STRICHARTZ ESTIMATES FOR SCHRÖDINGER OPERATORS WITH A NON-SMOOTH MAGNETIC POTENTIAL

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Abstract. We prove Strichartz estimates for the absolutely continuous evolution of a Schrödinger operator $H = (i\nabla + A)^2 + V$ in $\mathbb{R}^n$, $n \geq 3$. Both the magnetic and electric potentials are time-independent and satisfy pointwise polynomial decay bounds. The vector potential $A(x)$ is assumed to be continuous but need not possess any Sobolev regularity. This work is a refinement of previous methods, which required extra conditions on $\operatorname{div} A$ or $|\nabla|^{\frac{1}{2}}A$ in order to place the first order part of the perturbation within a suitable class of pseudo-differential operators.

1. Background

This paper continues an investigation into the dispersive properties of Schrödinger operators taking the form

$$H = (i\nabla + A(x))^2 + V(x) = -\Delta + i(A\cdot\nabla + \nabla\cdot A) + |A|^2 + V \\ = \Delta + L(x) \tag{1}$$

where $A(x)$ is a vector field in $\mathbb{R}^n$, $n \geq 3$, and $V(x)$ is a scalar function. The associated evolution equation

$$\begin{cases} iu_t(t,x) = Hu(t,x) \\ u(0,x) = u_0(x) \end{cases} \tag{2}$$

models the motion of a single charged particle within an ambient electromagnetic field, with V and A serving respectively as the electrostatic and magnetic potentials. Such operators also appear routinely when a nonlinear Schrödinger equation is linearized around a nonzero static solution.

The starting point for estimates in the free case ($L \equiv 0$) is an explicit formula for the propagator $e^{it\Delta}$ based on Fourier inversion.

$$u(t,x) = (4\pi it)^{-n/2}\int_{R^n} e^{i\frac{|x-y|^2}{4t}} u_0(x)\,dx$$

It is immediately clear that $u = e^{it\Delta}u_0$ satisfies a family of dispersive bounds

$$\|u(t,\cdot)\|_q \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{q})}\|u_0\|_{q'}, \qquad 2 \leq q \leq \infty, \tag{3}$$

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the $p = 2$ case following from Plancherel's identity and the $p = \infty$ case from evaluation of the convolution integral above. Each of the dispersive bounds with $2 \le q < \frac{2n}{n-2}$ can be incorporated into a TT^* argument (as in [19]) to prove one of the non-endpoint Strichartz inequalities,

$$\|e^{it\Delta}u_0\|_{L^p_t L^q_x} \lesssim \|u_0\|_2, \qquad \tfrac{2}{p} + \tfrac{n}{q} = \tfrac{n}{2}, \quad 2 \le p, q \le \infty. \tag{4}$$

This line of argument requires nothing more than the Hardy-Littlewood-Sobolev inequality, and the resulting estimates have played a central role in the well-posedness theory of nonlinear Schrödinger equations from their inception. It would therefore be desirable to prove an analogous statement for solutions of (2) with more general H. Unfortunately, it may not suffice to replace the Laplacian in (2) with another Schrödinger operator H because of the possible existence of point spectrum. Each eigenvalue of H gives rise to a solution of the form $u(t,x) = e^{-i\lambda t}\psi(x)$, negating the bound

$$\|e^{-itH}u_0\|_{L^p_t L^q_x} \lesssim \|u_0\|_2$$

for every $p < \infty$.

For short range self-adjoint perturbations of the Laplacian, the spectrum of H should remain absolutely continuous along the positive real axis, with discrete negative eigenvalues possibly accumulating at zero. No eigenvalues are embedded within the continuous spectrum [13]. In many cases it is possible to prove that the linear Schrödinger evolution decomposes into a discrete sum of bound states, plus a radiation term that enjoys the same dispersive properties as a free wave. Our goal is to expand the class of potentials for which this behavior is known to occur.

