A Limiting Absorption Principle for the three-dimensional Schrödinger equation with L^p potentials

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1 Introduction

Agmon's fundamental work [Agm] establishes the bound, known as the limiting absorption principle,

$$\sup_{\lambda>\lambda_0,\,\varepsilon>0} \big\|\big(-\triangle + V - (\lambda^2+i\varepsilon)\big)^{-1}\big\|_{L^{2,\sigma}(\mathbb{R}^d)\to L^{2,-\sigma}(\mathbb{R}^d)} < \infty \tag{1}$$

provided that $\lambda_0 > 0$, $(1+|x|)^{1+}|V(x)| \in L^\infty$ and $\sigma > \frac{1}{2}$. Here

$$L^{2,\sigma}(\mathbb{R}^d) = \{(1+|x|)^{-\sigma} f \;:\; f \in L^2(\mathbb{R}^d)\}$$

is the usual weighted L^2. The bound (1) is obtained from the same estimate for $V = 0$ by means of the resolvent identity. This bound for the free resolvent is related to the so called trace lemma, which refers to the statement that for every $f \in L^{2,\frac{1}{2}+}$ there is a restriction of $\hat{f}$ to any (compact) hypersurface, and this restriction belongs to L^2 relative to surface measure. Note that this fact does not require any curvature properties of the hypersurface - in fact, it is proved by reduction to flat surfaces. Another fundamental restriction theorem is the Stein-Tomas theorem, see [Ste]. It requires the hypersurfaces $\mathcal{S} \subset \mathbb{R}^d$ with $d \geq 2$ to have non vanishing Gaussian curvature, and states that

$$\int_{\mathcal{S}} |\hat{f}(\omega)|^2\, \sigma(d\omega) \leq C\|f\|^2_{L^p(\mathbb{R}^d)} \quad \text{where} \quad p = \frac{2d+2}{d+3}. \tag{2}$$

It is not hard to see that the related estimate for the free resolvent in $\mathbb{R}^3$ is given by

$$\|R_0(\lambda^2+i0)\|_{\frac{4}{3}\to 4} = C\,\lambda^{-\frac{1}{2}} \quad \text{for} \quad \lambda > 0. \tag{3}$$

This fact depends on the oscillation in the resolvent, i.e., on the exponential in

$$R_0(\lambda^2+i0)(x,y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}. \tag{4}$$

In contrast, using the denominator alone one obtains that

$$\sup_{\lambda} \|R_0(\lambda^2+i0)\|_{\frac{6}{5}\to 6} \leq C \tag{5}$$

via fractional integration. In analogy with Agmon's work, it is natural to ask for which potentials (3) can be extended to the perturbed operators $H = -\triangle + V$. In this paper we show that this is the case for real-valued $V \in L^p(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3), p > \frac{3}{2}$, and suggest two possible extensions.

Theorem 1. *Let $V \in L^p(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3), p > \frac{3}{2}$ be real-valued. Then for every $\lambda_0 > 0$, one has*

$$\sup_{0<\varepsilon<1,\, \lambda\geq\lambda_0} \left\| (-\triangle + V - (\lambda^2 + i\varepsilon))^{-1} \right\|_{\frac{4}{3}\to 4} \leq C(\lambda_0, V)\, \lambda^{-\frac{1}{2}}. \tag{6}$$

In particular, the spectrum of $-\triangle + V$ is purely absolutely continuous on $(0, \infty)$.

