DISPERSIVE ESTIMATES FOR SCHRÖDINGER OPERATORS WITH MEASURE-VALUED POTENTIALS IN $\mathbb{R}^3$

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Abstract. We prove dispersive estimates for the linear Schrödinger evolution associated to an operator $-\Delta+V$ in $\mathbb{R}^3$, where the potential is a signed measure with fractal dimension at least 3/2.

1. Introduction

The dispersive properties of the free Schrödinger semigroup $e^{it\Delta}$ as a map between $L^p(\mathbb{R}^n)$ and its dual space are well understood, thanks to Plancherel's identity (or more generally the Spectral Theorem) in the case $p = 2$, and Fourier inversion in the case $p = 1$. On one endpoint, the L^2 conservation laws extend readily to any self-adjoint perturbation $H = -\Delta + V$ taking the place of $-\Delta$ as the infinitesimal generator. Our goal in this paper is to establish a corresponding $L^1 \mapsto L^\infty$ estimate in three dimensions for a class of short-range potentials $V(x)$ that include measures as admissible local singularities.

Measure-valued potentials are quite common in one dimension; the operator $-\frac{d^2}{dx^2} + c\delta_0$ is often the subject of exercises in a first quantum mechanics course. In higher dimensions there are several plausible generalizations of this example. Dispersive estimates are known in the case where $V(x)$ consists of a finite collection of point masses in $\mathbb{R}^3$ [3]. In these results the spaces L^1 and L^∞ are modified by a set of local weights because the domain of the associated Schrödinger operator consists of functions that vanish at each point mass. Here we preserve the idea of the potential describing an infinitesimally thin barrier and show that dispersive estimates are valid in unweighted $L^p(\mathbb{R}^3)$ when $V(x)$ is supported on a compact two-dimensional surface $\Sigma \subset \mathbb{R}^3$. In fact we will consider all compactly supported fractal measures of sufficiently high dimension. For many purposes the threshold dimension is 1 (in $\mathbb{R}^n$ it would be $n-2$) so that multiplication by V is compact relative to the Laplacian. We are forced to increase the threshold dimension to 3/2 in the proof of the Schrödinger dispersive estimate in order to use the best available Fourier restriction theorems.

In this paper, a compactly supported signed measure μ is called *α-dimensional* if it satisfies

$$|\mu|(B(x,r)) \le C_\mu r^\alpha \quad \text{for all } r > 0 \text{ and } x \in \mathbb{R}^3 \tag{1}$$

Nontrivial α-dimensional measures exist for any $\alpha \in [0,3]$. We also characterize potentials in terms of the *global Kato norm*, defined on signed measures in $\mathbb{R}^3$ by

Date: August 3, 2011.
This work is supported in part by NSF grant DMS-1002515.

the quantity

$$\|\mu\|_{\mathcal{K}} = \sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\mu|(dx)}{|x-y|} \tag{2}$$

Every element with finite global Kato norm is a 1-dimensional measure with $C_\mu \le \|\mu\|_{\mathcal{K}}$, by comparing $|x-y|^{-1}$ to the characteristic function of a ball. The converse is not quite true, however the Kato class contains all compactly supported measures of dimension $\alpha > 1$. We will examine this relationship more carefully in Proposition 2.

Remark 1. Kato's work [12] is more closely associated with the local uniform integrability condition (6); the first true norm of this type (integrating over $|x-y| < 1$ only) is due to Schechter [18]. We follow the naming convention in Rodnianski-Schlag [17] where the global Kato norm is applied to dispersive estimates in $\mathbb{R}^3$.

For the free Schrödinger equation in $\mathbb{R}^n$, the standard dispersive bound is

$$\|e^{it\Delta} f\|_\infty \le (4\pi|t|)^{-n/2}\|f\|_1 \tag{3}$$

In three dimensions this inequality is stable under small perturbations of the Laplacian. Once the negative part of V is sufficiently large (e.g. $\|V_-\|_{\mathcal{K}} > 4\pi$) it becomes possible for $H = -\Delta + V$ to acquire one or more bound states that evolve in place without time-decay according to a law $e^{-itH}\psi_j = e^{-it\lambda_j}\psi_j$. We wish to show that with the exception of bound states, the Schrödinger propagator of H still satisfies an estimate of the form (3).

Our main result imposes an additional spectral assumption that all eigenvalues of H be strictly negative, and that there is no resonance at zero. In this context a resonance occurs when the equation

$$\psi + (-\Delta - (\lambda \pm i0))^{-1} V\psi = 0$$

has nontrivial solutions belonging to the weighted space $\langle x\rangle^s L^2(\mathbb{R}^3)$ for each $s > \frac{1}{2}$ but not to L^2 itself. Such functions also solve $H\psi = \lambda\psi$, however the lack of square-integrability gives resonances different spectral properties from a true eigenvalue. Forbidding eigenvalues and resonances at zero is a common practice, as it is known that the continuous part of the linear Schrödinger evolution may have leading-order decay of $|t|^{-1/2}$ if zero is not a regular point of the spectrum [6], [22]. The necessity of a spectral assumption over the interval $\lambda > 0$ is uncertain but it is included here for the sake of simplicity.

Theorem 1. *Let V be a compactly supported signed measure on $\mathbb{R}^3$ of dimension $d > \frac{3}{2}$. If the Schrödinger operator $-\Delta + V$ has no resonance at zero and no eigenvalues at any $\lambda \ge 0$, then the dispersive estimate*

$$\|e^{-it(-\Delta+V)} P_{ac} f\|_\infty \lesssim |t|^{-3/2}\|f\|_1 \tag{4}$$

holds for every $f \in L^1(\mathbb{R}^3)$. The symbol P_{ac} denotes projection onto the continuous spectrum of $-\Delta + V$.

The dispersive estimate is also valid if, for a fixed $d > \frac{3}{2}$, V can be expressed as the Kato-norm limit of compactly supported d-dimensionsal measures.

Remark 2. During the course of the proof we demonstrate that resonances cannot exist at any $\lambda > 0$ (Lemma 6), and that embedded eigenvalues also cannot exist provided the dimension of V is greater than 2 (Proposition 7). The uniform resolvent bounds that are central to the argument also suffice to prove the absence of singular continuous spectrum by applying Theorem XIII.20 of [16].

Dispersive estimates with a time decay rate of $|t|^{-3/2}$ were found by Rauch [14] and Jensen-Kato [10] for initial data belonging to weighted $L^2(\mathbb{R}^3)$. The first statement of type (4) was proved by Journé-Soffer-Sogge [11] for potentials satisfying both $\hat{V} \in L^1$ and $|V(x)| \lesssim |x|^{-7-\epsilon}$. Here the effects of the perturbation are computed directly onto the Schrödinger propagator using Duhamel's formula. Several authors have since refined the older spectral methods to reproduce (4) with less restrictive conditions on the potential ([21], [17], [8], [7], [1]). This has been particularly effective in three dimensions, thanks to a convenient expression for the resolvent of the Laplacian as an integral operator. Progress along these lines in other dimensions (with time decay $|t|^{-n/2}$ for estimates on $\mathbb{R}^n$) can be found in [19], [2], [5], along with the results in [11].