Following the notation in Chapter XIV of [9], define the function space

$$\|f\|_B := \sum_{j=0}^{\infty} 2^{\frac{j}{2}}\|f\|_{L^2(D_j)}, \qquad \|f\|_{B^*} := \sup_{j\ge 0} 2^{-\frac{j}{2}}\|f\|_{L^2(D_j)}, \tag{5}$$

where $D_j = \{x \in R^n : |x| \sim 2^j\}$ is a decomposition of $\mathbb{R}^n$ into dyadic shells for $j \ge 1$ and $D_0 = \{|x| \in \mathbb{R}^n : |x| \le 1\}$. These share numerous characteristics with weighted L^2, and we designate B_s to be the associated homogeneous Sobolev spaces with norm

$$\|f\|_{B_s} := \|\,|\nabla|^s f\|_B, \qquad \|f\|_{B_s^*} := \|\,|\nabla|^s f\|_{B^*}. \tag{6}$$

One other Banach space related to B will also play an important role during the discussion. Let

$$\|g\|_X := \Big(\sum_{j=0}^{\infty} 2^{3j}\|g\|^2_{L^\infty(D_j)}\Big)^{1/2} \tag{7}$$

Observe that pointwise multiplication by a function in X maps B^* to the weighted space $\langle x\rangle^{-1}L^2(\mathbb{R}^n)$, and also maps $\langle x\rangle L^2$ to B.

Theorem 1.1. *Let $H = (i\nabla + A)^2 + V$ be a magnetic Schrödinger operator on $\mathbb{R}^n$, $n \ge 3$, whose scalar and magnetic potentials are bounded functions satisfy the conditions*

$$\langle x\rangle^2 V \in L^\infty(\mathbb{R}^n), \quad \text{and} \quad \limsup_{|x|\to\infty} |x|^2|V(x)| = 0 \tag{C1}$$

$$A \text{ is continuous}, \quad \text{and} \quad \|A\|_X < \infty \tag{C2}$$

Let $P_{ac}(H)$ represent the orthogonal projection onto the absolutely continuous spectrum of H. If zero is not an eigenvalue or a resonance of H, then the Strichartz inequalities

$$\left\| e^{-itH} P_{ac}(H) u_0 \right\|_{L^p_t L^q_x} \lesssim \|u_0\|_2, \qquad \tfrac{2}{p} + \tfrac{n}{q} = \tfrac{n}{2} \tag{8}$$

are valid for each exponent $p > 2$, as well as the Kato smoothing bound

$$\left\| g(x) |\nabla|^{1/2} e^{-itH} P_{ac}(H) u_0 \right\|_{L^2_t L^2_x} \lesssim \|g^2\|_X^{\frac{1}{2}} \|u_0\|_2. \tag{9}$$

Remark 1. The magnetic field strength associated with a potential $A(x)$ is given by $B(x) = \operatorname{curl} A(x)$, so there is a large family of magnetic potentials that generate equivalent dynamics. The decay condition contained in (C2) is best suited to the Coulomb gauge ($\operatorname{div} A \equiv 0$) in that every smooth compactly supported magnetic field $B(x)$ is generated by a Coulomb-gauge magnetic potential satisfying (C2).

It is not possible to prove Theorem 1.1 by following precisely in the footsteps of the free case. In particular, no replacement for (3) is known in the case $A \neq 0$, and to achieve the full range of exponents likely requires additional regularity assumptions [8], [4]. There are two alternative approaches to this problem, both of which attempt to confine perturbative arguments to the more forgiving L^2 setting.

The first method is rooted in pseudo-differential calculus, decomposing the solution into wave packets and finding suitable parametrices to describe their respective trajectories. This techniques were applied to magnetic Schrödinger operators as early as [15]. More recent advances have highlighted the flexibility to work on manifolds other than $\mathbb{R}^n$ (as in. [3]) and with time-varying coefficients [14].

We instead adopt the framework outlined in [16], where Strichartz estimates follow directly from a pair of Kato smoothing bounds, one each for the free and perturbed propagators. The starting point is to express the perturbed evolution according to Duhamel's formula, with $H = -\Delta + L$.

$$e^{-itH} u_0 = e^{it\Delta} u_0 - i \int_0^t e^{i(t-s)\Delta} L e^{-isH} u_0 \, ds \tag{10}$$

The free evolution term is controlled by (4). To prove a Strichartz estimate for the integral term, it typically suffices to find a factorization

$$L = \sum_{j=1}^{J} Y_j^* Z_j$$

for which Z_j is a smooth perturbation relative to H on its absolutely continuous subspace and Y_j is smooth relative to the Laplacian. Following Kato's [12] theory of smoothing, this reduces to proving uniform estimates for the resolvents of both H and $-\Delta$.

It is well known (e.g. [6]) that Y_j must not have order greater than $\frac{1}{2}$, and one expects the same to be true of Z_j. The natural factorization of $L = A \cdot \nabla$ would therefore distribute half of a derivative to each of Y_j and Z_j. Some regularity of the coefficients $A(x)$ is needed in order to commute half a derivative across them.