We also formulate dynamical consequences of this result, in particular the existence and completeness of the wave operators. This theorem is the analogue of the classical Kato-Agmon-Kuroda theorem, see [ReeSim], Theorem XIII.33. It of course requires the absence of imbedded eigenvalues. In the classical context one uses Kato's theorem for that purpose. Here we wish to use a result on the absence of imbedded eigenvalues that only requires an integrability condition on V. One such result was obtained by Ionescu and Jersion [IonJer], namely:

Theorem 2. *Let $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$. Suppose $u \in W^{1,2}_{\rm loc}(\mathbb{R}^3)$ satisfies $(-\triangle + V)u = \lambda^2 u$ where $\lambda \neq 0$ in the sense of distributions. If, moreover, $\|(1+|x|)^{\delta-\frac{1}{2}} u\|_2 < \infty$ for some $\delta > 0$, then $u \equiv 0$.*

The weighted L^2-condition with $\delta > 0$ is natural in view of the Fourier transform of the surface measure of S^2, which is a generalized eigenfunction of the free case and decays like $(1+|x|)^{-1}$. As far as local regularity of the potential is concerned, the requirement that $V \in L^{3/2}_{\rm loc}$ is essentially optimal. There exist examples of $V \in L^{3/2}_{\rm weak}$ for which $-\Delta + V$ admits compactly supported eigenfunctions [KoTa]. The necessary decay condition on V is less clearly delineated: Ionescu and Jerison found a smooth real-valued potential V which lies in $L^q(\mathbb{R}^3)$ for all $q > 2$ but such that for $-\triangle + V$ imbedded eigenvalues exist. Their example decays like r^{-1} in some directions, and like r^{-2} in other directions. They further conjectured that their main result (Theorem 2.1 in [IonJer]) remains valid for potentials $V \in L^2(\mathbb{R}^3)$. Recent work by Koch and Tataru appears to verify this conjecture [KoTa2], and futher refinements which allow potentials to exhibit both $L^{3/2}_{\rm loc}$ singularities and L^2 decay seem possible as well. The proof of any such conjecture would immediately increase the scope of Theorem 1, as described below.

Proposition 3. *The following inferences are valid:*
1. If the conclusion of Theorem 2 holds for all $V \in L^p(\mathbb{R}^3), \frac{3}{2} \leq p < 2$, as is suggested by [KoTa2], then the conclusion of Theorem 1 also holds for all $V \in L^p(R^3)$.
2. More generally, if the conclusion of Theorem 2 holds for some $V \in L^p(\mathbb{R}^3) + L^q(\mathbb{R}^3), \frac{3}{2} < p, q < 2$, then the conclusion of Theorem 1 also holds for this V.

By Kato's theory of H-smoothing operators, see [Kat], it is well-known that the limiting absorption principle for the resolvent gives rise to estimates for the evolution e^{itH} known as smoothing estimates. This is a much studied class of bounds, see [Sjo], [Veg], [ConSau1], [ConSau2], [BenKla], [Doi], [Sim]. In fact, the Fourier transform establishes a link between the resolvent and the evolution that in a precise sense allows one to state that a certain class of estimates on the evolution is equivalent to corresponding ones for the resolvent, see [Kat]. In the free case, the $\frac{4}{3} \to 4$ bound for the resolvent corresponds to the following smoothing bound for the free evolution:

$$\sup_{\|F\|_4\leq 1} \int_{-\infty}^{\infty} \left\| F(-\triangle)^{\frac{1}{8}} e^{it\triangle} f \right\|_2^2 dt \leq C\, \|f\|_2^2.$$

However, this bound is known, see the work of Ruiz and Vega [RuiVeg]. For the perturbed evolution, $H = -\Delta + V$, one can prove similar estimates by means of Theorem 1, but we do not pursue this here. See the work of Ionescu and the second author [IonSch] for statements of this type.

This paper is organized as follows: In Section 2 we prove the bounds on the free resolvent that are needed in order to prove Theorem 1. Our main new bounds involve $R_0(\lambda^2 + i0)$ acting on functions whose Fourier transform vanish on λS^2. In Section 3 we apply these bounds in the context of the usual resolvent identity/Fredholm alternative type arguments to deal with $-\triangle + V$. This of course requires Theorem 2. Finally, in Section 4 we return to the free resolvent and prove some end point results.