We follow the same procedure as in [1], where the dispersive estimate is derived from an integrability property of a family of operators that describes the difference between the free and perturbed spectral measures. The desired integrability follows in turn from a Wiener L^1 inversion theorem involving Fourier analysis of operator-valued functions on the real line.

There are two main difficulties with extending previous work to the class of measure-valued potentials. The first is to verify that that multiplication by V has small form-bound relative to the Laplacian so that one can speak freely regarding the self-adjointness of $-\Delta + V$ and its essential spectrum. The second is to identify function spaces on which multiplication by V is well defined (which excludes any $L^p(\mathbb{R}^3)$) and the resolvent of the Laplacian has suitable asymptotics. Most of the analysis takes place in $L^2(V)$ for this reason. The embedding $\dot{H}^1(\mathbb{R}^3) \subset L^2(V)$ plays a key role mediating between the two types of operators and insuring that the end result is still translation-invariant.

Section 2 addresses the properties of V as a quadratic form over $\dot{H}^1(\mathbb{R}^3)$ and spells out basic relations between this Sobolev space and the global Kato norm. These results are not surprising but we are unaware of a careful treatment in the literature. The proof of Theorem 1 unfolds over the course of Section 3. We recall the reduction argument and abstract Wiener theorem from [1] then show that each one of its hypotheses are satisfied for the class of potentials under consideration. The high energy resolvent bounds (Theorem 8) may shed light on other scattering phenomena beyond the scope of the current paper.

2. Self-Adjointness

For any potential which is not a bounded function of x there are well known difficulties identifying the domain of $-\Delta + V$ and its adjoint operator. We can take advantage of the KLMN theorem [15, Theorem X.17] to produce a unique self-adjoint operator with the correct quadratic form on $\dot{H}^1(\mathbb{R}^3)$ provided V satisfies the form bound

$$\left| \int_{\mathbb{R}^3} |\varphi(x)|^2 \, dV \right| \leq a\|\varphi\|^2_{\dot{H}^1} + b\|\varphi\|^2_{L^2} \tag{5}$$

for some $a < 1$. It will suffice to assume that V satisfies the"local Kato condition"

$$\lim_{r\to 0^+} \sup_{y\in\mathbb{R}^3} \int_{|x-y|<r} |x-y|^{-1}\,|V|(dx) \;=\; 0. \tag{6}$$

Measures that are α-dimensional for some $\alpha > 1$ automatically satisfy (6) with an explicit modulus of continuity as r approaches zero.

Proposition 2. *Suppose μ is an α-dimensional measure, $\alpha > 1$, with support in the ball $B(0, 2^M)$. Then $\mu \in \mathcal{K}$ with the global and local estimates*

$$\|\mu\|_{\mathcal{K}} \lesssim \frac{C_\mu}{\alpha-1} 2^{(\alpha-1)M}$$
$$\text{and} \quad \sup_{y\in\mathbb{R}^3} \int_{|x-y|<r} \frac{|\mu|(dx)}{|x-y|} \lesssim \frac{C_\mu}{\alpha-1} r^{\alpha-1} \quad \text{for all } r>0.$$

Consequently, if V can be approximated in $\mathcal{K}$ by a sequence of measures μ_j with dimension $\alpha_j > 1$, then V satisfies (6).

Proof. For each point $y \in B(0, 2^{M+1})$,

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{|\mu|(dx)}{|x-y|} &\le \sum_{k=-\infty}^{\infty} 2^{-k} |\mu|(B(y, 2^k)) \\ &\lesssim C_\mu 2^{(M+2)(\alpha-1)} \Big(\frac{1}{1-2^{(1-\alpha)}} + 1 \Big) \end{aligned} \tag{7}$$

by estimating $|\mu|(B(y, 2^k)) \le C_\mu 2^{\alpha \max(k, M+2)}$. To integrate over the region of finite radius r, the sum in (7) is taken over $k \le \lceil \log r \rceil$ instead.

If $|y| > 2^{M+1}$ then the integral in (7) is easily bounded by $2|y|^{-1}|\mu|(B(0, 2^M))$ by observing that $|x-y| \sim |y|$ within the support of μ.

Convergence of μ_j in the global Kato norm forces the collection of functions $\eta_j(r) = \sup_y \int_{|x-y|<r} |x-y|^{-1}\, d|\mu_j|$ to converge uniformly in r. The property $\lim_{r\to 0} \eta_j(r) = 0$ is preserved by uniform convergence. □

The class of functions $V(x)$ (i.e. absolutely continuous measures $V(x)\,dx$) defined by property (6) is considered at length in [20]. It is suggested there that singular measures satisfying (6) may be approximated by a bounded function via convolution with smooth mollifiers. While this approach is indeed useful we emphasize that convergence in the Kato norm generally fails because of the placement of absolute values. The weak convergence argument that takes its place is detailed below.

Note that (6) implies that $\Delta^{-1}V$ is a uniformly continuous function. Moreover the entire family $\Delta^{-1}(V\omega)$ is equicontinuous, where ω ranges over the bounded measurable functions of unit norm. Both these claims are proved as part of Lemma 5 in the next section. Then we have the estimates

$$\|\tau_z \Delta^{-1} V f\|_{L^\infty(V)} \lesssim \|V\|_{\mathcal{K}} \|f\|_{L^\infty(V)} \quad \text{for all } z \in \mathbb{R}^3$$
$$\|(\mathbf{1}-\tau_z)\Delta^{-1} V f\|_{L^\infty(V)} \lesssim o(1) \|f\|_{L^\infty(V)} \qquad \text{as } |z| \to 0.$$

The vanishing rate $o(1)$ depends on the specific profile of V but is independent of the choice of $f \in L^\infty(V)$.

By duality, and the fact that translations commute with Δ^{-1}, the same operator estimates hold for $L^1(V)$ as well. Applying the Schur test to the integral kernels of these operators extends the result to all $L^p(V)$, $1 \le p \le \infty$. This gives an embedding of $\dot{H}^1(\mathbb{R}^3)$ into $L^2(V)$ because by a TT^* argument it suffices to show that $(-\Delta)^{-1}V$ is a bounded map from $L^2(V)$ to itself.

The function space $L^2(V)$ is not preserved by translations, however the embedded subspace $\dot{H}^1(\mathbb{R}^3) \subset L^2(V)$ is translation invariant. The fact that the action of translations on this subspace is continuous with respect to the $L^2(V)$ norm is also verified by a TT^* argument. Let $T = (\mathbf{1} - \tau_z)|\nabla|^{-1} : L^2(\mathbb{R}^3) \to L^2(V)$. Then

$$\begin{aligned}\|T\| &= \big(\|TT^*V\|_{L^2(V)\to L^2(V)}\big)^{1/2} \\ &= \Big(\big\|\big((\mathbf{1}-\tau_z) + (\mathbf{1}-\tau_{-z})\big)\Delta^{-1}V\big\|_{L^2(V)\to L^2(V)}\Big)^{1/2} \\ &= o(1)\end{aligned}$$

with the end result that $\|f - \tau_z f\|_{L^2(V)} \le o(1)\|f\|_{\dot{H}^1}$ for all $f \in \dot{H}^1(\mathbb{R}^3)$.