This regularity is not always present even in simple models. To give one example, the magnetic field produced by current flow along a 1-dimensional circuit is (locally) inversely proportional to the distance from the circuit. Its associated potential then has a logarithmic singularity along the length of the circuit.

To handle less smooth magnetic potentials, we take advantage of the parabolic nature of the Schrödinger equation to replace powers of the gradient with derivatives in the time direction. The key benefit is that a time-independent function $A(x)$ commutes with operators such as $\langle\partial_t\rangle$ without apparent difficulty. We are able to arrange the calculations so that $A(x)$ is touched exclusively by time-derivatives, hence it is necessary to assume only minimal spatial regularity. There is a small price to be paid for this convenience: some of the precise structure of Duhamel's formula (i.e. $0 < s < t$) is lost in the process. Another apparent sacrifice is that the polynomial decay condition (C2) falls one-half a power short of the natural scaling law $A(x) \sim |x|^{-1}$.

In the next section we choose a factorization for the first-order perturbation found in (1) and prove preliminary smoothing bounds for each term. The task is not complicated, as many of the underlying resolvent estimates appear in the literature in precisely the form needed. Section 3 contains the direct proof of Theorem 1.1. The key ingredient is a set of modified smoothing estimates whose form is dictated by the domain of integration in (10). The technical issues raised by our use of fractional time-derivatives are addressed here.

2. Resolvent and Smoothing Estimates

The magnetic and scalar potentials collectively perturb the Laplacian with a differential operator of the form

$$\begin{aligned} L &= i(A\cdot\nabla + \nabla\cdot A) + |A|^2 + V \\ &= \sum_{j=1}^{3} Y_j^* Z_j \end{aligned}$$

where our factorization of choice is to take

$$\begin{aligned} &Y_1 = \langle x\rangle^{-1}\langle\partial_t\rangle^{\frac14}, && Y_2 = i\langle x\rangle\langle\partial_t\rangle^{-\frac14}A\cdot\nabla, && Y_3 = \big|V + |A|^2\big|^{1/2}, \\ &Z_1 = Y_2, && Z_2 = Y_1, && Z_3 = \operatorname{sign}(V + |A|^2)Y_3. \end{aligned} \tag{11}$$

Operators Y_1 and Y_2 produce vector-valued functions, and could be further split (if necessary) into their three coordinate directions. The first set of smoothing estimates for Y_j relative to the Laplacian are now easy to obtain.

Lemma 2.1. *Suppose $V(x)$ and $A(x)$ are bounded and satisfy (C1), (C2). Each of the operators Y_j, $j = 1, 2, 3$, satisfies the bound*

$$\|Y_j e^{it\Delta} f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_2 \tag{12}$$

for all functions $f \in L^2(\mathbb{R}^n)$. To prevent ambiguity, the first two of these mapping estimates are interpreted as follows.

$$\int_{\mathbb{R}^n} \langle x\rangle^{-2}\big\|e^{it\Delta}f(\,\cdot\,,x)\big\|^2_{H^{\frac14}(\mathbb{R})}\,dx \lesssim \|f\|_2^2 \tag{13}$$

$$\int_{\mathbb{R}^n} \langle x\rangle^{2}\big\|A\cdot\nabla\big(e^{it\Delta}f\big)(\,\cdot\,,x)\big\|^2_{H^{-\frac14}(\mathbb{R})}\,dx \lesssim \|A\|_X^2\|f\|_2^2 \tag{14}$$

with the intermediate function $e^{it\Delta}f$ taking values at all times $t \in \mathbb{R}$, both positive and negative.

Proof. Let λ represent the Fourier variable dual to t. After setting up the TT^* operator and taking the Fourier transform in T, the smoothing bounds for Y_1, Y_2,

Y_3 are essentially equivalent to these properties of the free resolvent.

