STRICHARTZ AND SMOOTHING ESTIMATES FOR SCHRÖDINGER OPERATORS WITH LARGE MAGNETIC POTENTIALS IN $\mathbb{R}^3$

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ABSTRACT. We present a novel approach for bounding the resolvent of

$$H = -\Delta + i(A \cdot \nabla + \nabla \cdot A) + V =: -\Delta + L \tag{1}$$

for large energies. It is shown here that there exist a large integer m and a large number λ_0 so that relative to the usual weighted L^2-norm,

$$\|(L(-\Delta + (\lambda + i0))^{-1})^m\| < \frac{1}{2} \tag{2}$$

for all $\lambda > \lambda_0$. This requires suitable decay and smoothness conditions on A, V. (2) is trivial when $A = 0$, but difficult for large A since the gradient term exactly cancels the natural decay of the free resolvent. To obtain (2), we introduce a conical decomposition of the resolvent and then sum over all possible combinations of cones. Chains of cones that all point in the same direction lead to a Volterra-type gain of the form $(m!)^{-\varepsilon}$ with $\varepsilon > 0$ fixed. On the other hand, cones that are not aligned contribute little due to the assumed decay of $\widehat{A}$. We make no use of micro-local analysis, but instead rely on classical phase space techniques. As a corollary of (2), we show that the time evolution of the operator in $\mathbb{R}^3$ satisfies global Strichartz and smoothing estimates without any smallness assumptions. We require that zero energy is neither an eigenvalue nor a resonance.

1. INTRODUCTION

Magnetic Schrödinger operators on $L^2(\mathbb{R}^d)$ are of the form

$$H = -\Delta + i(A \cdot \nabla + \nabla \cdot A) + V = -\Delta + L \tag{3}$$

They model non-relativistic magnetic effects in quantum mechanics and have been much studied in the physics literature. The seminal paper [2] discusses the case of constant magnetic fields, see also [6]. The scattering theory for decaying magnetic fields is discussed in [19]. More recent results on scattering by magnetic potentials are [24] and [30]. The review [7] contains a long list of references.

There has been much activity surrounding dispersive estimates for the case $A = 0$ under suitable decay (and also regularity when $d \geq 4$) assumptions on V. In fact, in that case the harder $L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ estimate is

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now known in all dimensions $d \geq 1$ under the condition that zero energy is neither an eigenvalue nor a resonance (and there are now also results in the case when the latter assumption does not hold). The seminal paper for this class of estimates is [14] and we refer the reader to [25] for a survey of more recent work.

On the other hand, much less is known when $A \not\equiv 0$. In [27] and [9] Strichartz and smoothing estimates were obtained for small A and V. In this paper we prove the following theorem:

Theorem 1. *Let A and V be real-valued such that for all $x, \xi \in \mathbb{R}^3$*

$$\langle x\rangle|A(x)| + |DA(x)| + |V(x)| \lesssim \langle x\rangle^{-8-\varepsilon} \tag{4}$$

$$\sum_{|\alpha|\leq 2} |D^\alpha \widehat{A}(\xi)| \lesssim \langle \xi\rangle^{-3-\varepsilon} \tag{5}$$

for some $\varepsilon > 0$. Furthermore, assume that zero energy is neither an eigenvalue nor a resonance of H. Then, with P_c being the projection onto the continuous spectrum,

$$\|e^{itH}P_c f\|_{L^q_t(L^p_x)} \lesssim \|f\|_{L^2(\mathbb{R}^3)} \tag{6}$$

provided (p,q) are admissible, i.e., $\frac{2}{q} + \frac{3}{p} = \frac{3}{2}$ and $2 \leq p < 6$. Moreover, the inhomogeneous Strichartz estimates

$$\Big\|\int_{-\infty}^t e^{i(t-s)H} P_c F(s)\, ds\Big\|_{L^q_t(L^p_x)} \lesssim \|F\|_{L^{\widetilde{q}'}_t(L^{\widetilde{p}'}_x)} \tag{7}$$

hold, where (p,q) and $(\widetilde{p},\widetilde{q})$ are admissible as above. Finally, the Kato smoothing estimate

$$\int_0^\infty \big\|\langle x\rangle^{-\sigma}\langle\nabla\rangle^{\frac{1}{2}} e^{itH}P_c f\big\|_2^2\, dt \leq C\|f\|_2^2 \tag{8}$$

holds with $\sigma > 4$.

It is well-known that Strichartz estimates are basic to the scattering theory of nonlinear equations. In this case, an immediate application would be to the nonlinear Schrödinger equation with a magnetic potential which has neither eigenvalues nor a zero energy resonance. See [13] for such an application when $A = 0$ but V is present.

The definition of zero energy being neither an eigenvalue nor a resonance is the usual one: there does not exist $f \in \cap_{\tau>\frac{1}{2}} L^{2,-\tau}(\mathbb{R}^3)$, $f \not\equiv 0$ such that $Hf = 0$. In a sequel to this paper the authors will weaken the conditions on A and V — in fact, for the sake of simplicity we have chosen to impose somewhat stronger conditions on A and V than the methods of this paper actually require.

In the free case, Strichartz inequalities are typically proven by interpolating between the L^2-mass conservation law and an $L^1 \to L^\infty$ dispersive estimate. Dispersive bounds are currently unknown for any $A \neq 0$, so a different approach is required here. We adopt an argument introduced in [23], where the validity of Strichartz inequalities is instead derived from Kato's theory of smooth perturbations.

The approach in this work is perturbative around the free case despite the fact that we make no smallness assumption. Fredholm theory provides some of the necessary analytical tools, and in other cases one is required to show that the perturbation series has an appropriately large radius of convergence. The main novel ingredient in this paper is a limiting absorption estimate for large energies. It is possible, for example, to argue that the perturbative effect of a scalar potential V is small at high energies because it can be expressed in terms of an oscillatory integral. The same methods do not apply the first-order perturbation L because the presence of an additional gradient negates every benefit arising from oscillation.

More precisely, recall that in [1] and [11] it is proved that for H as in (3) under suitable decay conditions on A and V and with $\tau > \frac{1}{2}$,

$$\sup_{\lambda\in[\delta,\delta^{-1}]} \|\langle\nabla\rangle\langle x\rangle^{-\tau}(H-(\lambda^2+i0))^{-1}\langle x\rangle^{-\tau}\langle\nabla\rangle\|_{2\to 2} \le C(\delta) < \infty \tag{9}$$

provided there are no imbedded eigenvalues in the continuous spectrum. However, this is known due to recent work by Koch and Tataru [18]. It is well–known that this *limiting absorption principle* is of fundamental importance for proving dispersive estimates, at least for the case of large potentials. However, one needs to consider all real λ instead of restricting to a compact interval in the positive halfline. To extend (9) toward zero energies is similar to the case $A = 0$. This step requires the assumption on zero energy.

Note that (9) as stated cannot be extended to a semi-infinite interval since it would fail even for the free resolvent. Indeed, with $\tau > \frac{1}{2}$

$$\|\langle\nabla\rangle^{\alpha}\langle x\rangle^{-\tau}(H_0-(\lambda^2+i0))^{-1}\langle x\rangle^{-\tau}\langle\nabla\rangle^{\alpha}\|_{2\to 2} \sim \lambda^{2\alpha-1} \tag{10}$$

for any $\alpha \in [0,1]$ and all $\lambda > 1$. This shows that no more than one derivative in total can be gained here while still preserving a uniform upper bound. Furthermore, in the borderline case $\alpha = \frac{1}{2}$ there is no decay of the operator norm in the limit $\lambda \to \infty$.

We will adopt the shorthand notation

$$R_0(z) := (H_0 - z)^{-1}$$

for the resolvent of the Laplacian. The resolvent of a general operator H will be indicated by $R_H(z)$, or else $R_L(z)$ in the case where H is specifically of the form $H_0 + L$. Formally, the relationship between R_L and R_0 is captured in the identity

$$R_L(z) = (I + R_0(z)L)^{-1}R_0(z).$$

In this paper we extend (10) to $H = H_0 + L$ for the class of first-order perturbations described in Theorem 1. A unified statement of the mapping properties of the resolvent of H over the entire spectrum $\lambda > 0$ is as follows.

Theorem 2. *Suppose H is a magnetic Schrödinger operator whose potentials satisfy the conditions (4), (5). Then for $\tau > 4$ and $\alpha \in [0,1]$,*

$$\sup_{\lambda>1} \lambda^{1-2\alpha} \|\langle\nabla\rangle^\alpha \langle x\rangle^{-\tau} (H - (\lambda^2 + i0))^{-1} \langle x\rangle^{-\tau} \langle\nabla\rangle^\alpha\|_{2\to2} \lesssim 1. \tag{11}$$

If one further assumes that zero is not an eigenvalue or resonance of H, then this bound can be extended to

$$\sup_{\lambda\geq 0} \langle\lambda\rangle^{1-2\alpha} \|\langle\nabla\rangle^\alpha \langle x\rangle^{-\tau} (H - (\lambda^2 + i0))^{-1} \langle x\rangle^{-\tau} \langle\nabla\rangle^\alpha\|_{2\to2} \lesssim 1. \tag{12}$$

As a consequence, the spectrum of H is purely absolutely continuous over the entire interval $[0,\infty)$.

Remark 3. A result of type (11), in the case $\alpha = 0$, is proved in [22] using the method of Mourre commutators and micro-local analysis. In that work the potentials require only very slight polynomial decay, however they are also assumed to be infinitely differentiable, with the derivatives satisfying a symbol-like decay condition.

Clearly it would suffice to construct the operator inverse of $I+R_0(\lambda^2+i0)L$ in a suitable weighted space $L^{2,-\sigma}$, with uniform control over its norm. In the scalar ($A = 0$) case, this becomes easy for large λ as the norm of $R_0(\lambda^2+i0)V$ decreases to zero.