For the purposes of (5) there is not much distance between V and its close translates because

$$\begin{aligned}\int_{\mathbb{R}^3} |\varphi(x)|^2\,dV - \int_{\mathbb{R}^3} |\varphi(x)|^2\,dV(x-z) &= \int_{\mathbb{R}^3} \big(|\varphi(x)|^2 - |\varphi(x+z)|^2\big)\,dV \\ &= \int_{\mathbb{R}^3} \big(\varphi(x) - \varphi(x+z)\big)\bar{\varphi}(x)\,dV \\ &\quad + \int_{\mathbb{R}^3} \varphi(x+z)\big(\bar{\varphi}(x) - \bar{\varphi}(x+z)\big)\,dV \\ &\le \|(\mathbf{1}-\tau_{-z})\varphi\|_{L^2(V)}\big(\|\varphi\|_{L^2(V)} + \|\tau_{-z}\varphi\|_{L^2(V)}\big) \\ &\le o(1)\|\varphi\|_{\dot{H}^1}^2\end{aligned}$$

Let $V^r(x)$ be the quantity $r^{-3}\int_{B(x,r)} dV$, which represents the "average value" of V over a ball radius r. For fixed $r > 0$, $V^r(x)$ is a continuous function bounded by $r^{-2}\|V\|_{\mathcal{K}}$. Then by splitting $V = (V - V^r) + V^r$ and averaging the above inequality over all $|z| < r$ we see that

$$\begin{aligned}\Big|\int_{\mathbb{R}^3} |\varphi(x)|^2\,dV\Big| &\le o(1)\|\varphi\|_{\dot{H}^1}^2 + \Big|\int_{\mathbb{R}^3} |\varphi(x)|^2\,dV^r\Big| \\ &\le o(1)\|\varphi\|_{\dot{H}^1}^2 + r^{-2}\|V\|_{\mathcal{K}}\|\varphi\|_{L^2}^2\end{aligned}$$

so the coefficient on $\|\varphi\|_{\dot{H}^1}$ becomes smaller than 1 provided r is sufficiently close to zero.

3. The Dispersive Estimate

The proof of the dispersive estimate for the Schrödinger operator $-\Delta + V$ follows the same road-map and technical machinery as in [1]. First one represents the propagator $e^{it(-\Delta+V)}$ as an integral over the spectral measure, which is expressed in terms of resolvents via the Stone formula. Start with the expression

$$e^{-itH}P_{ac}f = \frac{1}{2\pi i}\int_0^\infty e^{-it\lambda}[R_V^+(\lambda) - R_V^-(\lambda)]f\,d\lambda, \tag{8}$$

where $R_V^\pm(\lambda) := (H - (\lambda \pm i0))^{-1}$ are the perturbed resolvents. The relationship between $R_V^\pm(\lambda)$ and the corresponding free resolvent $R_0^\pm(\lambda) = (-\Delta - (\lambda \pm i0))^{-1}$ is given by the multiplicative identities

$$R_V^\pm(\lambda) = (I + R_0^\pm(\lambda)V)^{-1}R_0^\pm(\lambda) = R_0^\pm(\lambda)(I + VR_0^\pm(\lambda))^{-1}.$$

When (8) is subjected to a change of variable $\lambda \to \lambda^2$ and integration by parts, the end result is

$$\begin{aligned} e^{-itH}P_{ac}f &= \frac{1}{\pi i}\int_{-\infty}^{\infty} e^{-it\lambda^2}\lambda R_V^+(\lambda^2)f\,d\lambda \\ &= \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{-it\lambda^2}\frac{d}{d\lambda}R_V^+(\lambda^2)f\,d\lambda \\ &= \frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{-it\lambda^2}\big(I+R_0^+(\lambda^2)V\big)^{-1}\frac{d}{d\lambda}\Big[R_0^+(\lambda^2)\Big]\big(I+VR_0^+(\lambda^2)\big)^{-1}f\,d\lambda. \end{aligned}$$

We have made a slight shift in notation here, setting $R_0^+(\lambda^2) = (-\Delta - (\lambda+i0)^2)^{-1}$ to account for the fact that $(\lambda+i0)^2 = (\lambda^2 - i0)$ when $\lambda < 0$.

The explicit formula for the free resolvent kernel in three dimensions is

$$R_0^+(\lambda^2)(x,y) = (4\pi|x-y|)^{-1}e^{i\lambda|x-y|}. \tag{9}$$

Apply Parseval's identity to the last integral in λ, taking $e^{-it\lambda^2}\frac{d}{d\lambda}\big[R_0^+(\lambda^2)\big]$ to be one of the factors. It is the Fourier transform of a bounded function in all variables, with upper bound controlled by $|t|^{-1/2}$.

Our remaining task is to show that $(I+VR_0^+(\lambda^2))^{-1}f$ is the Fourier transform of a measure on $\mathbb{R}^{1+3}$ whose total variation norm is bounded by $\|f\|_1$. This is done by applying an operator-valued Wiener L^1 Inversion Theorem [1, Theorem 3], and taking care to recognize where signed measures occur in lieu of integrable functions. The dispersive estimate then follows by integration in absolute value.

To set the background for the Wiener theorem, let X be a Banach space and $\mathcal{W}_X$ the space of bounded linear maps $T : X \to L^1(\mathbb{R}; X)$ with associated norm

$$\|T\|_{\mathcal{W}_X} = \sup_{\|f\|_X=1}\int_R \|Tf(\rho)\|_X\,d\rho. \tag{10}$$

This becomes an algebra under the product

$$S * Tf(\rho) = \int_{\mathbb{R}} S\big(Tf(\sigma)\big)(\rho-\sigma)\,d\sigma \tag{11}$$

and we use $\overline{\mathcal{W}}_X$ to denote the unital extension of $\mathcal{W}_X$.

Remark 3. In the case where there exists a family of bounded linear operators $S(\rho) : X \to X$ satisfying $S(\rho)f = Sf(\rho)$, the product formula can be restated as a convolution

$$S * T(\rho)f = \int_{\mathbb{R}} S(\rho-\sigma)Tf(\sigma)\,d\sigma.$$

More generally the associated "cross-section"operators $S(\rho)$ may be unbounded at each $\rho \in \mathbb{R}$. One such example with $X = L^1(\mathbb{R})$ is to fix an integrable function η and set $Sf(\rho) = f(\rho)\eta(x)$.