$$\begin{aligned} \big\|\langle x\rangle^{-1}\big(R_0^+(\lambda)-R_0^-(\lambda)\big)\langle x\rangle^{-1}f\big\|_2 &\lesssim \langle\lambda\rangle^{-\frac{1}{2}}\|f\|_2 \\ \big\|\langle x\rangle g\nabla\big(R_0^+(\lambda)-R_0^-(\lambda)\big)\nabla\langle x\rangle gf\big\|_2 &\lesssim \langle\lambda\rangle^{\frac{1}{2}}\|g\|_X^2\|f\|_2 \end{aligned} \tag{15}$$

uniformly over $\lambda \geq 0$ with any exponent $\sigma \geq 1$. We have adopted the shorthand notation for resolvents

$$\begin{aligned} R_L^{\pm}(\lambda) &= \lim_{\varepsilon\downarrow 0}\big(H-(\lambda\pm i\varepsilon)\big)^{-1} \\ R_0^{\pm}(\lambda) &= \lim_{\varepsilon\downarrow 0}\big(-\Delta-(\lambda\pm i\varepsilon)\big)^{-1} \end{aligned} \tag{16}$$

The first inequality can be found in [1], and a particularly sharp version appears in [17]. The second is a standard result in the analysis of resolvents (see Theorem 5.1 in [2] for a more general statement), paired with the observation that $-\Delta R_0^{\pm}(\lambda) = I + \lambda R_0^{\pm}(\lambda)$. □

In order to show that each Z_j is a smooth perturbation relative to (the absolutely continuous part of) H, it suffices to verify that the resolvent estimates in (15) continue to hold when $R_0^{\pm}(\lambda)$ is replaced by $R_L^{\pm}(\lambda)$. It is convenient to break the problem into three separate regimes according to whether $\lambda \ll 1$, $\lambda \sim 1$, or $\lambda \gg 1$. The latter two cases have been considered at length elsewhere, so we are content to collect these results into a single statement.

Lemma 2.2. *Let A be a vector field and V a scalar function satisfying conditions (C1)-(C2). Given any number $\lambda_0 > 0$, there exists a constant $C(L,\lambda_0) < \infty$ such that*

$$\begin{aligned} \big\|\langle x\rangle^{-1}\big(R_L^+(\lambda)-R_L^-(\lambda)\big)\langle x\rangle^{-1}f\big\|_2 &\leq C(L,\lambda_0)\langle\lambda\rangle^{-\frac{1}{2}}\|f\|_2 \\ \big\|\langle x\rangle g\nabla\big(R_L^+(\lambda)-R_L^-(\lambda)\big)\nabla\langle x\rangle gf\big\|_2 &\leq C(L,\lambda_0)\langle\lambda\rangle^{\frac{1}{2}}\|g\|_X^2\|f\|_2 \end{aligned} \tag{17}$$

uniformly over all $\lambda > \lambda_0$.

Proof. According to Proposition 4.3 of [7], there exists a number $\lambda_1(L) < \infty$ such that the perturbed resolvents $R_L^{\pm}(\lambda)$ satisfy the bounds

$$\begin{aligned} \|R_L^{\pm}(\lambda)\|_{B\to B^*} &\leq C\lambda^{-\frac{1}{2}} \\ \|R_L^{\pm}(\lambda)\|_{B_{-1}\to B_1^*} &\leq C\lambda^{\frac{1}{2}} \end{aligned}$$

for all $\lambda > \lambda_1$. The proof is based on the convergence of a Born series expansion at high energy, however the presence of a large first-order differential operator in L makes these estimates quite delicate as compared to the scalar ($A \equiv 0$) case. A uniform bound over the interval $\lambda \in [\lambda_0, \lambda_1]$ is proved as part of Theorem 1.3 in [10]. It is assumed there that no eigenvalues are embedded in the positive half-line, a condition which was subsequently shown in [13] to hold for the entire class of potentials under consideration. □

Remark 2. The mapping bound from B to B^* is considerably stronger than what is needed to prove Lemma 2.2. Alternatively, by projecting onto the range $\lambda \in [\lambda_0, \infty)$ rather than the entire continuous spectrum, one can obtain smoothing estimates while assuming less decay of the potentials. It should suffice to let A, V belong to the space

$$\tilde{X} := \Big\{ g \in L^\infty : \sum_{j=0}^{\infty} \|g\|_{L^\infty(D_j)} < \infty \Big\}.$$

The remaining short interval $\lambda \in [0, \lambda_1]$ is handled by a compactness argument. By applying resolvent identities, the difference $R_L^+(\lambda) - R_L^-(\lambda)$ can be expressed as

$$R_L^+(\lambda) - R_L^-(\lambda) = (I + R_0^+(\lambda)L)^{-1}\Big(R_0^+(\lambda) - R_0^-(\lambda)\Big)(I + LR_0^-(\lambda))^{-1}$$

It is already established (see (15)) that the difference of free resolvents maps the space $Y := \langle x\rangle^{-1}L^2(\mathbb{R}^n) + B_{-1}$, equipped with the norm

$$\|f\|_Y = \inf\Big\{\|\langle x\rangle f_1\|_2 + \|f_2\|_{B_{-1}} : \ f_1 + f_2 = f\Big\} \tag{18}$$

to its dual $Y^* = \langle x\rangle L^2 \cap B_1^*$ with uniformly bounded operator norm. We therefore need only to show that $(I + R_0^+(\lambda)L)^{-1}$ exists as a uniformly bounded family of operators on Y^* over this range of λ. The same will be true of the other inverse by duality.