2 The free resolvent

This section develops some estimates on the free resolvent given by (4). These estimates are motivated on the one hand by the Stein-Tomas theorem (2), and on the other hand, by the applications to the perturbed operator $H = -\triangle + V$, see Theorem 2. For what follows, it will be helpful to keep in mind that for real λ,

$$[R_0(\lambda^2 + i0) - R_0(\lambda^2 - i0)]f = C(\lambda) \cdot (\widehat{\sigma_{\lambda S^2}} * f),$$

which is exactly of the form T^*T, T being the restriction operator to the sphere λS^2. Thus $T^*T : L^{\frac{4}{3}}(\mathbb{R}^3) \to L^4(\mathbb{R}^3)$ in view of (2).

We will denote by $\mathbb{H}$ the closed upper half-plane in $\mathbb{C}$, and state most of our results for $\lambda \in \mathbb{H}$. For any positive real number λ, we have the boundary identites

$$(\lambda + i0)^2 = \lambda^2 + i0 \quad \text{and} \quad (-\lambda + i0)^2 = \lambda^2 - i0,$$

therefore estimates which hold uniformly out to $\partial\mathbb{H}$ are of particular importance.

Lemma 4. *Let $\lambda \in \mathbb{H}$ be any nonzero element, and $p = \frac{4}{3}$. Then $R_0(\lambda^2) : L^p(\mathbb{R}^3) \to L^{p'}(\mathbb{R}^3)$, with operator norm bounded by $|\lambda|^{-\frac{1}{2}}$.*

As suggested above, the proof follows a complex-interpolation argument strongly reminiscent of the proof of (2). For full details see Theorem 2.3 in [KenRuiSog], which establishes this bound for a more general family of inverses of second-order differential operators.

Lemma 5. *Let $\lambda \in \mathbb{H}$ be any nonzero element. For each pair of exponents $1 < p \leq \frac{4}{3}$, $3p \leq q \leq \frac{3p}{3-2p}$ there exist constants $C_{p,q} < \infty$ such that*

$$\|R_0(\lambda^2)f\|_{L^q} \leq C_{p,q}|\lambda|^{3/p-3/q-2}\|f\|_{L^p}$$

For each exponent $\frac{4}{3} \leq p < \frac{3}{2}$, $\frac{p}{3-2p} \leq q \leq \frac{3p}{3-2p}$ there exist constants $C_{p,q} < \infty$ such that

$$\|R_0(\lambda^2)f\|_{L^{p*}} \leq C_{p,q}|\lambda|^{3/p-3/q-2}\|f\|_{L^p}$$

Proof. The case $p = \frac{4}{3}, q = 4$ is Lemma 4 above. Since $R_0(\lambda^2)$ is realized as a convolution with a kernel satisfying $|K_\lambda(x)| \leq |4\pi x|^{-1}$, the cases $q = \frac{3p}{3-2p}, 1 < p < \frac{3}{2}$ are precisely the Hardy–Littlewood–Sobolev inequality. Note that the scaling exponent for λ is zero for these pairs (p, q).

All intermediate cases (p, q) then follow by interpolation. At the endpoint $p = 1, q = 3$, we see that $R_0^\pm(\lambda^2)$ maps $L^1(\mathbb{R}^3)$ to weak-$L^3(\mathbb{R}^3)$ uniformly in λ, by considering the norm

$$\|f\|_{L^3_{\text{weak}}(\mathbb{R}^3)} = \sup_{A\subset\mathbb{R}^3,\,|A|<\infty} |A|^{-\frac{2}{3}} \int_A |f(x)|\,dx$$

which is equivalent to the usual weak-L^3 "norm" and satisfies a triangle inequality, see Lieb, Loss [LieLos], Section 4.3 The cases $q = 3p$, $1 < p < \frac{4}{3}$ follow by Marcinkiewicz interpolation, and $q = \frac{p}{3-2p}$, $\frac{4}{3} < p < \frac{3}{2}$ by duality. ◻

The following results deal with functions whose Fourier transform vanishes on S^2. The first lemma yields a Hölder bound for the L^2 norms of the restrictions to spheres close to S^2.