The Fourier transform of an element $T \in \mathcal{W}_X$ is computed by its action on test functions in X,

$$\hat{T}(\lambda)f = \int_{\mathbb{R}} e^{-i\lambda\rho}Tf(\rho)\,d\rho.$$

The family of operators $\hat{T}(\lambda)$ are bounded uniformly by $\|T\|_{W_X}$, continuous in λ with respect to the strong operator topology, and converge strongly to zero as $|\lambda| \to \infty$. The Fourier transform of the identity element in $\overline{\mathcal{W}}_X$ is $\hat{\mathbf{1}}(\lambda) = I$.

Products in $\overline{\mathcal{W}}_X$ correspond to pointwise composition of operators on the Fourier transform side, so an element $T \in \overline{\mathcal{W}}_X$ cannot be invertible unless $\hat{T}(\lambda)$ is invertible in $\mathcal{B}(X)$ for each λ. With two extra assumptions this pointwise condition is also sufficient.

Theorem 3 ([1, Theorem 3])**.** *Suppose T is an element of $\mathcal{W}_X$ satisfying the properties*

(C1) $\lim_{\delta\to 0} \|T(\rho) - T(\rho-\delta)\|_{\mathcal{W}_X} = 0.$
(C2) $\lim_{R\to\infty} \|\chi_{|\rho|\geq R} T\|_{\mathcal{W}_X} = 0.$

If $I + \hat{T}(\lambda)$ is an invertible element of $\mathcal{B}(X)$ for every $\lambda \in \mathbb{R}$, then $\mathbf{1} + T$ possesses an inverse in $\overline{\mathcal{W}}_X$ of the form $\mathbf{1} + S$.

In fact it is only necessary for some finite power $T^N \in \mathcal{W}_X$ (using the definition of products in $\mathcal{W}_X$ given by (11)*) to satisfy the translation-continuity condition* (C1) *rather than T itself.*

The proof is constructive, and it is important to note that the inverse of $\mathbf{1}+T$ has norm controlled by the following quantities: $\|T\|_{\mathcal{W}_X}$, $\sup_\lambda \|(I+\hat{T}(\lambda))^{-1}\|_{X\to X} =: \alpha$, the value of R for which the norm in (C2) is smaller than $1/(K\alpha)$, the exponent N, and the value of δ for which the norm in (C1) is smaller than $1/K$ when applied to T^N. The auxilliary constant K is determined by a choice of cutoff functions used during the construction.

Proposition 4. *Any subset $U \subset \mathcal{W}_X$ for which there is uniform control over these five parameters will admit a uniform bound $\sup_{T\in U} \|(\mathbf{1}+T)^{-1}\|_{\overline{\mathcal{W}}_X} \leq C < \infty$.*

For the application to Schrödinger dispersive bounds we would like to choose X to be the space $\mathcal{M}$ of finite complex Borel measures on $\mathbb{R}^3$ and $\hat{T}(\lambda) = VR_0^+(\lambda^2)$. There is a slight technical obstruction because the family of operators $T(\rho)$ have a distribution kernel

$$K(\rho, x, y) = \frac{V(x)}{4\pi|x-y|}\delta_0(\rho + |x-y|).$$

Given a measure $\mu \in \mathcal{M}$, the image Tf is a measure on $\mathbb{R}^4$ whose total variation satisfies $\|T\mu\| \leq (4\pi)^{-1}\|V\|_{\mathcal{K}}\|\mu\|_{\mathcal{M}}$ but it is not guaranteed to belong to $L^1(\mathbb{R};\mathcal{M})$. We work instead with a regularized version of T obtained by cutting off $\hat{T}(\lambda)$ to finite support. Let η be a standard cutoff function on the line, and define T_L so that $\hat{T}_L(\lambda) = \eta(\lambda/L)\hat{T}(\lambda) = \eta(\lambda/L)VR_0^+(\lambda^2)$.

For each $L > 0$ the integral kernel associated to T_L is given by

$$K_L(\rho, x, y) = L\frac{V(x)}{4\pi|x-y|}\check{\eta}(L(\rho + |x-y|)). \tag{12}$$

Therefore at a fixed value of ρ we have a bound

$$\int_{\mathbb{R}^3} |T_L\mu(\rho, \cdot\,)| \leq \frac{L}{4\pi}\|\check{\eta}\|_{\sup} \iint \frac{|V|(dx)\,|\mu|(dy)}{|x-y|} \leq \frac{L}{4\pi}\|V\|_{\mathcal{K}}\|\mu\|_{\mathcal{M}}.$$

Now that $T_L\mu$ is seen to be an $\mathcal{M}$-valued function, one can also check its L^1 norm by integrating

$$\begin{aligned}\int_{\mathbb{R}} \|T_L\mu(\rho,\,\cdot\,)\|_{\mathcal{M}}\,d\rho &\le \int_{\mathbb{R}^7} d\rho\, L\big|\check{\eta}(L(\rho+|x-y|))\big|\frac{|V|(dx)}{4\pi|x-y|}|\mu|(dy) \\ &\le \frac{\|\check{\eta}\|_1}{4\pi}\int_{\mathbb{R}^6}\frac{|V|(dx)}{|x-y|}|\mu|(dy) \\ &\le \frac{\|\check{\eta}\|_1}{4\pi}\|V\|_{\mathcal{K}}\|\mu\|_{\mathcal{M}}.\end{aligned} \tag{13}$$

This demonstrates that each $T_L \in \mathcal{W}_{\mathcal{M}}$, with $\|T_L\|$ bounded independently of L. In order to prove Theorem 1 it suffices to show that

$$\limsup_{L\to\infty} \|(\mathbf{1}+T_L)^{-1}\|_{\overline{\mathcal{W}}_{\mathcal{M}}} < \infty$$

as this will guarantee that $(\mathbf{1}+T)^{-1}f$ is a finite measure on $\mathbb{R}^{1+3}$ by taking a distributional limit. Proposition 4 provides a clear path for obtaining uniform estimates.

For the majority of the discussion we will assume that V is compactly supported and has dimension $d > \frac{3}{2}$. The list of modifications to accomodate Kato-norm limits of such potentials is given at the conclusion.

Already there is a uniform bound for $\|T_L\|_{\mathcal{W}_{\mathcal{M}}}$, the next step is to determine α by establishing a norm bound for $(I+\eta(\lambda/L)VR_0^+(\lambda^2))^{-1} : \mathcal{M} \to \mathcal{M}$ that is uniform over $\lambda \in \mathbb{R}$, $L \ge L_0$. There are separate arguments for low/intermediate and high energy. The lower-energy estimates are based on compactness and absence of embedded eigenvalues. The high-energy analysis is a decay estimate for oscillatory integrals of the type encountered in Fourier restriction operators. The technical work encountered in this step will then make it easy to set values for the remaining three parameters (R, N, and δ).