The case of a nontrivial magnetic potential is qualitatively different, in that the norm of $R_0(\lambda^2 + i0)L$ does not decay as $\lambda \to \infty$. This follows directly from the estimate (10), which is essentially constant when the free resolvent is paired with a full derivative. What we are able to show, via more delicate analysis, is that the spectral radius of these operators diminishes to zero even though their norm does not. This allows us to work instead with the inverse of $I-(-1)^m(R_0(\lambda^2+i0)L)^m$, which exists as a convergent power series for sufficiently large λ and m.

The estimate on $(R_0(\lambda^2+i0)L)^m$ is loosely based on principles of stationary phase. The region of criticality for the phase function is quite extensive, however it possesses a useful geometric structure reminiscent of Volterra operators. The remaining region can be handled by more conventional non-stationary phase methods, complicated slightly by the fact that we do not wish to assume much regularity of the potential. Full details of this argument are presented in Section 4.

2. The basic setup

The following result is proved in [23], see Theorem 4.1 in that paper. It is based on Kato's notion of smoothing operators, see [15]. We recall that for a self-adjoint operator H, an operator Γ is called H-smooth in Kato's sense if for any $f \in \mathcal{D}(H_0)$

$$\|\Gamma e^{itH} f\|_{L^2_t L^2_x} \leq C_\Gamma(H)\|f\|_{L^2_x} \tag{13}$$

or equivalently, for any $f \in L^2_x$

$$\sup_{\varepsilon>0} \|\Gamma R_H(\lambda \pm i\varepsilon) f\|_{L^2_\lambda L^2_x} \leq C_\Gamma(H)\|f\|_{L^2_x}. \tag{14}$$

We shall call $C_\Gamma(H)$ the smoothing bound of Γ relative to H. Let $\Omega \subset \mathbb{R}$ and let P_Ω be a spectral projection of H associated with a set Ω. We say that Γ is H-smooth on Ω if ΓP_Ω is H-smooth. We denote the corresponding smoothing bound by $C_\Gamma(H,\Omega)$. It is not difficult to show (see e.g. [21]) that, equivalently, Γ is H-smooth on Ω if

$$\sup_{\beta>0} \|\chi_\Omega(\lambda)\Gamma R_H(\lambda \pm i\beta) f\|_{L^2_\lambda L^2_x} \leq C_\Gamma(H,\Omega)\|f\|_{L^2_x}. \tag{15}$$

The first conclusion (6) of Theorem 1 is obtained by applying Proposition 4 below. The related inhomogeneous estimate (7) then follows from (6) in a standard fashion via the Christ-Kiselev lemma [3]. The remainder of the paper will therefore be devoted to verifying the conditions needed in Proposition 4. Along the way we establish the smoothing bound (8) as an additional consequence of these conditions.

Proposition 4. *Let $H_0 = -\Delta$ and $H = H_0 + L$ with $L = \sum_{j=1}^J Y_j^* Z_j$. We assume that each Y_j is H_0 smooth with a smoothing bound $C_B(H_0)$ and that for some $\Omega \subset \mathbb{R}$ the operators Z_j are H-smooth on Ω with the smoothing bound $C_A(H,\Omega)$. Assume also that the unitary semigroup e^{itH_0} satisfies the estimate*

$$\|e^{itH_0}\psi_0\|_{L^q_t L^r_x} \leq C_{H_0}\|\psi_0\|_{L^2_x} \tag{16}$$

for some $q \in (2,\infty]$ and $r \in [1,\infty]$. Then the semigroup e^{itH} associated with $H = H_0 + L$, restricted to the spectral set Ω, also verifies the estimate (16), *i.e.,*

$$\|e^{itH}P_\Omega\psi_0\|_{L^q_t L^r_x} \leq JC_{H_0}C_B(H_0)C_A(H,\Omega)\|\psi_0\|_{L^2_x} \tag{17}$$

We refer the reader to [23] for the proof. Note that this approach does not capture the Keel-Tao endpoint (which would correspond to $q = 2$) — the reason being the Christ-Kiselev lemma which is used in the proof of Proposition 4.

To apply this proposition we write, with a decreasing weight $w(x) = \langle x\rangle^{-\sigma}$, for some sufficiently large $\sigma > 0$,

$$\begin{aligned} L &= 2iA\cdot\nabla + i\mathrm{div}A + V \\ &= 2iAw^{-1}\cdot\nabla\langle\nabla\rangle^{-\frac{1}{2}}\langle\nabla\rangle^{\frac{1}{2}}w + 2iA\cdot\nabla(w^{-1})w + i\mathrm{div}A + V \\ &= \sum_{j=1}^2 Y_j^* Z_j \end{aligned} \tag{18}$$

where

$$\begin{aligned} &Y_1^* := 2iAw^{-1}\cdot\nabla\langle\nabla\rangle^{-\frac{1}{2}}, \quad Z_1 := \langle\nabla\rangle^{\frac{1}{2}}w \\ &Y_2^* := [2iA\cdot\nabla(w^{-1})w + i\mathrm{div}A + V]w^{-1}, \quad Z_2 := w \end{aligned} \tag{19}$$

Once the operators Z_1 and Z_2 are verified as being H-smooth on the interval $\Omega = [0, \infty)$, the Kato smoothing estimate (8) in Theorem 1 follows immediately.

Throughout this paper, we shall treat $\sigma > 0$ as a parameter. In various places we shall specify how large it needs to be chosen. Eventually, we shall require $\sigma > 4$, which will lead to the condition (4). It is standard that Y_1 and Y_2 are H_0-smooth provided

$$|A(x)| + |\mathrm{div} A(x)| + |V(x)| \lesssim \langle x \rangle^{-1-\sigma-\varepsilon} \tag{20}$$

We now start discussing the smoothing properties of Z_1 and Z_2 relative H. It will suffice to discuss Z_1.

Let us first consider intermediate energies λ^2, i.e., $\lambda \in [\lambda_0^{-1}, \lambda_0] = \mathcal{J}_0$ with λ_0 large. Then it was shown in [11], see also [1], that the resolvent of H satisfies the following bound

$$\sup_{\lambda \in \mathcal{J}_0} \|\langle x \rangle^{-\frac{1}{2}-\varepsilon} \langle \nabla \rangle R_L(\lambda^2 + i0) f\|_2 \leq C(\lambda_0) \, \|\langle x \rangle^{\frac{1}{2}+\varepsilon} \langle \nabla \rangle^{-1} f\|_2$$

(in fact, a stronger bound was proved in [11]). More precisely, this bound follows provided there are no eigenvalues of H in the interval $\mathcal{J}_0$. The latter property (absence of imbedded eigenvalues) is shown in [18] to hold for the entire family of potentials under consideration. An elementary proof, following [8], is also possible if one assumes that $\nabla V(x)$ exists and possesses moderate pointwise decay. It is therefore safe to conclude that

$$\sup_{\lambda \in \mathcal{J}_0} \|Z_1 R_L(\lambda^2 + i0) Z_1^*\|_{2\to 2} \leq C(\lambda_0) \|\langle \nabla \rangle^{\frac{1}{2}} w \langle \nabla \rangle^{-1} \langle x \rangle^{\frac{1}{2}+\varepsilon}\|_{2\to 2}^2 \leq C(\lambda_0) \tag{21}$$

since $\|\langle \nabla \rangle^{\frac{1}{2}} w \langle \nabla \rangle^{-1} \langle x \rangle^{\frac{1}{2}+\varepsilon}\|_{2\to 2} < \infty$ by pseudo-differential calculus. Finally, by Kato's smoothing theory, see [21] Theorem XIII.30, we conclude that Z_1 is H-smooth on $\Omega = \mathcal{J}_0$.

Note that this argument does not carry over to $\lambda \to \infty$ (in other words, for magnetic potentials, unlike the case of V alone, large energies are not easy). This is due to the fact that the limiting absorption principles in [11] and [1] do not yield a gain of one derivative uniformly in λ. We devote Section 4 to this issue.

Next, we turn to small energies.

3. Small energies

As usual, this is reduced to zero energy. For the latter, we need to impose an invertibility condition which amounts to boundedness of the resolvent $R_L(0)$ between suitable spaces. More precisely, by the resolvent identity,

$$R_L(\lambda^2 + i0) = (1 + R_0(\lambda^2 + i0)L)^{-1} R_0(\lambda^2 + i0)$$

provided the inverse on the right-hand side exists. Therefore,

$$\begin{aligned}&\|Z_1 R_L(\lambda^2+i0)Z_1^*\|_{2\to2}\\&=\|Z_1(1+R_0(\lambda^2+i0)L)^{-1}Z_1^{-1}Z_1R_0(\lambda^2+i0)Z_1^*\|_{2\to2}\\&\le\|Z_1(1+R_0(\lambda^2+i0)L)^{-1}Z_1^{-1}\|_{2\to2}\|Z_1R_0(\lambda^2+i0)Z_1^*\|_{2\to2}\end{aligned}$$

By the smoothing properties of Z_1 relative to H_0,

$$\sup_\lambda \|Z_1 R_0(\lambda^2+i0)Z_1^*\|_{2\to2}<\infty$$

provided $\sigma>1$. For $\lambda>1$ this follows from Agmon [1] with $\sigma>\frac12$, whereas for small λ this can be reduced to a Hilbert-Schmidt norm provided $\sigma>1$, see [12].

Thus, we need to verify that

$$\begin{aligned}&\sup_{|\lambda|<\lambda_0^{-1}}\|Z_1(1+R_0(\lambda^2+i0)L)^{-1}Z_1^{-1}\|_{2\to2}\\&=\sup_{|\lambda|<\lambda_0^{-1}}\|\langle\nabla\rangle^{\frac12}w(1+R_0(\lambda^2+i0)L)^{-1}w^{-1}\langle\nabla\rangle^{-\frac12}\|_{2\to2}<\infty\end{aligned}$$

for some choice of large λ_0. First, we consider the case $\lambda=0$. As usual, we let $G:=R_0(0)$.