Lemma 2.3. *Let V and A satisfy conditions (C1)-(C2) and suppose that the associated magnetic Schrödinger operator does not have an eigenvalue or resonance at zero. Then*

$$\sup_{\lambda\in[0,\lambda_0]} \big\|(I + R_0^+(\lambda)L)^{-1}\big\|_{Y^*\to Y^*} < \infty.$$

Proof. Any potential $V \in \langle x\rangle^{-2}L^\infty$ satisfying (C1) or $A \in X$ can be approximated in the appropriate norm by a function with compact support. As a consequence, L acts as a compact operator taking Y^* to Y. Meanwhile, the resolvent $R_0^\pm(\lambda)$ maps Y back to Y^*. The end result is that $I + R_0^+(\lambda)L$ is always a compact perturbation of the identity.

The Fredholm Alternative asserts that $(I + R_0^+(\lambda)L)$ must possess a bounded inverse unless there there is a nontrivial null-space consisting of functions $f \in Y^*$ that solve the equation $f = -R_0^+(\lambda)Lf$. Any such f would also be a distributional solution of the eigenfunction equation

$$-\Delta f + Lf = \lambda f.$$

The combined efforts of [13] and [10] rule out their existence for each $\lambda > 0$. To avoid them when $\lambda = 0$ we must make include an *a priori* assumption that zero is not an eigenvalue or resonance of H. The need to exclude resonances stems from the fact that Y^* includes functions that do not decay rapidly enough to belong to $L^2(\mathbb{R}^n)$.

Remark 3. In [11] the presence of an eigenvalue at $\lambda = 0$ is shown to nullify some Strichartz estimates even after projecting away from the associated eigenvector with $P_{ac}(H)$.

Now that $(I + R_0^+(\lambda)L)^{-1}$ exists at each $\lambda \in [0, \lambda_0]$, the uniform bound on their norms follows from continuity with respect to λ. More precisely, the resolvents $R_0^+(\lambda)$ enjoy the weak-* continuity property described below. Given two functions $f, g \in Y$ and $\Lambda \in [0, \infty)$,

$$\lim_{\lambda\to\Lambda} \langle R_0^+(\lambda)f, g\rangle = \langle R_0^+(\Lambda)f, g\rangle.$$

The main idea here is again that f and g can be approximated by compactly supported functions, and that the integration kernel of $R_0^+(\Lambda) - R_0^+(\lambda)$ vanishes pointwise on bounded sets.

Suppose there exists a sequence of values λ_n that allow the operator norm of $(I + R_0^+(\lambda_n)L)^{-1}$ to grow without bound. This would imply the presence of unit functions $f_n \in Y^*$ for which $\|f_n + R_0^+(\lambda_n)f_n\|_{Y^*} \to 0$. By passing to a subsequence we may assume that $\lambda_n \to \Lambda$ and that f_n converges in the weak-* topology to a function $f_\infty \in Y^*$.

Recall that L is a compact operator, which means that Lf_n should converge to $Lf_\infty \in Y$ in norm. By the weak-*continuity of resolvents, it follows that $R_0^+(\Lambda)Lf_\infty$ is the weak-* limit of $R_0^+(\lambda_n)Lf_n$. This sequence is sufficiently close to $-f_n$ that $-R_0^+(\Lambda)Lf_\infty$ must be a weak-* limit of f_n This makes f_∞ a solution to the eigenfunction equation, so $f_\infty \equiv 0$.

The contradiction arises because now we have $\|Lf_n\|_Y \to 0$, which would cause $\|f_n+R_0^+(\lambda_n)Lf_n\|_{Y^*} \to 1$. The part of the construction where $\|f_n+R_0^+(\lambda_n)Lf_n\| \to 0$ is thus violated. □

These resolvent estimates can now be combined to prove the second Kato smoothing estimate for our factorization of L.