Observe that the family of operators $VR_0^+(\lambda^2)$ is norm-continuous with respect to λ via the estimate

$$\begin{aligned}\|VR_0^+(\lambda_1^2) - VR_0^+(\lambda_2^2)\|_{\mathcal{M}\to\mathcal{M}} &\le \|V\|_{\mathcal{M}} \sup_{x,y\in\mathbb{R}^3}\Big|\frac{e^{i\lambda_1|x-y|}-e^{i\lambda_2|x-y|}}{4\pi|x-y|}\Big| \\ &\le \|V\|_{\mathcal{M}}\frac{|\lambda_1-\lambda_2|}{4\pi}.\end{aligned} \tag{14}$$

Then the norm of $(I+VR_0^+(\lambda^2))^{-1}$ is a continuous function of λ, and it is bounded on any finite interval $\lambda \in [-L_0, L_0]$ provided the operator $I+VR_0^+(\lambda^2)$ is invertible for each λ. The Fredholm Alternative argument behind pointwise (in λ) invertability is standard, however its details need to be checked in the function spaces under consideration.

Lemma 5. *Suppose V is a compactly supported measure satisfying* (6)*. Then for each $\lambda \in \mathbb{R}$, the operator $VR_0^+(\lambda^2) : \mathcal{M} \to \mathcal{M}$ is compact.*

Proof. It is easier to show that the operator $R_0^-(\lambda^2)V$ acts compactly on the space of bounded functions in $\mathbb{R}^3$. The stated result follows by duality.

Let g be any bounded measurable function with $\sup_{x\in\mathbb{R}^3}|g(x)| \le 1$. Then $|R_0^-(\lambda^2)Vg(x)| \le (4\pi)^{-1}\|V\|_{\mathcal{K}}$, so the image of the unit ball is bounded as expected. To verify equicontinuity, examine the difference

$$|R_0^-(\lambda^2)Vg(x_1) - R_0^-(\lambda^2)Vg(x_2)| \le \int_{\mathbb{R}^3} |g(y)|\left|\frac{e^{-i\lambda|x_1-y|}}{4\pi|x_1-y|} - \frac{e^{-i\lambda|x_2-y|}}{4\pi|x_2-y|}\right| |V|(dy)$$

Fix a value of $r > 0$ such that $\int_{|x-y|<2r} |x-y|^{-1}|V|(dy) < \varepsilon$ for every $x \in \mathbb{R}^3$. Assuming $|x_1 - x_2| < \frac{r}{2}$, the integral splits into the regions $|y - x_1| < r$ and $|y - x_1| > r$. In the former region there is little cancellation so the integral is maximized by

$$\int_{|y-x_1|<r} (4\pi|x_1-y|)^{-1}\,|V|(dy) + \int_{|y-x_2|<2r} (4\pi|x_2-y|)^{-1}\,|V|(dy) < (2\pi)^{-1}\varepsilon.$$

In the latter region the Mean value Theorem places a bound

$$\begin{aligned}\left|\frac{e^{-i\lambda|x_1-y|}}{4\pi|x_1-y|} - \frac{e^{-i\lambda|x_2-y|}}{4\pi|x_2-y|}\right| &\le \frac{1}{2\pi}\max(|\lambda|, \tfrac{2}{|y-x_1|})\frac{|x_1-x_2|}{|y-x_1|}\\ &< (2\pi)^{-1}\max(|\lambda|, 2r^{-1})\frac{|x_1-x_2|}{|y-x_1|}\end{aligned}$$

where we have used the geometric property $|y - x_2| > \frac{1}{2}|y - x_1|$. It follows that $|R_0^-(\lambda^2)Vg(x_1) - R_0^-(\lambda^2)Vg(x_2)| < C\varepsilon$ provided $|x_1 - x_2| < \min(r, |\lambda|^{-1})\varepsilon$.

Furthermore, for all x well outside the support of V, there is the decay estimate $|R_0^-(\lambda^2)Vg(x)| < 2|x|^{-1}\|V\|_{\mathcal{M}}$. Compactness of the operator $R_0^-(\lambda^2)V$ now follows from the Arzelà-Ascoli theorem. □

Lemma 6. *Suppose $V \in \mathcal{M}$ (with no support assumption) satisfies* (6)*, and for some $\lambda \neq 0$ there exists a nonzero solution $\mu \in \mathcal{M}$ to the eigenvalue equation $(I+VR_0^+(\lambda^2))\mu = 0$. Then $R_0^+(\lambda^2)\mu$ is an L^2 eigenfunction of the operator $-\Delta+V$ with eigenvalue λ^2.*

Proof. Based on the one-sided inverse $(-\Delta - \lambda^2)R_0^+(\lambda^2) = I$ acting on $\mathcal{M}$, any measure satisfying $\mu = -VR_0^+(\lambda^2)\mu$ gives rise to the identity

$$(-\Delta - \lambda^2)R_0^+(\lambda^2)\mu = -VR_0^+(\lambda^2)\mu.$$

Then $R_0^+(\lambda^2)\mu$ belongs to the null-space of $(-\Delta + V - \lambda^2)$. The image of a typical element of $\mathcal{M}$ under $R_0^+(\lambda^2)$ belongs to the weighted space $|x|^{\frac{1}{2}+\varepsilon}L^2$, which would correspond to a resonance of $-\Delta+V$ rather than an eigenvalue. We show next that if $\lambda \neq 0$ then in fact $\mu \in \mathcal{M}$ has special mapping properties that place $R_0^+(\lambda^2)\mu \in L^2(\mathbb{R}^3)$.

Split the free resolvent into the sum of its local and nonlocal parts. The local part R_1^+ is convolution against $(4\pi|x|)^{-1}e^{i\lambda|x|}\chi_{|x|<r}$, and the nonlocal part R_2^+ is convolution against the bounded function $(4\pi|x|)^{-1}e^{i\lambda|x|}\chi_{|x|\ge r}$. The value of r is chosen so that $\sup_{y\in\mathbb{R}^3}\int_{|x-y|<r}|x-y|^{-1}\,|V|(dx) < 1$.

Each solution of the eigenvalue equation satisfies $R_0^+(\lambda^2)\mu = -R_0^+(\lambda^2)VR_0^+(\lambda^2)\mu$, which splits into $-(R_1^+ + R_2^+)VR_0^+(\lambda^2)\mu$ leading to the identity

$$R_0^+(\lambda^2)\mu = -(I + R_1^+V)^{-1}R_2^+VR_0^+(\lambda^2)\mu = (I + R_1^+V)^{-1}R_2^+\mu$$

The integration kernel of R_2^+ is bounded everywhere by r^{-1}, so $R_2^+\mu$ belongs to the space of bounded functions on $\mathbb{R}^3$. Meanwhile the splitting radius was chosen so that the operator inverse $(I+R_1^+V)^{-1}$ acting on bounded functions has a convergent Neumann series expansion. Thus $\sup_x |R_0^+(\lambda^2)\mu(x)| < \infty$.