Lemma 5. *Assume that $L=2i\nabla\cdot A-i\mathrm{div}A+V$ satisfies $|A(x)|\lesssim\langle x\rangle^{-\sigma-1-\varepsilon}$, $|\mathrm{div}A(x)|+|V(x)|\lesssim\langle x\rangle^{-2\sigma}$ with $\sigma>1$. Then $Z_1GLZ_1^{-1}$ is a compact operator on L^2.*

Proof. First, we consider only the $2i\nabla\cdot A$ part of L. We claim that

$$\|\langle\nabla\rangle G\nabla\cdot Aw^{-1}f\|_2\lesssim\|f\|_2 \tag{22}$$

To see this, observe that by Plancherel

$$\|D^\alpha G\nabla\cdot Aw^{-1}f\|_2\lesssim\|Aw^{-1}f\|_2\lesssim\|f\|_2$$

provided $|\alpha|=1$. On the other hand, we will show that

$$\|G\nabla\cdot Aw^{-1}f\|_2\lesssim\|Aw^{-1}f\|_{L^{2,1+\varepsilon}}\lesssim\|f\|_2 \tag{23}$$

It suffices to prove that multiplication by $\frac{\xi}{|\xi|^2}$ maps $H^{1+\varepsilon}$ to L^2. Let $\chi(\xi)$ be a smooth cut-off around zero. Then $(1-\chi(\xi))\frac{\xi}{|\xi|^2}$ maps $H^{1+\varepsilon}$ to itself which is even stronger. Moreover, by Hölder's inequality and Sobolev imbedding,

$$\|\chi(\xi)|\xi|^{-1}g\|_2\le\|\chi(\xi)|\xi|^{-1}\|_{L^{3-}}\|g\|_{L^{6+}}\lesssim\|g\|_{H^{1+\varepsilon}}$$

which implies (23). In conclusion, we have proved (22).

Thus,

$$\langle\nabla\rangle^{\frac12}wG\nabla\cdot Aw^{-1}\langle\nabla\rangle^{-\frac12}=\langle\nabla\rangle^{\frac12}w\langle\nabla\rangle^{-1}\langle\nabla\rangle G\nabla\cdot Aw^{-1}\langle\nabla\rangle^{-\frac12}$$

is compact in L^2, since $\langle\nabla\rangle^{\frac12}w\langle\nabla\rangle^{-1}$ is compact in L^2.

Second, we discuss the $\widetilde{V} := -i\mathrm{div}A + V$ part of L. It will suffice to show that

$$\|\langle\nabla\rangle^{\frac{1}{2}} wG\widetilde{V}w^{-1}\langle x\rangle^{\varepsilon} f\|_2 \lesssim \|f\|_2 \tag{24}$$

since then

$$\langle\nabla\rangle^{\frac{1}{2}} wG\widetilde{V}w^{-1}\langle\nabla\rangle^{-\frac{1}{2}} = \langle\nabla\rangle^{\frac{1}{2}} wG\widetilde{V}w^{-1}\langle x\rangle^{\varepsilon}\langle x\rangle^{-\varepsilon}\langle\nabla\rangle^{-\frac{1}{2}}$$

is compact. To prove (24), we argue as before:

$$\|\langle\nabla\rangle^{\frac{1}{2}} wG\widetilde{V}w^{-1}\langle x\rangle^{\varepsilon} f\|_2 \lesssim \|\nabla wG\widetilde{V}w^{-1}\langle x\rangle^{\varepsilon} f\|_2 + \|wG\widetilde{V}w^{-1}\langle x\rangle^{\varepsilon} f\|_2$$

The second summand on the right-hand side is controlled by the Hilbert-Schmidt norm provided $\sigma > 1$. The first summand is similar to the proof of (23). □

The following remark will be used to analyze the condition at energy zero.

Remark 6. Combining (22) with the usual boundedness properties of G on weighted L^2 spaces (i.e., $G : L^{2,\beta_1} \to L^{2,-\beta_2}$, provided $\beta_1 + \beta_2 > 2$ and $\beta_1, \beta_2 > \frac{1}{2}$, see [12] or [10]) yields

$$\|GLh\|_{L^{2,-\tau+\varepsilon/2}(\mathbb{R}^3)} \le \|h\|_{L^{2,-\tau}(\mathbb{R}^3)} \tag{25}$$

for any $\tau > (1+\varepsilon)/2$ provided $|\mathrm{div}A(x)| + |V(x)| \lesssim \langle x\rangle^{-2-\varepsilon}$ and $|A(x)| \lesssim \langle x\rangle^{-\tau-1-\varepsilon}$.

As an immediate consequence we arrive at the following.

Corollary 7. *Assume that* $\ker(I + Z_1GLZ_1^{-1}) = \{0\}$ *as an operator on* $L^2(\mathbb{R}^3)$. *Then* $I + Z_1GLZ_1^{-1}$ *is invertible on* L^2. *Moreover,*

$$\|Z_1(I + R_0(\lambda^2 + i0)L)^{-1}Z_1^{-1}\|_{2\to 2} < \infty \tag{26}$$

uniformly for small λ. *An analogous statement holds with* Z_2 *instead of* Z_1.

Proof. The first statement is Fredholm's alternative. Note that

$$(I + Z_1GLZ_1^{-1})^{-1} = Z_1(I + GL)^{-1}Z_1^{-1}$$

where GL on the right-hand side is an operator on $Z_1^{-1}(L^2(\mathbb{R}^3))$. By the same token, (26) is the same as

$$\|(I + Z_1R_0(\lambda^2 + i0)LZ_1^{-1})^{-1}\|_{2\to 2} < \infty$$

uniformly for small λ. To prove this, we write

$$I + Z_1R_0(\lambda^2 + i0)LZ_1^{-1} = I + Z_1GLZ_1^{-1} + Z_1B_\lambda LZ_1^{-1}$$

where $B_\lambda = R_0(\lambda^2 + i0) - G$. By a Neumann series argument, it suffices to prove that

$$\sup_{|\lambda|<\lambda_0^{-1}} \|Z_1B_\lambda LZ_1^{-1}\|_{2\to 2} \to 0 \tag{27}$$

as $\lambda_0 \to \infty$. We have the following bounds on the kernel of $B_\lambda(x,y)$:

$$\begin{aligned} |B_\lambda(x,y)| &\lesssim \frac{|\lambda|^\gamma}{|x-y|^{1-\gamma}}, \qquad 0\le\gamma\le 1 \\ |\nabla_x B_\lambda(x,y)\nabla_y| &\lesssim \frac{\lambda}{|x-y|^2}+\frac{\lambda^2}{|x-y|} \\ |\nabla_x B_\lambda(x,y)|+|B_\lambda(x,y)\nabla_y| &\lesssim \frac{\lambda}{|x-y|} \end{aligned} \tag{28}$$

To prove (27), we estimate

$$\begin{aligned} \|Z_1 B_\lambda L Z_1^{-1}\|_{2\to2} &\lesssim \|\nabla w B_\lambda L w^{-1}\|_{2\to2} + \|w B_\lambda L w^{-1}\|_{2\to2} \\ &\lesssim \|w \nabla B_\lambda L w^{-1}\|_{2\to2} + \|w B_\lambda L w^{-1}\|_{2\to2} \end{aligned}$$

As before, we write $L = 2i\nabla\cdot A + \widetilde{V}$. To conclude the argument, one now uses (28) together with Schur's lemma (for the $\frac{\lambda}{|x-y|^2}$ term) as well as the Hilbert-Schmidt norm (for the others). □

We now relate the condition in Corollary 7 to the notion of resonance and/or eigenvalue at zero.

Lemma 8. *Suppose that zero is neither an eigenvalue nor a resonance of H. Then under the conditions of Lemma 5 one has*

$$\ker(I + Z_j G L Z_j^{-1}) = \{0\} \quad \text{on} \quad L^2(\mathbb{R}^3)$$

for $j = 1, 2$. In particular, (26) *holds for small λ.*

Proof. Suppose $f \in L^2(\mathbb{R}^3)$ satisfies

$$f + Z_1 G L Z_1^{-1} f = 0$$

Set $h := Z^{-1} f$. Then $h = -GLh \in L^{2,-\sigma}(\mathbb{R}^3)$. Applying Remark 6 we see that $h \in L^{2,-(\sigma-\frac{\varepsilon}{2})}(\mathbb{R}^3)$. Repeating this process shows that $h \in \cap_{\tau>\frac12} L^{2,-\tau}(\mathbb{R}^3)$. It follows, see [12] and [10] that $Hh = 0$ in the distributional sense. However, by our assumption on zero energy it follows that $h = 0$ and therefore $f = 0$ as desired. The argument for Z_2 is analogous. □

4. Large energies

The goal of this section is to prove the bound

$$\sup_{\lambda>\lambda_0} \|Z_1 R_L(\lambda^2 + i0) Z_1^*\|_{2\to2} < \infty \tag{29}$$

with some large λ_0 and similarly with Z_2. Here Z_1, Z_2 are as in (19) with $w(x) = \langle x\rangle^{-\sigma}$. Note that in combination with the previous sections this will finish the proof of Theorem 1. In order to establish (29) we introduce some notations: for any $\lambda > 1$ define

$$\widehat{T_\lambda f}(\xi) = \langle \xi/\lambda\rangle^{-1}\,\widehat{f}(\xi)$$

as well as

$$S_\lambda := T_\lambda^{-1} R_0(\lambda^2 + i0)$$

It is clear that for any τ one has

$$T_\lambda : L^{2,\tau} \to L^{2,\tau} \tag{30}$$

with a bound independent of λ. Indeed, by the Fourier transform this is equivalent to

$$\langle \xi/\lambda \rangle^{-1} \; : \; H^\tau \to H^\tau$$

as a multiplication operator with norm independent of λ. The decay in large $|\xi|$ suggests that T_λ also improves local regularity. More precisely,

$$\| \langle \nabla \rangle^\alpha T_\lambda f \|_{L^{2\tau}} \lesssim \langle \lambda \rangle^\alpha \|f\|_{L^{2,\tau}}$$

for any α in the range $[0, 1]$.

The Fourier multiplier associated to S_λ is less well behaved, however we still have the following bound:

Lemma 9. *With S_λ as before*

$$\| \langle \nabla \rangle^\alpha S_\lambda f \|_{L^{2,-\tau}} \lesssim \lambda^{\alpha - 1} \|f\|_{L^{2,\tau}}$$

provided $\tau > \frac{1}{2}$ and $\alpha \in [0, 1]$.

