays rapidly when $|\xi| \geq 1$, we can conclude that

$$\begin{aligned}\sup_{[|v|\sim 2^{-j}]} \|P_k(w_1 e^{iv\cdot})\|_{L^{2,\sigma_2}} &\lesssim 2^{k\frac{n}{2}}2^{-kN}\\ \sup_{[|v|\sim 2^{-j}]} |G_{-k,f}(v)| &\lesssim 2^{k\frac{n}{2}}2^{-kN}\|f\|_2\\ \|P_{-j}\big(w_1 P_k(w_2^{-1}f)\big)\|_2 &\lesssim 2^{(k-j)\frac{n}{2}-kN}\|f\|_2\end{aligned}$$

leading to the summation

$$\begin{aligned}&\sum_{j,k\geq 0} 2^{\frac{j}{2}+\frac{k}{2}}\|P_{-j}\big(w_1 P_k(w_2^{-1}f)\big)\|_2\\ &\lesssim \sum_{j,k\geq 0} 2^{-j((n-1)/2)}2^{-k(N-(n+1)/2)}\|f\|_2 \lesssim \|f\|_2\end{aligned}$$

□

Finally, we state a fractional Leibniz rule which is used in the proof of Lemma 2.2

Lemma 6.3. *For any $\alpha \geq 0$, $1 < p < \infty$, and arbitrarily small $\gamma > 0$,*

$$\|\,|\nabla|^\alpha (fg)\|_p \leq C_1\Big[\|\,|\nabla|^\alpha f\|_{p_1}\|g\|_{q_1} + \|\,|\nabla|^\alpha g\|_{p_2}\|f\|_{q_2}\Big]$$

provided $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$, $p \leq p_1, p_2 < \infty$, $p < q_1, q_2 \leq \infty$. The constant C_1 depends on $n, \alpha, p, p_1, p_2, q_1, q_2$.

Proof. This is standard para-differential calculus. See for example [26], page 105. □

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