Observe that the duality pairing $\langle R_0^+(\lambda^2)\mu, \mu\rangle$ is well defined, with the property

$$\langle R_0^+(\lambda^2)\mu, \mu\rangle = -\langle R_0^+(\lambda^2)\mu, VR_0^+(\lambda^2)\mu\rangle = -\int_{\mathbb{R}^3} |R_0^+(\lambda^2)\mu(x)|^2\, V(dx) \in \mathbb{R}$$

because V is assumed to be a real-valued signed measure. Consequently

$$\mathrm{Im}\,\langle R_0^+(\lambda^2)\mu, \mu\rangle = C\lambda^{-1}\int_{\lambda\mathbb{S}^2} |\hat{\mu}(\xi)|^2\, dS(\xi) = 0$$

by Parseval's identity, which implies that the Fourier transform of μ vanishes on the sphere radius $|\lambda|$. It follows from Proposition 4.1 of [9] that $R_0^+(\lambda^2)\mu \in L^2(\mathbb{R}^3)$. If V is assumed to have compact support then direct examination of $\hat{\mu}$ in the neighborhood of the sphere $|\xi| = |\lambda|$ shows that $R_0^+(\lambda^2)\mu$ also has rapid polynomial decay at infinity. □

Remark 4. The proposition cited shows that $R_0^+(\lambda^2)f \in L^2(\mathbb{R}^3)$ provided $f \in L^1(\mathbb{R}^3)$ and $\hat{f}\big|_{\lambda\mathbb{S}^2} = 0$. It can be extended to finite measures with only superficial changes to the proof.

Lemma 6 eliminates the possibility of embedded resonances. Pointwise existence of $(I + VR_0^+(\lambda^2))^{-1}$ still requires that $-\Delta + V$ have no embedded eigenvalues, and no resonance or eigenvalue at $\lambda = 0$. These properties are incorporated into the spectral assumptions of Theorem 1. For a large class of potentials the embedded eigenvalue condition is automatically satisfied.

Proposition 7. *Suppose V is a compactly supported measure of dimension $d > 2$. Then $-\Delta + V$ has no embedded eigenvalues $\lambda^2 > 0$.*

Proof. The unique continuation theorems in [13] asserting the absence of embedded eigenvalues apply here provided multiplication by V is a bounded map from $\dot{W}^{\frac{1}{4},4}(\mathbb{R}^3)$ to its dual space. An equivalent condition is that $\dot{W}^{\frac{1}{4},4}(\mathbb{R}^3)$ embeds as a subspace of $L^2(V)$.

If V is supported in $B(0, 2^M)$ with dimesnion $d > 2$ then V also satisfies the Kato-type condition $\sup_y \int_{\mathbb{R}^3} |x-y|^{-\gamma}\,|V|(dx) < \infty$ for any $\gamma < d$, by imitating the proof of Proposition 2. Applying the argument for Kato-class potentials in Section 2 leads to the conclusion here that $(-\Delta)^{\frac{\gamma-3}{2}}V$ is a bounded map on $L^p(V)$ for all $1 \le p \le \infty$, and in particular that $\dot{H}^{\frac{3-\gamma}{2}}(\mathbb{R}^3)$ embeds into $L^2(V)$. The same is certainly true for the non-homogeneous space $H^{\frac{3-\gamma}{2}}$ as well.

A straightforward T^*T argument shows that $(1-\Delta)^{\frac{2-\gamma}{2}}L^\infty(\mathbb{R}^3)$ is also contained in $L^2(V)$ for any $\gamma > 2$. Noting that $(1-\Delta)^{\frac{2-\gamma}{2}}$ has an integrable convolution kernel $K_\gamma(x)$ there is an immediate bound

$$\|(1-\Delta)^{\frac{2-\gamma}{2}}V(1-\Delta)^{\frac{2-\gamma}{2}}f\|_1 \le \|K_\gamma\|_1^2\|V\|_{\mathcal{M}}\|f\|_\infty.$$

Finally, the fact that $W^{\frac{1}{4},4}(\mathbb{R}^3) \subset L^2(V)$ follows by choosing $2 < \gamma < d$ and applying Riesz-Thorin interpolation to the L^2 and L^∞ estimates. The assumption that V is compactly supported permits an extension to the homogeneous Sobolev space $\dot{W}^{\frac{1}{4},4}(\mathbb{R}^3)$ as desired.

□

At this point we have shown that $\|(I+\eta(\lambda/L)VR_0^+(\lambda^2))^{-1}\|$ is uniformly bounded on any compact set $\lambda \in [-L_0, L_0]$ for all $L \geq L_0$. A separate high-energy argument is required to set a value for L_0 for which the operator inverse can be controlled independent of $|\lambda|, L > L_0$. We will show that, under the assumption that V has dimension $d > \frac{3}{2}$, that $\lim_{\lambda\to\pm\infty} \|(VR_0^+(\lambda^2))^2\| = 0$ as a bounded operator on $\mathcal{M}$. Then the operator inverse is controlled by $1 + \|V\|_{\mathcal{K}}$ for all sufficiently large λ and L, by applying (13) and summing the Neumann series.

Our calculations rely on a resolvent estimate at high energy relating $L^2(V)$ to its dual space.

Theorem 8. *Suppose V is a compactly supported measure of dimension $d > \frac{3}{2}$. There exists $\varepsilon > 0$ so that the free resolvent satisfies*

$$\|R_0^+(\lambda^2)Vf\|_{L^2(V)} \lesssim \langle\lambda\rangle^{-\varepsilon}\|f\|_{L^2(V)}. \tag{15}$$

There are close connection between the free resolvent $R_0^+(\lambda^2)$ and the restriction of Fourier transforms to the sphere $\lambda\mathbb{S}^2$. We make use of a Fourier restriction estimate proved by Erdogan [4], with the specific case of interest in three dimensions extracted below.

Theorem 9 ([4], Equation (15))**.** *Let A_R denote the annulus $|x - R| < 1$ inside $\mathbb{R}^3$, with $R > 1$. Suppose V is a compactly supported measure of dimension $d \in (\frac{3}{2}, \frac{5}{2})$. Then functions g supported in A_R satisfy an inequality*

$$\|g^\vee\|_{L^2(V)} \lesssim R^\beta \|g\|_2 \tag{16}$$

for each $\beta > \frac{7}{8} - \frac{d}{4}$. In particular it is possible to choose $\beta < \frac{1}{2}$.

Proof of Theorem 8. The specific inequality we derive has the form

$$\|R_0^+(\lambda^2)Vf\|_{L^2(V)} \lesssim \lambda^{2\beta-1}\log\lambda\|f\|_{L^2(V)}$$

uniformly over $\lambda > 4$. The logarithmic factor is most likely an artifact of the method of estimation. The case $\lambda < -4$ is identical up to complex conjugation.