Proof. By algebra of operators,

$$\langle \nabla/\lambda \rangle^2 R_0(\lambda^2 + i0) = 2R_0(\lambda^2 + i0) - \lambda^{-2} I \tag{31}$$

Therefore, if $\tau > \frac{1}{2}$ and $\lambda > 1$, then

$$\begin{aligned} \| \langle \nabla/\lambda \rangle^2 R_0(\lambda^2 + i0) f \|_{L^{2,-\tau}} &\le 2 \|R_0(\lambda^2 + i0) f\|_{L^{2,-\tau}} + \lambda^{-2} \|f\|_{L^{2,-\tau}} \\ &\lesssim \lambda^{-1} \|f\|_{L^{2,\tau}} \end{aligned}$$

by Agmon's limiting absorption principle [1]. Finally, we bound

$$\| \langle \nabla \rangle^\alpha S_\lambda f \|_{L^{2,-\tau}} \le \| \langle \nabla \rangle^\alpha T_\lambda \|_{L^{2,-\tau} \to L^{2,-\tau}} \| \langle \nabla/\lambda \rangle^2 R_0(\lambda^2 + i0) f \|_{L^{2,-\tau}}$$

which finishes the proof. □

Remark 10. The resolvent estimate that we used above,

$$\|R_0(\lambda^2 + i0) f\|_{L^{2,-\tau}} \lesssim \lambda^{-1} \|f\|_{L^{2,\tau}}$$

follows directly from the calculations in [1], but only appears as a separately stated theorem in later works such as [12].

Next, we combine T_λ and S_λ with Z_1 (in what follows, we will treat Z_1, the case of Z_2 being easier):

Lemma 11. *Using the previous notations,*

$$\|Z_1 T_\lambda f\|_2 \lesssim \lambda^{\frac{1}{2}} \|f\|_{L^{2,-\sigma}}, \qquad \|S_\lambda Z_1^* f\|_{L^{2,-\sigma}} \lesssim \lambda^{-\frac{1}{2}} \|f\|_2$$

for all $\lambda > 1$.

Proof. First,

$$Z_1 T_\lambda = w\langle\nabla\rangle^{\frac12} T_\lambda + [\langle\nabla\rangle^{\frac12}, w]T_\lambda \tag{32}$$

Now, by the same Fourier argument as above,

$$\|\langle\nabla\rangle^{\frac12} T_\lambda f\|_{L^{2,-\sigma}} \lesssim \lambda^{\frac12}\|f\|_{L^{2,-\sigma}}$$

Hence, the first term on the right-hand side of (32) satisfies the desired bound. On the other hand, the commutator term in (32) can be written as

$$\|[\langle\nabla\rangle^{\frac12}, w]T_\lambda\|_{L^{2,-\sigma}\to L^2} \le \|[\langle\nabla\rangle^{\frac12}, w]w^{-1}\|_{L^2\to L^2}\|wT_\lambda\|_{L^{2,-\sigma}\to L^2} \lesssim 1$$

uniformly in λ. Indeed, $[\langle\nabla\rangle^{\frac12}, w]w^{-1}$ is a pseudo-differential operator of order zero and is therefore L^2 bounded, whereas

$$\|wT_\lambda\|_{L^{2,-\sigma}\to L^2} \lesssim 1$$

by the preceding. Next, we claim that

$$\|Z_1 S_\lambda^* f\|_2 \lesssim \lambda^{-\frac12}\|f\|_{L^{2,\sigma}} \tag{33}$$

which will finish the proof by duality. To prove (33), we write

$$Z_1 S_\lambda^* = Z_1 T_\lambda T_\lambda^{-2} R_0(\lambda^2 - i0)$$

¿From (31),

$$\|T_\lambda^{-2} R_0(\lambda^2 - i0) f\|_{L^{2,-\sigma}} \lesssim \lambda^{-1}\|f\|_{L^{2,\sigma}}$$

provided $\sigma > \frac12$. Secondly, we have already shown that

$$Z_1 T_\lambda \;:\; L^{2,-\sigma} \to L^2$$

with bound $\lambda^{\frac12}$. Thus, (33) follows and we are done. □

Now we continue with the proof of (29). By the resolvent identity, we have

$$Z_1 R_L(\lambda^2 + i0) Z_1^* = Z_1 T_\lambda (I + S_\lambda L T_\lambda)^{-1} S_\lambda Z_1^*$$

provided $I + S_\lambda L T_\lambda$ is invertible as an operator on $L^{2,-\sigma}$. This invertibility will follow by means of a partial Neumann series via the following lemma. The proof of this lemma, which is the crucial technical ingredient in this paper, will be given in the next section.

Lemma 12. *Given A and V as in Theorem 1 as well as a positive constant $c > 0$, there exist sufficiently large $m = m(c)$ and $\lambda_0 = \lambda_0(c)$ such that*

$$\sup_{\lambda > \lambda_0} \|(R_0(\lambda^2 + i0)L)^m\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \le c \tag{34}$$

Here $\sigma > 4$.

In view of Lemmas 11, the estimate in (29) follows from the following result:

Corollary 13. *With the notation from above and for $\sigma > 4$, we have*

$$(I + S_\lambda L T_\lambda)^{-1} \;:\; L^{2,-\sigma} \to L^{2,-\sigma}$$

with a uniform norm for all large λ.

Proof. We write the partial Neumann series, with m as in Lemma 12,

$$(I + S_\lambda L T_\lambda)^{-1} = \Big(\sum_{k=0}^{m} (-1)^k (S_\lambda L T_\lambda)^k\Big)(I + (-1)^{m+1}(S_\lambda L T_\lambda)^{m+1})^{-1}$$

By Lemma 12, the inverse on the right-hand side exists on $L^{2,-\sigma}$ with a uniform bound for all $\lambda > \lambda_0$. Indeed, one has

$$(S_\lambda L T_\lambda)^{m+1} = S_\lambda L (R_0(\lambda^2 + i0)L)^m T_\lambda$$

so that, with some constant C_1 that only depends on A and V,

$$\begin{aligned}
&\|(S_\lambda L T_\lambda)^{m+1}\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \\
&\le \|S_\lambda L\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \|(R_0(\lambda^2+i0)L)^m\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \|T_\lambda\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \\
&\le C_1\, c < \frac{1}{2}
\end{aligned}$$

provided c was chosen sufficiently small. Furthermore,

$$S_\lambda L T_\lambda = 2i S_\lambda A \cdot \nabla T_\lambda + S_\lambda (i\mathrm{div} A + V) T_\lambda$$

By (30) and Lemma 9,

$$\|S_\lambda (i\mathrm{div} A + V) T_\lambda f\|_{L^{2,-\sigma}} \lesssim \|f\|_{L^{2,-\sigma}}$$

Furthermore, again from (30) and Lemma 9,

$$\|S_\lambda A \cdot \nabla T_\lambda\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \lesssim \|S_\lambda A\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \|\nabla T_\lambda\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \lesssim \lambda^{-1}\lambda \lesssim 1$$

which means the finite sum of terms $k = 0, \ldots, m$ can be controlled with a bound independent of λ. □

At this point the proof of Theorem 2 is essentially complete, thanks to the identity

$$\begin{aligned}
\|\langle\nabla\rangle^\alpha R_L(\lambda^2 + i0)\langle\nabla\rangle^\alpha f\|_{L^{2,-\sigma}} &= \|\langle\nabla\rangle^\alpha T_\lambda (I + S_\lambda L T_\lambda)^{-1} S_\lambda \langle\nabla\rangle^\alpha f\|_{L^{2,-\sigma}} \\
&\le \|\langle\nabla\rangle^\alpha T_\lambda\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \|(I + S_\lambda L T_\lambda)^{-1}\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \|\langle\nabla\rangle^\alpha S_\lambda f\|_{L^{2,-\sigma}} \\
&\lesssim \langle\lambda\rangle^{2\alpha-1} \|(I + S_\lambda L T_\lambda)^{-1}\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \|f\|_{L^{2,\sigma}}
\end{aligned}$$

For large λ, the desired operator bound for $(I + S_\lambda L T_\lambda)^{-1}$ is given by Corollary 13. For small λ, it follows from the Fredholm theory arguments in Section 3. One needs only to repeat the steps taken in that section using the operator T_λ^{-1} in place of Z_1.

5. The proof of Lemma 12

We start with the following observation: since $L = 2i\nabla \cdot A - i\mathrm{div} A + V$,

$$(R_0(\lambda^2 + i0)L)^m = (2i)^m (R_0(\lambda^2 + i0)\nabla \cdot A)^m + E_m(\lambda^2) \tag{35}$$

where the error $E_m(\lambda^2)$ satisfies

$$\|E_m(\lambda^2)\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \le C(m, V, A)\, \lambda^{-1}$$

provided

$$|A(x)| + |\mathrm{div} A(x)| + |V(x)| \lesssim \langle x\rangle^{-1-\varepsilon}$$

This follows from Agmon's limiting absorption principle [1].

Thus, we are reduced to $L = \nabla \cdot A$. To deal with this case, we shall perform a conical decomposition of the free resolvent. Let $\{\chi_{\mathcal{S}}\}_{\mathcal{S}\in\Sigma}$ be a smooth partition of unity on the sphere S^2 which is adapted to a family of caps Σ of diameter δ (which is a small parameter to be specified later). For the most part, we shall drop the subscript $\mathcal{S}$ so that χ will denote any one of these cut-offs and $\widetilde{\chi}$ will typically denote a cut-off associated to χ but with a dilated cap as support. We write

$$R_0(\lambda^2+i0)(x) = \sum_{\mathcal{S}\in\Sigma} \frac{e^{i\lambda|x|}}{4\pi|x|}\chi_{\mathcal{S}}(x/|x|) =: \sum_{\mathcal{S}\in\Sigma} R_{\mathcal{S}}(\lambda^2+i0)(x) \tag{36}$$

We begin by studying the multiplier associated with $R_{\mathcal{S}}$.