Lemma 6. *Let $1 \le p < \frac{4}{3}$ and set $\gamma = \frac{2}{p} - \frac{3}{2}$. Then for all $|\delta| < \frac{1}{2}$ one has*

$$\|\hat{f}((1+\delta)\cdot)\|_{L^2(S^2)} \lesssim |\delta|^\gamma \|f\|_{L^p(\mathbb{R}^3)} \tag{7}$$

for all $f \in L^p(\mathbb{R}^3)$ with $\hat{f} = 0$ on S^2.

Proof. Let $\sigma_{(1+\delta)S^2}$ be the normalized measure on $(1+\delta)S^2$. Then one has

$$\begin{aligned}\|\hat{f}((1+\delta)\cdot)\|^2_{L^2(S^2)} &= \langle f * \widehat{\sigma_{(1+\delta)S^2}}, f\rangle = \langle f * [\widehat{\sigma_{(1+\delta)S^2}} - \widehat{\sigma_{S^2}}], f\rangle \\ &= \sum_{j=0}^{\infty} \langle f * K_j, f\rangle\end{aligned}$$

where $K_j(x) = \Big(\widehat{\sigma_{(1+\delta)S^2}} - \widehat{\sigma_{S^2}}\Big)\chi_j$ and $\{\chi_j\}_{j\ge0}$ are a standard dyadic partition of unity. Since $\|\widehat{\sigma_{(1+\delta)S^2}} - \widehat{\sigma_{S^2}}\|_\infty \lesssim \delta$, it follows that

$$\|K_j\|_\infty \lesssim \begin{cases} \delta & \text{if } 2^j < \delta^{-1} \\ 2^{-j} & \text{if } 2^j \ge \delta^{-1} \end{cases}$$

Thus $\|K_j\|_\infty \lesssim \min(\delta, 2^{-j}) := \alpha_j$. Moreover,

$$\begin{aligned}\|\widehat{K_j}\|_\infty &= \big\|\big(\sigma_{(1+\delta)S^2} - \sigma_{S^2}\big) * \widehat{\chi_j}\big\|_\infty \\ &= \Big|\int \widehat{\chi_j}(\xi-\eta)\,\sigma_{(1+\delta)S^2}(d\eta) - \int \widehat{\chi_j}(\xi-\eta)\,\sigma_{S^2}(d\eta)\Big| \\ &= \Big|\int \Big[\widehat{\chi_j}(\xi-(1+\delta)\eta) - \widehat{\chi_j}(\xi-\eta)\Big]\,\sigma_{S^2}(d\eta)\Big| \\ &\lesssim \min(2^{2j}\delta, 2^j) := \beta_j.\end{aligned}$$

If $1 < p < \frac{4}{3}$, let $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$ so that $\theta > \frac{1}{2}$. Then $\|K_j * f\|_{p'} \lesssim \alpha_j^\theta \beta_j^{1-\theta}\|f\|_p$ for all $j \ge 0$. Summing over j yields the desired bound. In the case $p = 1$, the estimate $\|\widehat{\sigma_{(1+\delta)S^2}} - \widehat{\sigma_{S^2}}\|_\infty \lesssim \delta$ mentioned above suffices to show that $\|\hat{f}((1+\delta)\cdot)\|_{L^2(S^2)} \lesssim \delta^{\frac{1}{2}}$.

◻

The point of the following proposition is that one can take $\delta > 0$ in (9). In the following section, this will allow us to apply Theorem 2.

Proposition 7. *Let $1 \le p < \frac{4}{3}$. Then for any $\delta < \frac{1}{2} - \frac{2}{p'}$ one has*

$$\sup_{\varepsilon>0} \Big\| (1+|x|)^{\delta-\frac{1}{2}} R_0(1 \pm i\varepsilon) f \Big\|_2 \lesssim \|f\|_p \tag{9}$$

for any $f \in L^p(\mathbb{R}^3)$ so that $\hat{f} = 0$ on S^2.