Lemma 2.4. *Suppose $V(x)$ and $A(x)$ are bounded and satisfy (C1), (C2). Each of the operators Z_j, $j = 1, 2, 3$, satisfies the bound*

$$\|Z_j e^{-itH} P_{ac}(H) f\|_{L^2(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|f\|_2 \tag{19}$$

for all functions $f \in L^2(\mathbb{R}^n)$. To prevent ambiguity, the first two of these mapping estimates are interpreted as follows.

$$\int_{\mathbb{R}^n} \langle x\rangle^{-2} \big\|e^{-itH}P_{ac}(H)f(\,\cdot\,,x)\big\|^2_{H^{\frac14}(\mathbb{R})}\,dx \lesssim \|f\|_2^2$$
$$\int_{\mathbb{R}^n} \langle x\rangle^{2} \big\|A\cdot\nabla\big(e^{-itH}P_{ac}(H)f\big)(\,\cdot\,,x)\big\|^2_{H^{-\frac14}(\mathbb{R})}\,dx \lesssim \|A\|_X^2\|f\|_2^2$$

with the intermediate function $e^{-itH}P_{ac}(H)f$ taking values at all times $t \in \mathbb{R}$, both positive and negative.

The Kato smoothing conclusion (9) in Theorem 1.1 is a direct interpolation of the two inequalities above.

3. Proof of Theorem 1.1

The intended purpose of Lemmas 2.1 and 2.4 was to control the integral term in Duhamel's formula (10). For each term in the factorization of L, one would like to apply (19) to obtain a function in $L^2(\mathbb{R}\times\mathbb{R}^n)$, followed by the dual form of (12) to return it to $L^2(\mathbb{R}^n)$, with the free Strichartz estimate (4) then giving a map into $L^p_tL^q_x$.

The domain of integration in (10) is clearly stated as $0 \le s \le t$. Unfortunately, the interpretation of fractional derivatives and time-propagation in both (12) and (19) is just as clearly two-sided in s. The desired Strichartz estimate (8) therefore follows from two modified smoothing bounds which include a sharp cutoff to preserve the triangular region of integration.

Lemma 3.1. *Suppose $V(x)$ and $A(x)$ are bounded and satisfy (C1), (C2). Each of the operators Z_j, $j = 1, 2, 3$, satisfies the bound*

$$\big\|Z_j\big(\mathbf{1}_{\{s\ge0\}}e^{-isH}\big)P_{ac}(H)f\big\|_{L^2_sL^2_x} \lesssim \|f\|_2 \tag{20}$$

for all functions $f \in L^2(\mathbb{R}^n)$.

Lemma 3.2. *Under the same hypotheses, each of the operators Y_j, $j = 1, 2, 3$, satisfies a bound*

$$\Big\| \int_{\mathbb{R}} \big(\mathbf{1}_{\{t\geq s\}} e^{i(t-s)\Delta}\big) Y_j^* g(s,x)\, ds \Big\|_{L^p_t L^q_x} \lesssim \|g\|_{L^2_s L^2_x} \tag{21}$$

for each Strichartz pair $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$, $p > 2$, and every $g \in L^2(\mathbb{R} \times \mathbb{R}^n)$.

Proof of Lemma 3.1. For Z_3 this is a trivial consequence of (19). For both Z_1 and Z_2 it is also an immediate consequence, as pointwise multiplication by $\mathbf{1}_{\{s\geq 0\}}$ is a bounded operator on every Sobolev space H^α, $|\alpha| < \frac{1}{2}$. ☐

Proof of Lemma 3.2. For the pointwise multiplication operator Y_3, the argument in [16] requires no modification. The dual form of (12) combines with (4) to yield a bound

$$\Big\| \int_{\mathbb{R}} e^{i(t-s)\Delta} Y_3 g(s,x)\, ds \Big\|_{L^p_t L^q_x} \lesssim \|g\|_{L^2_s L^2_x}$$

for all Strichartz pairs (p, q) including the endpoint $(2, \frac{2n}{n-2})$. The Christ-Kiselev lemma [5] (in particular the version appearing in [18]) confirms that the same bound is true if the integral is only taken over the triangular region $t \geq s$, provided there is a strict inequality $p > 2$ in the exponents.