Proposition 14. *Let χ be a cut-off supported in a δ-cap on S^2 where $\delta > 0$ is a small parameter. Let K_λ be defined as*

$$K_\lambda(\xi) := \mathcal{F}\Big[\frac{e^{i\lambda|x|}}{4\pi|x|}\chi(x/|x|)\Big](\xi)$$

where $\mathcal{F}$ denotes the Fourier transform. Then

$$K_\lambda(\xi) := \begin{cases} O(\lambda^{-2}\delta^2) & \text{if } |\xi| < \frac{\lambda}{2} \\ O(|\xi|^{-2}) & \text{if } |\xi| > 10\lambda \end{cases}$$

and for $\frac{\lambda}{2} \le |\xi| \le 10\lambda$

$$K_\lambda(\xi) = O(\delta^{-2}\lambda^{-2}) + \lambda^{-1}\widetilde{\chi}(\xi/|\xi|) f_\delta(\xi/\lambda)\big[d\sigma_{\lambda S^2}(\xi) + i\text{P.V.}\frac{1}{\lambda-|\xi|}\big] \tag{37}$$

where $\widetilde{\chi}$ is a modified cut-off supported in twice the cap of χ and $\|f_\delta\|_\infty \lesssim 1$, $\|f_\delta\|_{C^\alpha} \lesssim \delta^{-2\alpha}$ for any $\alpha < 1$.

Proof. By scaling, it suffices to set $\lambda = 1$. Let

$$K(\xi) = K_{\varepsilon,\delta}(\xi) = \int e^{-\varepsilon|x|}\frac{e^{i|x|}}{4\pi|x|}\chi(x/|x|)e^{-ix\cdot\xi}\,dx$$

We assume that $\chi(x)$ is smooth and supported in a δ-neighborhood of $(0,0,1)$. Furthermore, by symmetry we can assume that $\xi_2 = 0$. We shall use the identity

$$\begin{aligned} K(\xi) &= \int_{S^2}\int_0^\infty e^{-\varepsilon r}e^{ir}r\chi(\omega)e^{-ir\omega\cdot\xi}\,drd\sigma(\omega) \\ &= \int_{S^2}(\varepsilon - i(1-\omega\cdot\xi))^{-2}\chi(\omega)\,d\sigma(\omega) \end{aligned} \tag{38}$$

Case 1: $\xi_3 \le \frac{1}{2}$ and $|\xi| \le 10$.

Then, from (38) we infer that

$$K(\xi) = O(\delta^2)$$

Case 2: $|\xi_3| \ge \frac{|\xi|}{2}$ and $|\xi| > 10$.

In this case $|1 - \omega \cdot \xi| \gtrsim |\xi|$ so that

$$|K(\xi)| \lesssim \frac{\delta^2}{|\xi|^2}$$

from (38).

Cases 3 and 4 deal with $|\xi| > 10$, $|\xi_3| < \frac{|\xi|}{2}$. Note that then

$$\{\omega \cdot \xi \ : \ \omega \in 2\mathcal{S}\} = [a(\xi), b(\xi)]$$

where $\mathcal{S} := \mathrm{supp}(\chi) \subset S^2$ and $b(\xi) - a(\xi) \lesssim \delta$. Moreover, $2\mathcal{S}$ denotes the twice dilated set $\mathcal{S}$.

Case 3: $|\xi_3| \le \frac{|\xi|}{2}$ and $|\xi| > 10$, with $1 \notin [|\xi|a(\xi), |\xi|b(\xi)]$.

Then

$$\begin{aligned}
|K(\xi)| &\lesssim \int_{a(\xi)+\delta}^{b(\xi)-\delta} \frac{\delta\, ds}{(1 - s|\xi|)^2} \lesssim \frac{1}{|\xi|} \int_{1-(b(\xi)-\delta)|\xi|}^{1-(a(\xi)+\delta)|\xi|} \frac{\delta}{u^2}\, du \\
&\lesssim \frac{\delta}{|\xi|}\big(|1 - (b(\xi) - \delta)|\xi||^{-1} + |1 - (a(\xi) + \delta)|\xi||^{-1}\big) \\
&\lesssim \frac{\delta}{|\xi|} \frac{1}{\delta|\xi|} \lesssim |\xi|^{-2}
\end{aligned}$$

as claimed

Case 4: $|\xi_3| \le \frac{|\xi|}{2}$ and $|\xi| > 10$, with $1 \in [|\xi|a(\xi), |\xi|b(\xi)]$.

Here we write

$$K(\xi) = \int_I \frac{\delta\psi(s)}{(s|\xi| - 1 - i\varepsilon)^2}\, ds$$

where I is an interval of size $\sim \delta$ centered at $|\xi|^{-1}$ and $|\psi^{(\ell)}(s)| \lesssim \delta^{-\ell}$. Shifting the center of ψ to 0 and abusing notation, we obtain

$$\begin{aligned}
K(\xi) &= \int_{-c\delta}^{c\delta} \frac{\delta\psi(s)}{(s|\xi| - i\varepsilon)^2}\, ds = \frac{\delta}{|\xi|} \int_{-c\delta}^{c\delta} \frac{\psi'(s)\, ds}{s|\xi| - i\varepsilon} \\
&= \frac{\delta}{|\xi|} \int_{-c\delta}^{c\delta} \frac{\psi'(s) - \psi'(0)}{s|\xi| - i\varepsilon} + \frac{\delta}{|\xi|} \int_{-c\delta}^{c\delta} \frac{\psi'(0)\, ds}{s|\xi| - i\varepsilon} \\
&= O(|\xi|^{-2})
\end{aligned}$$

using the bounds on ψ' and ψ''.

Case 5: $\xi_3 \ge \frac{1}{2}$ and $\frac{1}{2} \le |\xi| \le 10$.

In this case we write

$$K(\xi) = O(\delta^{-2}) + \int_{\delta^{-2}}^{\infty} e^{-\varepsilon r} e^{ir} r a(r\xi)\, dr$$

where

$$a(r\xi) = \int_{S^2} \chi(\omega) e^{-ir\omega\cdot\xi}\, d\sigma(\omega)$$

By stationary phase

$$a(r\xi) = \frac{e^{-ir|\xi|}}{r|\xi|}\Big(\chi(\xi/|\xi|) + \widetilde{\chi}(\xi/|\xi|)\frac{\delta^{-2}}{|\xi|r}\Big) + O\Big(\frac{\delta^{-4}}{|\xi|^3 r^3}\Big)$$

Therefore, with $e := \frac{\xi}{|\xi|}$,

$$\begin{aligned}
K(\xi) &= O(\delta^{-2}) + \frac{\chi(e)}{|\xi|}\frac{e^{[-\varepsilon+i(1-|\xi|)]\delta^{-2}}}{\varepsilon+i(1-|\xi|)} + \frac{\widetilde{\chi}(e)}{|\xi|^2\delta^2}\int_{\delta^{-2}}^{\infty}\frac{e^{[-\varepsilon+i(1-|\xi|)]r}}{r}\,dr \\
&= O(\delta^{-2}) + \frac{1}{\varepsilon-i(1-|\xi|)}\Big[\frac{\chi(e)}{|\xi|}e^{[-\varepsilon+i(1-|\xi|)]\delta^{-2}} + \frac{\widetilde{\chi}(e)}{|\xi|^2}e^{[-\varepsilon+i(1-|\xi|)]\delta^{-2}} \\
&\qquad - \frac{\widetilde{\chi}(e)}{|\xi|^2\delta^2}\int_{\delta^{-2}}^{\infty}\frac{e^{[-\varepsilon+i(1-|\xi|)]r}}{r^2}\,dr\Big] \\
&=: O(\delta^{-2}) + \frac{\widetilde{\chi}(e)}{\varepsilon-i(1-|\xi|)}f_{\varepsilon,\delta}(\xi)
\end{aligned}$$

Note that, as $\varepsilon \to 0$, $f_\delta := \lim_{\varepsilon\to0} f_{\varepsilon,\delta}$ satisfies

$$\|f_\delta\|_\infty \lesssim 1,\ \|f_\delta\|_{C^\alpha} \lesssim \delta^{-2\alpha}$$

for any $\alpha < 1$. Furthermore, in the sense of distributions,

$$\lim_{\varepsilon\to0}\frac{\widetilde{\chi}(e)}{\varepsilon-i(1-|\xi|)} = \widetilde{\chi}(e)\big[d\sigma_{S^2}(\xi) + i\mathrm{P.V.}\frac{1}{1-|\xi|}\big]$$

Here $\widetilde{\chi}$ on the right-hand side is modified to absorb any needed constants. □

We shall use this result to prove Proposition 16 below, which is a version of the limiting absorption principle. First, we prove a lemma about the action of the singular part in (37) on functions.

Lemma 15. *Given a function φ in $\mathbb{R}^3$ and $0 < \alpha < 1$, define*

$$[\varphi]_\alpha(\xi) := \sup_{|h|<1}\frac{|\varphi(\xi)-\varphi(\xi+h)|}{|h|^\alpha}$$

Then

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^3}\varphi(\xi)\Big[\sigma_{\lambda S^2}(d\xi) + i\,\mathrm{P.V.}\frac{d\xi}{\lambda-|\xi|}\chi_{[\lambda-1<|\xi|<\lambda+1]}\Big]\Big| \\
&\lesssim \|\varphi\|_{L^1(\lambda S^2)} + C_\alpha\,\|[\varphi]_\alpha\|_{L^1(\lambda S^2)}
\end{aligned}$$

provided the right-hand side is finite.