Proof. We first consider the case where

$$\operatorname{supp}(\hat{f}) \subset \{\xi \in \mathbb{R}^3 \,:\, \frac{1}{2} < |\xi| < 2\}. \tag{10}$$

Let χ be a smooth, radial, bump function around zero so that $\hat{\chi}$ is compactly supported. Let $R \gg 1$. Then

$$\begin{aligned} \|\chi(\frac{\cdot}{R}) R_0(1+i\varepsilon) f\|_2^2 &= R^6 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \hat{\chi}(R(\xi-\eta)) \frac{\hat{f}(\eta)}{|\eta|^2 - 1 - i\varepsilon}\, d\eta \int_{\mathbb{R}^3} \hat{\chi}(R(\xi-\tilde{\eta})) \frac{\overline{\hat{f}(\tilde{\eta})}}{|\tilde{\eta}|^2 - 1 + i\varepsilon}\, d\tilde{\eta} \\ &= R^3 \int_{\mathbb{R}^6} \rho(R(\eta-\tilde{\eta}))\, \frac{\hat{f}(\eta)}{|\eta|^2-1-i\varepsilon} \frac{\overline{\hat{f}(\tilde{\eta})}}{|\tilde{\eta}|^2-1+i\varepsilon}\, d\eta\, d\tilde{\eta}, \end{aligned} \tag{11}$$

where we have set

$$\begin{aligned} \int_{\mathbb{R}^3} \hat{\chi}(R(\xi-\eta))\hat{\chi}(R(\xi-\tilde{\eta}))\, d\xi &= R^{-3} \int_{\mathbb{R}^3} \hat{\chi}(\zeta - R\eta)\hat{\chi}(\zeta - R\tilde{\eta})\, d\zeta \\ &= R^{-3} \int_{\mathbb{R}^3} \hat{\chi}(\zeta - R(\eta-\tilde{\eta}))\hat{\chi}(\zeta)\, d\zeta \\ &=: R^{-3} \rho(R(\eta-\tilde{\eta})). \end{aligned}$$

Note that ρ is a compactly supported smooth bump-function. Introducing polar coordinates in (11) yields uniformly in $\varepsilon \neq 0$ (recall (10))

$$\begin{aligned} (11) \;&=\; R^3 \int_{\mathbb{R}^3} \int_0^\infty \int_{S^2} \rho(R(\eta - \tilde{r}\tilde{\omega})) \frac{\hat{f}(\eta)}{|\eta|^2-1-i\varepsilon} \frac{\overline{\hat{f}(\tilde{r}\tilde{\omega})}}{|\tilde{r}\tilde{\omega}|^2 - 1 + i\varepsilon}\, d\tilde{\omega} \tilde{r}^2\, d\tilde{r}\, d\eta \\ &\lesssim R^3 \int_{\mathbb{R}^3} \int_{|\eta|-R^{-1}}^{|\eta|+R^{-1}} \int_{[S^2 : |\tilde{\omega} - \frac{\eta}{|\eta|}| < R^{-1}]} \frac{|\hat{f}(\eta)|}{||\eta|-1|} \frac{|\hat{f}(\tilde{r}\tilde{\omega})|}{|\tilde{r}-1|}\, d\tilde{\omega}\, d\tilde{r}\, d\eta \\ &\lesssim R^2 \int_{\mathbb{R}^3} \frac{|\hat{f}(\eta)|}{||\eta|-1|} \int_{|\eta|-R^{-1}}^{|\eta|+R^{-1}} \left(\int_{[S^2 : |\tilde{\omega} - \frac{\eta}{|\eta|}| < R^{-1}]} |\hat{f}(\tilde{r}\tilde{\omega})|^2\, d\tilde{\omega} \right)^{\frac{1}{2}} \frac{d\tilde{r}}{|\tilde{r}-1|}\, d\eta \\ &\lesssim R^2 \int_0^\infty \frac{dr}{|r-1|} \int_{r-R^{-1}}^{r+R^{-1}} \frac{d\tilde{r}}{|\tilde{r}-1|} \int_{S^2} |\hat{f}(r\omega)| \left(\int_{[S^2 : |\tilde{\omega}-\omega| < R^{-1}]} |\hat{f}(\tilde{r}\tilde{\omega})|^2\, d\tilde{\omega} \right)^{\frac{1}{2}} \end{aligned}$$