When the same logic is applied to Y_1, the first integral estimate takes the form

$$\Big\| \int_{\mathbb{R}} \big(\langle \partial_s\rangle^{\frac{1}{4}} e^{i(t-s)\Delta}\big) \langle x\rangle^{-1} g(s,x)\, ds \Big\|_{L^p_t L^q_x} \lesssim \|g\|_{L^2_s L^2_x}.$$

Directly applying the Christ-Kiselev lemma leads to the further bound

$$\Big\| \int_{\mathbb{R}} \mathbf{1}_{\{t\geq s\}} \big(\langle \partial_s\rangle^{\frac{1}{4}} e^{i(t-s)\Delta}\big) \langle x\rangle^{-1} g(s,x)\, ds \Big\|_{L^p_t L^q_x} \lesssim \|g\|_{L^2_s L^2_x}.$$

The statement of (21) for Y_1 instead concerns the operator inequality

$$\Big\| \int_{\mathbb{R}} \langle \partial_s\rangle^{\frac{1}{4}} \big(\mathbf{1}_{\{t\geq s\}} e^{i(t-s)\Delta}\big) \langle x\rangle^{-1} g(s,x)\, ds \Big\|_{L^p_t L^q_x} \lesssim \|g\|_{L^2_s L^2_x}$$

so we are left to bound the difference between the two. This consists of two terms,

$$\int_{\mathbb{R}} \Big(\mathbf{1}_{\{t<s\}} \langle \partial_s\rangle^{\frac{1}{4}} \big(\mathbf{1}_{\{t\geq s\}} e^{i(t-s)\Delta}\big) \\ - \mathbf{1}_{\{t\geq s\}} \langle \partial_s\rangle^{\frac{1}{4}} \big(\mathbf{1}_{\{t<s\}} e^{i(t-s)\Delta}\big)\Big) \langle x\rangle^{-1} g(s,x)\, ds.$$

It suffices to control the first term, as the second will behave identically by symmetry. After the initial multiplication by $\langle x\rangle^{-1}$, the rest of the operator is a convolution in $\mathbb{R} \times \mathbb{R}^n$ with a kernel $K_1(t,x)$ satisfying the bounds

$$|K_1(t,x)| \lesssim \begin{cases} |x|^{-n-\frac{1}{2}} \langle t|x|^{-2}\rangle^{-\frac{5}{4}}, & \text{if } t \leq 1, \\ e^{-t}\, |x|^{-n+2} \langle x\rangle^{-\frac{5}{2}}, & \text{if } t > 1. \end{cases} \tag{22}$$

A direct computation then shows that K_1 belongs comfortably to both $L^1_t L^{n/2}_x$ and $L^2_t L^1_x$, thus convolution against K_1 maps $L^2(\mathbb{R} \times \mathbb{R}^n)$ (even without the extra weight $\langle x\rangle^{-1}$) to any of the mixed-norm Strichartz spaces.

The analysis for Y_2 is quite similar. Once again there is a bulk term that can be controlled by combining the dual form of (12) with (4) and the Christ-Kiselev lemma. There is a nonzero remainder generated by commuting multiplication by the cutoff $\mathbf{1}_{t\geq s}$ with fractional integration in s. In this case its precise form is

$$\int_{\mathbb{R}} \Big(\mathbf{1}_{\{t<s\}} \langle \partial_s \rangle^{-\frac{1}{4}} \big(\mathbf{1}_{\{t\geq s\}} \nabla e^{i(t-s)\Delta} \big) \\ - \mathbf{1}_{\{t\geq s\}} \langle \partial_s \rangle^{-\frac{1}{4}} \big(\mathbf{1}_{\{t<s\}} \nabla e^{i(t-s)\Delta} \big) \Big) \cdot A(x)\langle x\rangle g(s,x)\, ds.$$

As before, multiplication by $A\langle x\rangle \lesssim 1$ is composed with a convolution operator whose kernel $K_2(t,x)$ satisfies the bounds

$$|K_2(t,x)| \lesssim \begin{cases} |x|^{-n-\frac{1}{2}} \langle t|x|^{-2}\rangle^{-\frac{3}{4}}, & \text{if } t \leq 1, \\ e^{-t}\,|x|^{-n+1}\langle x\rangle^{-\frac{3}{2}}, & \text{if } t > 1. \end{cases} \tag{23}$$

The kernel K_2 belongs to both $L^2_t L^1_x$ and $L^1_t L^{n/2,\infty}$, which is still sufficient to generate a bounded map from $L^2_t L^2_x$ to each of the Strichartz spaces.