Proof. It suffices to consider the principal value part. Thus,

$$
\begin{aligned}
&\Big|\mathrm{P.V.}\int_{||\xi|-\lambda|<1}\frac{\varphi(\xi)}{|\xi|-\lambda}\,d\xi\Big| = \Big|\mathrm{P.V.}\int_{\lambda-1}^{\lambda+1}\frac{\beta^2\int_{S^2}\varphi(\beta\theta)d\sigma(\theta)}{\beta-\lambda}\,d\beta\Big| \\
&\lesssim \int_{\lambda-1}^{\lambda+1}\frac{\beta^2\int_{S^2}|\varphi(\beta\theta)-\varphi(\lambda\theta)|d\sigma(\theta)}{|\beta-\lambda|}\,d\beta \\
&+\Big|\mathrm{P.V.}\int_{\lambda-1}^{\lambda+1}\frac{\beta^2\int_{S^2}\varphi(\lambda\theta)d\sigma(\theta)}{\beta-\lambda}\,d\beta\Big| \qquad (39)
\end{aligned}
$$

The second term in (39) satisfies

$$\lesssim \lambda\int_{S^2}|\varphi(\lambda\theta)|\,d\sigma(\theta)\lesssim \lambda^{-1}\|\varphi\|_{L^1(\lambda S^2)}$$

whereas the first term is

$$\lesssim \int_{\lambda-1}^{\lambda+1}\beta^2|\beta-\lambda|^{\alpha-1}[\varphi]_\alpha(\lambda\theta)\,d\sigma(\theta)d\beta\le C_\alpha\,\|[\varphi]_\alpha\|_{L^1(\lambda S^2)}$$

as claimed. □

We now turn to the limiting absorption principle. Note the decay λ^{-1} on the right-hand side which corresponds to a gain of a derivative on the left-hand side. Also, note that the constant does not depend on δ at least if $\lambda>\delta^{-2}$.

Proposition 16. *Let $w=\langle x\rangle^{-\sigma}$ with $\sigma>4$. For $\lambda>\delta^{-2}$ define the kernels*

$$
\begin{aligned}
\widetilde{Q}_\lambda(x,y) &:= w(x)\frac{e^{i\lambda|x-y|}}{|x-y|}\chi\Big(\frac{x-y}{|x-y|}\Big)w(y)\\
Q_\lambda(x,y) &:= w(x)\nabla_x\frac{e^{i\lambda|x-y|}}{|x-y|}\chi\Big(\frac{x-y}{|x-y|}\Big)w(y)
\end{aligned}
$$

Then,

$$\|\widetilde{Q}_\lambda\|_{2\to2}\le C_0\lambda^{-1},\qquad \|Q_\lambda\|_{2\to2}\le C_0$$

The constant C_0 does not depend on δ.

Proof. It will suffice to treat Q_λ. We apply Schur's lemma. Thus, using the notation of Proposition 14 (and assuming that w is real-valued)

$$
\begin{aligned}
&\int Q_\lambda(x,y)\overline{f}(y)g(x)\,dxdy\\
&=\int \xi K_\lambda(\xi)\overline{\widehat{w}*\widehat{f}}(\xi)\widehat{w}*\widehat{g}(\xi)\,d\xi\\
&=\int\int \xi K_\lambda(\xi)\overline{\widehat{w}}(\xi-\xi_1)\widehat{w}(\xi-\xi_2)\,d\xi\overline{\widehat{f}}(\xi_1)\widehat{g}(\xi_2)\,d\xi_1d\xi_2
\end{aligned}
$$

The theorem follows provided we can show that

$$\sup_{\xi_2}\int\Big|\int \xi K_\lambda(\xi)\widehat{w}(\xi_1-\xi)\widehat{w}(\xi-\xi_2)\,d\xi\Big|\,d\xi_1\lesssim 1 \qquad (40)$$

First, note the bounds

$$|\widehat{w}(\xi)| \lesssim \langle\xi\rangle^{-3-\varepsilon}, \qquad |\nabla\widehat{w}(\xi)| \lesssim \langle\xi\rangle^{-3-\varepsilon} \tag{41}$$

In fact, one has rapid decay here but it is not needed. Second, it follows from Proposition 14 that $K_\lambda := K_1 + K_2 + K_3$ where

$$\begin{aligned} K_1(\xi) &= O(\delta^{-2}\lambda^{-2})\chi_{[|\xi|<10\lambda]} \\ K_2(\xi) &= O(|\xi|^{-2})\chi_{[|\xi|>10\lambda]} \\ K_3(\xi) &= \lambda^{-1}\chi(e)f_\delta(\xi/\lambda)\Big[d\sigma_{\lambda S^2}(\xi) + i\mathrm{P.V.}\frac{1}{\lambda-|\xi|}\chi_{[\lambda-1<|\xi|<\lambda+1]}\Big] \end{aligned} \tag{42}$$

The cut-offs here are understood to be smooth. It is easy to see that K_1 and K_2 contribute $O(\delta^{-2}\lambda^{-1})$ and $O(\lambda^{-1})$ to (40), respectively. To bound the contribution of K_3, we use Lemma 15. Thus, define

$$\varphi(\xi) := \xi\chi(\xi/|\xi|)f_\delta(\xi/\lambda)\widehat{w}(\xi_1-\xi)\widehat{w}(\xi-\xi_2)$$

Then

$$\|\varphi\|_{L^1(\lambda S^2)} \lesssim \lambda\int_{\lambda S^2}\chi(\xi/|\xi|)\langle\xi_1-\xi\rangle^{-3-\varepsilon}\langle\xi-\xi_2\rangle^{-3-\varepsilon}\,d\sigma(\xi) =: J_\lambda(\xi_1,\xi_2) \tag{43}$$

as well as

$$\|[\varphi]_\alpha\|_{L^1(\lambda S^2)} \lesssim \Big((\lambda\delta)^{-1} + (\delta^2\lambda)^{-\alpha}\Big)J_\lambda(\xi_1,\xi_2) \lesssim J_\lambda(\xi_1,\xi_2) \tag{44}$$

provided $\lambda > \delta^{-2}$. In view of Lemma 15 the contribution by K_3 to (40) is bounded by

$$\sup_{\xi_2}\lambda^{-1}\int J_\lambda(\xi_1,\xi_2)d\xi_1 \lesssim 1$$

and the proposition follows. □

Next, we study the effect of composing two resolvents which have been restricted to disjoint conical regions.

Proposition 17. *Assume that $\sigma > 4$ and*

$$\sum_{|\alpha|\le 2}|D^\alpha\widehat{A}(\xi)| \lesssim \langle\xi\rangle^{-3-\varepsilon} \qquad \forall\xi\in\mathbb{R}^3 \tag{45}$$

where $\varepsilon > 0$. Let $\mathcal{S}_1,\mathcal{S}_2 \subset S^2$ with $\mathrm{dist}(\mathcal{S}_1,\mathcal{S}_2) > 5\delta$ *where* dist *is the distance on S^2. Let $R_1(\lambda^2)$ and $R_2(\lambda^2)$ be the free resolvents which have been restricted to conical regions corresponding to $\mathcal{S}_1,\mathcal{S}_2$, respectively. Then*

$$\|wR_1(\lambda^2)\nabla\cdot AR_2(\lambda^2)\nabla w\|_{2\to2} \lesssim \delta^{-2}\lambda^{-1}$$

provided $\lambda > \delta^{-2}$.

Proof. We use Schur's lemma as in the proof of Proposition 16. Thus, we write

$$\int\int\int g(x)w(x)\nabla_z R_1(\lambda^2)(x-z)A(z)\cdot\nabla_y R_2(\lambda^2)(z-y)w(y)\overline{f}(y)\,dxdydz$$
$$=\int\int\widehat{g}(\xi)U(\xi,\eta)\widehat{\overline{f}}(\eta)\,d\xi d\eta$$

where (with real-valued w)

$$U(\xi,\eta):=\int\widehat{w}(\xi-\xi_1)\xi_1\widehat{R_1(\lambda^2)}(\xi_1)\widehat{A}(\xi_2-\xi_1)\xi_2\widehat{R_2(\lambda^2)}(\xi_2)\widehat{w}(\eta-\xi_2)\,d\xi_1 d\xi_2$$

We claim that

$$\sup_\eta\int_{\mathbb{R}^3}|U(\xi,\eta)|\,d\xi\lesssim\delta^{-2}\lambda^{-1}\tag{46}$$

By symmetry, this will imply the proposition. Next, we write as in (42) for the Fourier transforms $K_\lambda^{(j)}=\widehat{R_j(\lambda^2)}$ with $j=1,2$

$$K_\lambda^{(j)}=K_1^{(j)}+K_2^{(j)}+K_3^{(j)}$$

The integral on the left-hand side of (46) is bounded by

$$\sum_{i,j=1}^{3}\int\Big|\int\widehat{w}(\xi-\xi_1)\xi_1K_i^{(1)}(\xi_1)\widehat{A}(\xi_2-\xi_1)\xi_2K_j^{(2)}(\xi_2)\widehat{w}(\eta-\xi_2)\,d\xi_1d\xi_2\Big|\,d\xi\tag{47}$$

Of the nine different combinations here all but $i=j=3$ are easy. Indeed, if $i=1,2$ and for any $j=1,2,3$,

$$\int\Big|\int\widehat{w}(\xi-\xi_1)\xi_1K_i^{(1)}(\xi_1)\widehat{A}(\xi_2-\xi_1)\xi_2K_j^{(2)}(\xi_2)\widehat{w}(\eta-\xi_2)\,d\xi_1d\xi_2\Big|\,d\xi$$
$$\lesssim\delta^{-2}\lambda^{-1}\int|\widehat{w}(\eta-\xi_1)|\,d\xi_1\int\Big|\int\widehat{A}(\xi_2-\xi_1)\xi_2K_j^{(2)}(\xi_2)\widehat{w}(\eta-\xi_2)\,d\xi_2\Big|\,d\xi d\xi_1$$
$$\lesssim\delta^{-2}\lambda^{-1}$$

by the discussion following (40) (in particular, recall (41)). It remains to consider $i=j=3$. For this we shall use Lemma 15. Let

$$G_\lambda(\xi_1,\eta):=\int\widehat{A}(\xi_2-\xi_1)\xi_2K_3^{(2)}(\xi_2)\widehat{w}(\eta-\xi_2)\,d\xi_2$$
$$=\lambda^{-1}\int\varphi(\xi_2)\Big[\sigma_{\lambda S^2}(d\xi_2)+i\,\mathrm{P.V.}\frac{d\xi_2}{\lambda-|\xi_2|}\chi_{[\lambda-1<|\xi_2|<\lambda+1]}\Big]$$

with

$$\varphi(\xi_2):=\widehat{A}(\xi_2-\xi_1)\xi_2\chi_2(\xi_2/|\xi_2|)f_\delta(\xi_2/\lambda)\widehat{w}(\eta-\xi_2)$$