The free resolvent $R_0^\pm(\lambda^2)$ acts by multiplying Fourier transforms pointwise by the distribution

$$\frac{1}{|\xi|^2 - \lambda^2} \pm i\frac{\pi}{\lambda}\, d\sigma(|\xi| = |\lambda|).$$

For the surface measure term it suffices to note that since V has compact support the dual statement to (16) implies that

$$\begin{aligned} \big\|(Vf)^\wedge\big\|_{L^2(A_R)} &\lesssim R^\beta\|f\|_{L^2(V)} \\ \text{and} \quad \big\|\nabla_\xi(Vf)^\wedge\big\|_{L^2(A_R)} &\lesssim R^\beta\|f\|_{L^2(V)} \end{aligned}$$

from which it follows that $\big\|(Vf)^\wedge\big|_{|\xi|=R}\big\|_{L^2(d\sigma)} \lesssim R^\beta\|f\|_{L^2(V)}$. The same estimates hold for higher derivatives $D_\xi^\alpha(Vf)^\wedge$ by considering $x^\alpha f \in L^2(V)$ instead of f. In particular there is control of the outward normal gradient

$$\Big\|\tfrac{\xi}{|\xi|}\cdot\nabla_\xi(Vf)^\wedge(\xi)\big|_{|\xi|=R}\Big\|_{L^2(d\sigma)} \lesssim R^\beta\|f\|_{L^2(V)}$$

which will come in handy in the next step.

Let ϕ be a smooth function supported in the annulus $\frac{1}{2} \leq |\xi| < 2$ that is identically 1 when $\frac{3}{4} \leq |\xi| \leq \frac{3}{2}$. First use ϕ to cut the Fourier multiplier away from the sphere of radius λ, with the result

$$K_\lambda(x) := \Big(\frac{1-\phi(\xi/\lambda)}{|\xi|^2-\lambda^2}\Big)^\vee(x) \sim \begin{cases} |x|^{-1} & \text{if } |x| < \lambda^{-1} \\ \lambda\, O(\lambda|x|)^{-N} & \text{if} |x| \geq \lambda^{-1} \end{cases}$$

A slightly modified version of (7) shows that $\int_{\mathbb{R}^3} |K_\lambda(x-y)|\,|V|(dx) \lesssim \lambda^{1-d}$ uniformly in y. The Schur test then shows that integration against $K_\lambda(x-y)V(y)$ defines a bounded operator on $L^p(V)$, $1 \leq p \leq \infty$, with norm comparable to λ^{1-d}. This part of the free resolvent, with frequencies removed from λ, is bounded on $L^2(V)$ and enjoys relatively rapid polynomial decay since $d > \frac{3}{2}$.

For each $\frac{\lambda}{2} < s < 2\lambda$ define $F_s(x)$ to be $Vf * \frac{\sin(s|x|)}{|x|}$ so that $s\hat{F}_s(\xi)$ is the restriction of $4\pi(Vf)^\wedge(\xi)$ to the sphere $|\xi| = s$. Based on the preceding estimates, both F_s and $\frac{d}{ds}F_s$ belong to $L^2(V)$ with norms bounded by $\lambda^{2\beta-1}$ uniformly over the interval $s \sim \lambda$.

The remaining part of the free resolvent appears as a principal value integral, with the derived bound

$$\Big\| p.v. \int_{\lambda/2}^{2\lambda} \Big(\frac{s\phi(\frac{s}{\lambda})}{s+\lambda}F_s\Big)\frac{1}{s-\lambda}\,ds\Big\|_{L^2(V)} \lesssim \lambda^{2\beta-1}\log\lambda.$$

The size and smoothness of F_s make it possible to bring the norm inside when $|s-\lambda| > 1$, and to lessen the singularity via integration by parts when $|s-\lambda| \leq 1$. □

Corollary 10. *Suppose V is a compactly supported measure of dimension $d > \frac{3}{2}$. Then there exists $\varepsilon > 0$ so that*

$$\|(VR_0^+(\lambda^2))^k\mu\|_{\mathcal{M}} \lesssim \langle\lambda\rangle^{-\varepsilon(k-1)}C(V)^k\|\mu\|_{\mathcal{M}} \tag{17}$$

Proof. The convolution kernel of $R_0^+(\lambda^2)$ is dominated by $|x-y|^{-1}$, which belongs to $L^p(V)$ uniformly over $y \in \mathbb{R}^3$ for each $1 \leq p < d$. Then $R_0^+(\lambda^2)$ maps $\mathcal{M}$ to $L^p(V)$. Interpolation between Theorem 8 and the elementary $L^1(V)$ bounds yields an operator bound

$$\|R_0^+(\lambda^2)Vf\|_{L^p(V)} \leq C(V)\langle\lambda\rangle^{-\frac{2}{p'}\varepsilon}\|f\|_{L^p(V)}$$

which can be applied $(k-1)$ times, followed by an inclusion map $L^p(V) \hookrightarrow L^1(V)$. □

For our application, choose L_0 large enough so that (17) ensures the operator norm of $(VR_0^+(\lambda^2))^2$ is less than $\frac{1}{2}$ for all $|\lambda| > L_0$. Then

$$\big\|(I + \eta(\lambda/L)VR_0^+(\lambda^2))^{-1}\big\|_{\mathcal{M}\to\mathcal{M}} < C(1 + \|V\|_{\mathcal{K}})$$

for all $\lambda, L > L_0$. We previously showed that $(I + VR_0^+(\lambda^2))^{-1}$ is bounded and continuous over the interval $\lambda \in [-L_0, L_0]$, and the introduction of a cutoff $\eta(\lambda/L)$ has no effect there once $L > L_0$. The combined bounds show that

$$\alpha_L := \sup_{\lambda\in\mathbb{R}} \|(\mathbf{1} + \hat{T}_L(\lambda))^{-1}\|_{\mathcal{M}\to\mathcal{M}} \leq \alpha < \infty$$

uniformly for $L > L_0$.

The choice of N is governed by Corollary 10. Set N to be the first integer large enough so that $(N-1)\varepsilon > 2$. This number depends only on the dimension of V without regard to any measures of its size.

The next parameter to consider is R_L, which governs the inequality

$$\|\chi_{|\rho|\geq R_L}T_L\|_{\mathcal{W}_{\mathcal{M}}} \leq (K\alpha)^{-1}$$

This is controlled by direct examination of the integral kernel of T_L in (12). More specifically,

$$
\begin{aligned}
\|\chi_{|\rho|\geq R} T_L\|_{\mathcal{W}_{\mathcal{M}}} &= \sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3}\int_{|\rho|>R} L\frac{|\check{\eta}(L(\rho+|x-y|))|}{4\pi|x-y|}\,d\rho|V|(dx) \\
&\lesssim \sup_{y\in\mathbb{R}^3}\int_{|x-y|>R-1}\frac{|V|(dx)}{|x-y|} + L^{-1}\int_{|x-y|<R-1}\frac{|V|(dx)}{|x-y|}
\end{aligned} \tag{18}
$$

The first integral is bounded by $R^{-1}\|V\|_{\mathcal{M}}$ for any $R > 2$. Meanwhile the second integal is less than $L^{-1}\|V\|_{\mathcal{K}}$ so for all $L \gtrsim \alpha\|V\|_{\mathcal{K}}$ it suffices to choose R to be the larger of 2 and $C\alpha\|V\|_{\mathcal{M}}$.