The only task remaining is to verify the asserted size estimates (22) and (23). Each one involves a fractional derivative or integral, so they are best handled via the Fourier transform. Recall that the convolution kernel for the Schrödinger propagator can be expressed as

$$\mathbf{1}_{\{t\geq 0\}} e^{it\Delta}(x) = \int_{\mathbb{R}} e^{-it\lambda} R_0^-(\lambda)(x)\, d\lambda$$

therefore the kernel $K_1(t,x)$ is defined to be

$$K_1(t,x) = \mathbf{1}_{\{t<0\}} \int_{\mathbb{R}} \langle \lambda \rangle^{\frac{1}{4}} e^{-it\lambda} R^-(\lambda)(x)\, d\lambda. \tag{24}$$

For any $t < 0$, the free resolvent $R^-(\lambda)$ has an analytic continuation in λ to the lower halfplane. Moreover, its kernel has the asymptotic bound

$$|D^\alpha R^-(\lambda)(x)| \lesssim |x|^{-n+2-|\alpha|}\, e^{|x|\Im[\lambda^{1/2}]} \langle |x|\lambda^{\frac{1}{2}}\rangle^{\frac{n-3+2|\alpha|}{2}}$$

for derivatives of order $|\alpha| = 0, 1$.

The smooth power function $\langle\lambda\rangle^{\frac{1}{4}}$ is holomorphic in the halfplane with a segment removed extending from $-i$ downward along the imaginary axis. This allows the contour of integration to be moved from the real axis to the two sides of this imaginary segment. To be precise, the original Fourier transform existed only in the distributional sense, so we must first mollify the behavior at infinity with a function such as $\frac{ki}{ki-\lambda}$ and take limits as $k \to \infty$.

On this new path, change variables so that $\lambda = -i\mu^2$. The result is that

$$\begin{aligned} |K_1(t,x)| &= C\Big| \int_1^\infty \mu(1-\mu^4)^{\frac{1}{8}} e^{-t\mu^2} R^-(-i\mu^2)(x)\, d\mu \Big| \\ &\lesssim |x|^{-n+2} \int_1^\infty \mu^{\frac{3}{2}} e^{-t\mu^2} e^{-c|x|\mu} \langle |x|\mu\rangle^{\frac{n-3}{2}}\, d\mu \\ |K_2(t,x)| &= C\Big| \int_1^\infty \mu(1-\mu^4)^{-\frac{1}{8}} e^{-t\mu^2} \nabla R^-(-i\mu^2)(x)\, d\mu \Big| \\ &\lesssim |x|^{-n+1} \int_1^\infty \mu^{\frac{1}{2}} e^{-t\mu^2} e^{-c|x|\mu} \langle |x|\mu\rangle^{\frac{n-1}{2}}\, d\mu \end{aligned}$$

where $c = \sqrt{1/2}$ is a fixed constant. One possible bound comes from dominating the factor $e^{-c|x|\mu}\langle |x|\mu\rangle^\beta$ by a constant, which leaves

$$|K_1(t,x)| \lesssim |x|^{-n+2}\int_1^\infty \mu^{\frac{3}{2}} e^{-t\mu^2}\,d\mu \lesssim \min\left(t^{-\frac{5}{4}}, e^{-t}\right)|x|^{-n+2}$$
$$|K_2(t,x)| \lesssim |x|^{-n+1}\int_1^\infty \mu^{\frac{1}{2}} e^{-t\mu^2}\,d\mu \lesssim \min\left(t^{-\frac{3}{4}}, e^{-t}\right)|x|^{-n+1}$$

These appear to be optimal when $|x|^2 < t < 1$, and are adequate for our purposes when $|x| < 1 < t$. Another alternative is to dominate $e^{-t\mu^2}$ by e^{-t}, in which case

$$|K_1(t,x)| \lesssim e^{-t}|x|^{-n+2}\int_1^\infty \mu^{\frac{3}{2}} e^{-c|x|\mu}\langle |x|\mu\rangle^{\frac{n-3}{2}}\,d\mu \lesssim e^{-t}|x|^{-n-\frac{1}{2}}$$
$$|K_2(t,x)| \lesssim e^{-t}|x|^{-n+1}\int_1^\infty \mu^{\frac{1}{2}} e^{-c|x|\mu}\langle |x|\mu\rangle^{\frac{n-1}{2}}\,d\mu \lesssim e^{-t}|x|^{-n-\frac{1}{2}}$$

These bounds appear to be sharp when $t < |x|^2 < 1$, but are also sufficient to complete the argument whenever $t, |x| \geq 1$. □

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