Here χ_2 is a cut-off adapted to $\mathcal{S}_2$. By Lemma 15, and (43), (44),

$$|G_\lambda(\xi_1,\eta)|\lesssim\int_{\lambda S^2}\chi_2(\xi_2/|\xi_2|)\langle\xi_2-\xi_1\rangle^{-3-\varepsilon}\langle\eta-\xi_2\rangle^{-3-\varepsilon}\,d\sigma(\xi_2)$$

Note that the same estimates hold if we replace $\widehat{A}$ with $\nabla\widehat{A}$. Therefore,

$$|\nabla_{\xi_1} G_\lambda(\xi_1,\eta)| \lesssim \int_{\lambda S^2} \chi_2(\xi_2/|\xi_2|)\langle \xi_2-\xi_1\rangle^{-3-\varepsilon}\langle \eta-\xi_2\rangle^{-3-\varepsilon}\, d\sigma(\xi_2)$$

In view of these estimates we can apply Lemma 15 again to obtain

$$\begin{aligned}
&\Big|\int \widehat{w}(\xi-\xi_1)\xi_1 K_3^{(1)}(\xi_1) G_\lambda(\xi_1,\eta)\, d\xi_1\Big| \\
&\lesssim \int_{\lambda S^2} \langle \xi-\xi_1\rangle^{-3-\varepsilon}\chi_1(\xi_1/|\xi_1|)\int_{\lambda S^2}\chi_2(\xi_2/|\xi_2|)\langle \xi_2-\xi_1\rangle^{-3-\varepsilon}\langle \eta-\xi_2\rangle^{-3-\varepsilon}\, d\sigma(\xi_2)\, d\sigma(\xi_1)
\end{aligned}$$

Hence the contribution of $i=j=3$ to (47) is bounded by

$$\begin{aligned}
&\int\int_{\lambda S^2}\int_{\lambda S^2} \langle \xi-\xi_1\rangle^{-3-\varepsilon}\chi_1(\xi_1/|\xi_1|)\chi_2(\xi_2/|\xi_2|)\langle \xi_2-\xi_1\rangle^{-3-\varepsilon}\langle \eta-\xi_2\rangle^{-3-\varepsilon}\, d\sigma(\xi_2) d\sigma(\xi_1)\, d\xi \\
&\lesssim \int_{\lambda S^2}\int_{\lambda S^2} \chi_1(\xi_1/|\xi_1|)\chi_2(\xi_2/|\xi_2|)\langle \xi_2-\xi_1\rangle^{-3-\varepsilon}\langle \eta-\xi_2\rangle^{-3-\varepsilon}\, d\sigma(\xi_2) d\sigma(\xi_1) \\
&\lesssim \frac{1}{\lambda \mathrm{dist}(\mathcal{S}_1,\mathcal{S}_2)} \lesssim \lambda^{-1}\delta^{-1}.
\end{aligned}$$

This is again smaller than $\delta^{-2}\lambda^{-1}$, as claimed. □

We now write the power on the right-hand side of (35) as a sum of products (dropping λ^2+i0 from the resolvent):

$$(R_0\nabla\cdot A)^m = \sum_{\mathcal{S}_1,\ldots,\mathcal{S}_m\in\Sigma} R_{\mathcal{S}_1}\nabla\cdot A\ldots\nabla\cdot AR_{\mathcal{S}_m}\nabla\cdot A \tag{48}$$

There are two types of chains $\mathcal{S}_1,\mathcal{S}_2,\ldots,\mathcal{S}_m$ in this sum:

- if $\mathrm{dist}(\mathcal{S}_i,\mathcal{S}_{i+1})\le 5\delta$ for all $1\le i\le m-1$, then we call this chain *directed*
- otherwise, we call it *undirected*

For the undirected chains there is the following corollary of the previous proposition.

Corollary 18. *If $\{\mathcal{S}_j\}_{j=1}^m$ is undirected, then for $\sigma>4$*

$$\|R_{\mathcal{S}_1}\nabla\cdot A\ldots\nabla\cdot AR_{\mathcal{S}_m}\nabla\cdot A\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \le C(m,A)\delta^{-2}\lambda^{-1} \tag{49}$$

provided $\lambda>\delta^{-2}$. In particular,

$$\Big\|\sum_{\substack{\mathcal{S}_1,\ldots,\mathcal{S}_m\in\Sigma\\ \text{undirected}}} R_{\mathcal{S}_1}\nabla\cdot A\ldots\nabla\cdot AR_{\mathcal{S}_m}\nabla\cdot A\Big\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \le C(m,A)\delta^{-2(m+1)}\lambda^{-1} \tag{50}$$

provided $\lambda>\delta^{-2}$.

Proof. This follows by applying Proposition 17 to one pair of resolvents where $\mathrm{dist}(\mathcal{S}_i,\mathcal{S}_{i+1})>5\delta$; for the others, use Proposition 16. More precisely, with i as specified, we write

$$AR_{\mathcal{S}_i}\nabla\cdot AR_{\mathcal{S}_{i+1}}\nabla\cdot A = Aw^{-1}wR_{\mathcal{S}_i}\nabla\cdot AR_{\mathcal{S}_{i+1}}\nabla\cdot ww^{-1}A \tag{51}$$

where as usual $w(x) = \langle x\rangle^{-\sigma}$. In view of $|A(x)| \lesssim \langle x\rangle^{-2\sigma}$ and by our assumptions on $\widehat{A}$, we apply Proposition 17 to the right-hand side of (51) to conclude that

$$\|wR_{\mathcal{S}_i}\nabla\cdot AR_{\mathcal{S}_{i+1}}\nabla\cdot w\|_{2\to 2} \lesssim \delta^{-2}\lambda^{-1} \tag{52}$$

To combine this with Proposition 16, we insert factors of ww^{-1} as follows: with $\widetilde{A} := w^{-1}Aw^{-1}$,

$$\prod_{j=1}^{m}(R_{\mathcal{S}_j}\nabla A) = w^{-1}\,(wR_{\mathcal{S}_1}\nabla w)\,\widetilde{A}\,(wR_{\mathcal{S}_2}\nabla w)\,\widetilde{A}\cdot\ldots$$
$$\ldots\cdot\widetilde{A}\,(wR_{\mathcal{S}_i}\nabla\cdot AR_{\mathcal{S}_{i+1}}\nabla\cdot w)\,\widetilde{A}\,(wR_{\mathcal{S}_{i+2}}\nabla w)\,\ldots\,(wR_{\mathcal{S}_m}\nabla w)\,\widetilde{A}w$$

Observe that

$$\sup_j \|wR_{\mathcal{S}_j}\nabla w\|_{2\to 2} \le C$$

uniformly in $\lambda > \delta^{-2}$ as well as $\|\widetilde{A}f\|_2 \lesssim \|f\|_2$. Combining this with (52) yields (49). To pass to (50) one sums over all possible choices of undirected chains of which there are no more than $(C/\delta)^{2m}$. $\square$

Remark 19. The summation over all possible paths is quite inefficient, as it does not take advantage of any orthogonality between different operators $R_{\mathcal{S}}$. However large the constants may be, once A, m, and δ are fixed, the bound in (50) still approaches zero in the limit $\lambda\to\infty$.

Finally, we turn to the directed chains. For these it will be important that $\delta m \ll 1$ to ensure that the composition of resolvents restricted to any directed chain remains outgoing. Moreover, we will need to distinguish the *near* and *far* parts of the free resolvent kernels which are defined as follows:

$$Q^0_{\mathcal{S}}(x,y) := w(x)[\nabla_y R_{\mathcal{S}}(x-y)]\chi(|x-y|<\rho)w(y)$$
$$Q^1_{\mathcal{S}}(x,y) := w(x)[\nabla_y R_{\mathcal{S}}(x-y)]\chi(|x-y|>\rho)w(y)$$

where $1 = \chi(|x-y|<\rho) + \chi(|x-y|>\rho)$ is a smooth partition of unity adapted to the indicated sets. The parameter ρ here is a small number depending on m. For the near part, we have the following refinement of Proposition 16.

Proposition 20. *Under the conditions of Proposition 16 one has*

$$\|Q^0_{\mathcal{S}}\|_{2\to 2} \le C_2\rho, \qquad \|Q^1_{\mathcal{S}}\|_{2\to 2} \le C_2$$

provided $\lambda > \delta^{-2}\rho^{-1}$. Here C_2 does not depend on δ.