It will be convenient in the next step to have additional control of $\|\chi_{|\rho|\geq R}T_L\|_{\mathcal{W}_{\mathcal{M}}}$. Using the same construction, one can find R so that

$$\|\chi_{|\rho|\geq R}T_L\|_{\mathcal{W}_{\mathcal{M}}} \leq (KN)^{-1}(C\|V\|_{\mathcal{K}})^{-(N-1)}$$

for any $L \gtrsim N(C\|V\|_{\mathcal{K}})^N$.

The purpose of N is to give $\|(\hat{T}_L)^N(\lambda)\|$ sufficiently rapid decay (in λ) so that Fourier inversion forces $(T_L)^N(\rho)$ to be a continuously differentiable operator-valued function, with derivative smaller in operator norm than $C(V)^N$. For each fixed $0 < \delta < 1$ one can estimate the size of the difference $(T_L)^N(\rho) - (T_L)^N(\rho-\delta)$ using the mean value theorem with the result

$$
\begin{aligned}
\|(T_L)^N(\,\cdot\,) - (T_L)^N(\,\cdot\, - \delta)\|_{\mathcal{W}_{\mathcal{M}}} &\leq \int_{|\rho|\leq NR+1} \|(T_L)^N(\rho) - (T_L)^N(\rho-\delta)\|_{\mathcal{B}(\mathcal{M})}\,d\rho \\
&\qquad + 2\|\chi_{|\rho|\geq NR}(T_L)^N\|_{\mathcal{W}_{\mathcal{M}}} \\
&\leq \delta(C(V))^N NR + 2N\|\chi_{|\rho|\geq R}T_L\|_{\mathcal{W}_{\mathcal{M}}}\|T_L\|_{\mathcal{W}_{\mathcal{M}}}^{N-1}
\end{aligned}
$$

Based on the prior estimates the choice of $\delta < (NRK)^{-1}C(V)^{-N}$ will cause

$$\|T_L(\,\cdot\,) - T_L(\,\cdot\, - \delta)\|_{\mathcal{W}_{\mathcal{M}}} \leq 3/K$$

for all $L > \max(L_0, \alpha\|V\|_{\mathcal{K}}, N(C\|V\|_{\mathcal{K}})^N)$. This concludes the proof of Theorem 1 in the case where V is a compactly supported measure of dimension $d > \frac{3}{2}$.

The extension of dispersive estimates to potentials V which are the Kato-norm limit of compactly supported d-dimensional measures is more or less routine. As before the norm of T_L within $\mathcal{W}_{\mathcal{M}}$ has a uniform bound in terms of $\|V\|_{\mathcal{K}}$ from (13). No approximation properties are required in this step.

The arguments used to establish a finite value for α require more individual attention. Proposition 2 guarantees that V satisfies (6) so there are no complications regarding the self-adjointness of $-\Delta+V$. Elementary limiting arguments can be applied to (14) to show that $VR_0^+(\lambda^2)$ has continuous (but not necessarily Lipschitz) dependence on λ, and to Lemma 5 to show that each operator $VR_0^+(\lambda^2)$ is compact. Under the assumption that $-\Delta + V$ has no resonance at zero and no threshold or embedded eigenvalues, there is a uniform norm bound on $(I + VR_0^+(\lambda^2))^{-1}$ over any finite interval $\lambda \in [-L_0, L_0]$. Note that embedded resonances are still forbidden by Lemma 6.

For the low and intermediate energy estimates, it suffices for V to be the limit (in $\mathcal{K}$) of a sequence of compactly supported measures satisfying (6). Following Corollary 10, if each μ_j in the approximating sequence has dimension $d_j > \frac{3}{2}$ then

$$\lim_{\lambda\to\pm\infty}\|(VR_0^+(\lambda^2))^2\|_{\mathcal{M}\to\mathcal{M}} = 0.$$

Then L_0 can be set large enough for the Neumann series of $(I+\eta(\lambda/L)VR_0^+(\lambda^2))^{-1}$ to converge uniformly over the infinite intervals $|\lambda| > L_0$. Combined with the low-energy results this gives a finite bound for α_L once $L > L_0$.

The process for choosing N requires that $d = \inf d_j > \frac{3}{2}$, so that there is a uniform value of $\varepsilon > 0$ in Corollary 10. Then one can again declare N to be the smallest integer satisfying $(N-1)\varepsilon > 2$.

The selection criteria for R_L are little changed. So long as V is the Kato norm-limit of compactly supported potentials it is permissible to estimate

$$\lim_{R\to\infty} \sup_{y\in\mathbb{R}^3} \int_{|x-y|>R-1} \frac{|V|(dx)}{|x-y|} = 0$$

inside of (18), replacing the explicit decay rate of $R^{-1}\|V\|_{\mathcal{M}}$. Thus for some $R < \infty$ this integral term will be smaller than $(K\alpha)^{-1}$. The lower bounds placed on L are independent of the support of V and do not need further adjustment.

To find δ, first choose an approximating measure μ of dimension $d > \frac{3}{2}$ such that $\|V-\mu\|_{\mathcal{K}} \lesssim (KN)^{-1}\|V\|_{\mathcal{K}}^{-(N-1)}$. With $\tilde{T}_L$ denoting the element of $\mathcal{W}_{\mathcal{M}}$ generated by the potential μ and cutoff $\eta(\lambda/L)$, it follows from (13) that

$$\|T_L - \tilde{T}_L\|_{\mathcal{W}_{\mathcal{M}}} < (KN)^{-1}\|V\|_{\mathcal{K}}^{-(N-1)}$$

uniformly for all $L > 0$. Then the norm difference of their respective N^{th} powers is controlled by K^{-1}. Based on the properties of μ, one can choose $\delta > 0$ so that $\|(\tilde{T}_L)^N(\,\cdot\,) - (\tilde{T}_L)^N(\,\cdot - \delta)\|_{\mathcal{W}_{\mathcal{M}}} \le 3/K$ for all sufficiently large L. By the triangle inequality a similar translation bound holds for T_L as well, with

$$\|(T_L)^N(\,\cdot\,) - (T_L)^N(\,\cdot - \delta)\|_{\mathcal{W}_{\mathcal{M}}} \le 5/K.$$

All five of the parameters related to the construction of $(\mathbf{1}+T_L)^{-1}$ (namely $\|T_L\|_{\mathcal{K}}$, α_L, R, N, and δ) have an eventual uniform bound as $L \to \infty$. Therefore the operator inverses $\|(\mathbf{1}+T_L)^{-1}\|_{\overline{W}_{\mathcal{M}}}$ are also uniformly bounded, and their weak limit $(\mathbf{1}+T)^{-1}$ is a finite measure on $\mathbb{R}^{1+3}$ as desired.

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