Proof. Because of Proposition 16 it will suffice to prove the bound on $Q^0_{\mathcal{S}}$. In this proof, we shall write

$$\chi_\rho(x-y) := \chi(|x-y|<\rho)$$

Observe that $\widehat{\chi}_\rho$ is rapidly decaying outside of a ball of size $\lesssim \rho^{-1}$. Thus, as in the proof of Proposition 16, and with $\widetilde{K}_\lambda(\xi) := \xi K_\lambda(\xi)$,

$$\begin{aligned}
&\int Q^0_{\mathcal{S}}(x,y)\overline{f}(y)g(x)\,dxdy \\
&= \int [\widetilde{K}_\lambda * \widehat{\chi_\rho}](\xi)\overline{\widehat{w} * \widehat{f}}(\xi)\widehat{w} * \widehat{g}(\xi)\,d\xi \\
&= \int\int [\widetilde{K}_\lambda * \widehat{\chi_\rho}](\xi)\overline{\widehat{w}}(\xi-\xi_1)\widehat{w}(\xi-\xi_2)\,d\xi\overline{\widehat{f}}(\xi_1)\widehat{g}(\xi_2)\,d\xi_1 d\xi_2
\end{aligned}$$

The theorem follows provided we can show that

$$\sup_{\xi_2}\int\Big|\int [\widetilde{K}_\lambda * \widehat{\chi_\rho}](\xi)\widehat{w}(\xi_1-\xi)\widehat{w}(\xi-\xi_2)\,d\xi\Big|\,d\xi_1 \lesssim \rho \tag{53}$$

It follows from Proposition 14 that

$$\widetilde{K}_\lambda := \widetilde{K}_1 + \widetilde{K}_2 + \widetilde{K}_3$$

where (with smooth cut-offs)

$$[\widetilde{K}_1 * \widehat{\chi_\rho}](\xi) = O(\delta^{-2}\lambda^{-1}) \tag{54}$$

$$[\widetilde{K}_2 * \widehat{\chi_\rho}](\xi) = O(\lambda^{-1}) \tag{55}$$

$$\begin{aligned}
&\widetilde{K}_3 * \widehat{\chi_\rho} = \\
&= \lambda^{-1}\widehat{\chi_\rho} * \Big\{\chi_{\mathcal{S}}\, f_\delta(\cdot/\lambda)\Big[\lambda d\sigma_{\lambda S^2}(\eta) + i\mathrm{P.V.}\frac{\eta}{\lambda-|\eta|}\chi_{[\lambda-1<|\eta|<\lambda+1]}\Big]\Big\}
\end{aligned} \tag{56}$$

We also used there that $\lambda \gg \rho^{-1}$. The contributions of (54) and (55) to (53) are treated as in Proposition 16 and yield a bound of $\delta^{-2}\lambda^{-1} < \rho$ as desired. For the contribution of (56) we note that

$$|\widetilde{K}_3 * \widehat{\chi_\rho}|(\xi) \lesssim \rho$$

Hence, the contribution of (56) to (53) is controlled by

$$\lesssim \rho\sup_{\xi_2}\int\int |\widehat{w}(\xi_1-\xi)\widehat{w}(\xi-\xi_2)|\,d\xi d\xi_1 \lesssim \rho$$

as desired. □

Next, we write

$$\begin{aligned}
&\sum_{\substack{\mathcal{S}_1,\ldots,\mathcal{S}_m\in\Sigma\\ \text{directed}}} R_{\mathcal{S}_1}\nabla\cdot A\ldots\nabla\cdot AR_{\mathcal{S}_m}\nabla\cdot A \\
&= \sum_{\substack{\mathcal{S}_1,\ldots,\mathcal{S}_m\in\Sigma\\ \text{directed}}}\ \sum_{\varepsilon_1,\ldots,\varepsilon_m=0,1} w^{-1}\,Q^{\varepsilon_1}_{\mathcal{S}_1}\,\widetilde{A}\ldots\widetilde{A}\,Q^{\varepsilon_m}_{\mathcal{S}_m}\,\widetilde{A}\,w
\end{aligned} \tag{57}$$

Fix a directed chain and assume without loss of generality that it is directed along the positive x_1-axis. Since $\delta m \ll 1$, one has

$$Q^1_{\mathcal{S}_j}(x,y) = 0 \quad\text{unless}\quad x_1 - y_1 > \frac{\rho}{2}$$

for each $1 \le j \le m$. Next, we decompose

$$\widetilde{A} = \sum_{n\in\mathbb{Z}} \widetilde{A}_n, \qquad \widetilde{A}_n(x) := \widetilde{A}(x)\chi_{[n\rho/2<x_1<(n+1)\rho/2]}$$

We start by estimating the contribution of products consisting entirely of far kernels.

Lemma 21. *Suppose that $|A(x)| \le C_A\langle x\rangle^{-2\sigma-1-\varepsilon}$ with $\sigma > 4$. Then, using the previous notations,*

$$\Big\| Q^1_{\mathcal{S}_1}\,\widetilde{A}\ldots\widetilde{A}\,Q^1_{\mathcal{S}_m}\,\widetilde{A}\Big\|_{2\to 2} \le \frac{C_3^m}{m!\,\rho^m}$$

provided $\lambda > \delta^{-2} + \rho^{-1}$. The constant C_3 here depends only on A.

Proof. By our assumptions,

$$\|\widetilde{A}_n f\|_2 \le C_A(1+|n|\rho/2)^{-1-\varepsilon}\|f\|_2$$

Moreover, since $\sup_{1\le j\le m}\|Q^1_{\mathcal{S}_j}\|_{2\to2} \le C_2$,

$$\begin{aligned}
&\|Q^1_{\mathcal{S}_1}\,\widetilde{A}\ldots\widetilde{A}\,Q^1_{\mathcal{S}_m}\,\widetilde{A}\Big\|_{2\to2} \\
&\le \sum_{n_1>n_2>\ldots>n_m} \|Q^1_{\mathcal{S}_1}\,\widetilde{A}_{n_1}\ldots\widetilde{A}_{n_{m-1}}\,Q^1_{\mathcal{S}_m}\,\widetilde{A}_{n_m}\Big\|_{2\to2} \\
&\le C_2^m \sum_{n_1>n_2>\ldots>n_m}\prod_{j=1}^m \|\widetilde{A}_{n_j}\|_{2\to2} \\
&\le C_A^m C_2^m \sum_{n_1>n_2>\ldots>n_m}\prod_{j=1}^m (1+|n_j|\rho/2)^{-1-\varepsilon} \\
&\le \frac{C_A^m C_2^m}{m!} \sum_{n_1,n_2,\ldots,n_m\in\mathbb{Z}}\prod_{j=1}^m (1+|n_j|\rho/2)^{-1-\varepsilon} \\
&= \frac{C_3^m}{\rho^m m!}
\end{aligned}$$

as claimed. $\square$

Next, we turn to the general case.

Lemma 22. *Under the conditions of Lemma 21,*

$$\sum_{\varepsilon_1,\ldots,\varepsilon_m=0,1} \|Q^{\varepsilon_1}_{\mathcal{S}_1}\,\widetilde{A}\ldots\widetilde{A}\,Q^{\varepsilon_m}_{\mathcal{S}_m}\,\widetilde{A}\,\|_{2\to2} \le C_5^m m^{-\frac{m}{16}}$$

where C_5 only depends on A.

Proof. Let $\mu = \sum_{j=2}^m \varepsilon_j$. Then

$$\sum_{\varepsilon_1,\ldots,\varepsilon_m=0,1} \|Q_{\mathcal{S}_1}^{\varepsilon_1}\,\widetilde{A}\ldots\widetilde{A}\,Q_{\mathcal{S}_m}^{\varepsilon_m}\,\widetilde{A}\,\|_{2\to 2}$$

$$\leq \sum_{\varepsilon_1,\ldots,\varepsilon_m=0,1}\ \sum_{n_1}^{(\varepsilon_2)}\cdots\sum_{n_{m-1}}^{(\varepsilon_m)}\sum_{n_m} C_2^m \rho^{1-\varepsilon_1}\rho^{m-1-\mu}\prod_{j=1}^m \|\widetilde{A}_{n_j}\|_{2\to 2} \tag{58}$$

Here, for fixed n_{i+1},

$$\sum_{n_i}^{(\varepsilon_{i+1})} = \begin{cases} \sum_{n_i > n_{i+1}} & \text{if } \ \varepsilon_{i+1} = 1 \\ \sum_{n_{i+1}+3\geq n_i\geq n_{i+1}} & \text{if } \ \varepsilon_{i+1} = 0 \end{cases}$$

Now

$$(58) \leq 2 \sum_{\varepsilon_2,\ldots,\varepsilon_m=0,1}\ \sum_{n_1}^{(\varepsilon_2)}\cdots\sum_{n_{m-1}}^{(\varepsilon_m)}\sum_{n_m}(C_A\,C_2)^m \cdot \rho^{m-1-\mu}\prod_{j=1}^m (1+|n_j|\rho/2)^{-1-\varepsilon}$$

$$\leq (4C_A\,C_2)^m \sum_{\ell=1}^{m-1}\binom{m-1}{\ell}\frac{\rho^\ell}{(m-\ell-1)!}\Big(\frac{C}{\rho}\Big)^{m-\ell-1} \tag{59}$$

by counting and symmetry as in the proof of Lemma 21. Simplifying further, we conclude that

$$(59) \leq C_4^m \rho^{-(m-1)}\sum_{\ell=1}^{m-1}\binom{m-1}{\ell}\frac{\rho^{2\ell}}{(m-\ell-1)!} \tag{60}$$

The contribution of the sum over $\ell \geq \frac{m-1}{2}+\frac{m}{4}$ to the right-hand side of (60) is at most $(2C_4)^m\rho^{\frac{m}{2}}$. On the other hand, the sum over $\ell < \frac{m-1}{2}+\frac{m}{4}$ is bounded by

$$(2C_4)^m\ \frac{\rho^{-(m-1)}}{\lfloor m/4\rfloor!}$$

Setting $\rho := m^{-\frac{1}{8}}$ the lemma follows. □

Using (57), Lemma 22 and the observation that there are at most $\delta^{-2}C^m$ directed chains we conclude that

$$\Big\|\sum_{\substack{\mathcal{S}_1,\ldots,\mathcal{S}_m\in\Sigma\\ \text{directed}}} R_{\mathcal{S}_1}\nabla\cdot A\ldots\nabla\cdot A R_{\mathcal{S}_m}\nabla\cdot A\Big\|_{L^{2,-\sigma}\to L^{2,-\sigma}} \leq \delta^{-2}C_6^m m^{-\frac{m}{16}} \tag{61}$$

Recall that in Lemma 12 we are given an operator L (quickly reduced to the case $L = \nabla\cdot A$) and a small parameter $c > 0$. Based on the value of $C_6(A)$ from (61) we choose m and $\delta = (10m)^{-1}$ large enough so that the right side of (61) is less than $\frac{c}{2}$. The bound for directed chains is independent of λ.

For the undirected chains, we apply Corollary 18 directly. With the quantities m and δ already fixed, it is easy to find λ_0 so that the right side of (50) is less than $\frac{c}{2}$ whenever $\lambda > \lambda_0$. This finishes the proof of Lemma 12.

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