and therefore also

$$(11) \quad \lesssim \quad R^2 \int_0^\infty \frac{dr}{|r-1|} \int_{r-R^{-1}}^{r+R^{-1}} \frac{d\tilde{r}}{|\tilde{r}-1|} \left(\int_{S^2} |\hat{f}(r\omega)|^2 \, d\omega \right)^{\frac{1}{2}} \left(\int_{S^2} \int_{[S^2 : |\tilde{\omega}-\omega| < R^{-1}]} |\hat{f}(\tilde{r}\tilde{\omega})|^2 \, d\tilde{\omega} \, d\omega \right)^{\frac{1}{2}}$$

$$\lesssim \quad R \int_{\frac{1}{2}}^{2} \frac{dr}{|r-1|} \int_{r-R^{-1}}^{r+R^{-1}} \frac{d\tilde{r}}{|\tilde{r}-1|} |1-r|^\gamma |1-\tilde{r}|^\gamma \|f\|_p^2$$

$$\lesssim \quad R^{1-2\gamma} \|f\|_p^2 = R^{\frac{4}{p'}} \|f\|_p^2,$$

where the last two lines use (7). The lemma now follows by summing over dyadic R, at least provided (10) holds. Finally, if

$$\mathrm{supp}(\hat{f}) \subset \{\xi \in \mathbb{R}^3 \ : \ |\xi| \le \frac{1}{2} \text{ or } |\xi| \ge 2\},$$

then one notes that

$$\sup_{\varepsilon \neq 0} \|R_0(1 \pm i\varepsilon) f\|_2 \lesssim \|(1-\triangle)^{-1} f\|_2 \lesssim \|f\|_p$$

by the Sobolev imbedding theorem provided $1 \le p \le 2$ and we are done. □

In Section 4 we discuss further bounds on the free resolvent which are motivated by the previous proposition.

3 The perturbed resolvent

The goal of this section is to prove theorem 1. As in [Agm], the proof of Theorem 1 is based on the resolvent identity. This requires inverting the operator $I + R_0(\lambda^2 \pm i0)V$ on $L^4(\mathbb{R}^3)$. First, we check that this is a compact perturbation of the identity.

Lemma 8. *Let $V \in L^p(\mathbb{R}^3)$, $\frac{3}{2} \le p \le 2$. Then for any nonzero $\lambda \in \mathbb{H}$, the map $A(\lambda) := R_0(\lambda^2)V$ is a compact operator on $L^4(\mathbb{R}^3)$.*

Proof. Firstly, note that in view of Lemma 5 and because of $V \in L^p$, $A(\lambda)$ is bounded $L^4 \to L^4$. Secondly, observe that we may assume that $V \in L^\infty$ with compact support. Indeed, replace V with $V_n = V \chi_{[|V|<n]} \chi_{[|x|<n]}$. Then $\|V - V_n\|_p \to 0$ as $n \to \infty$ implies that $\|A(\lambda) - A_n\|_{4\to 4} \to 0$ as $n \to \infty$. If we can show that $A_n := R_0(\lambda^2)V_n$ are compact as operators $L^4 \to L^4$ for each n, it therefore follows that $A(\lambda)$ is also compact. So assume that V is bounded, and supported in the ball $\{|x| < R\}$. Fix λ and write $A = A(\lambda)$. We first claim that $A : L^4 \to W^{2,4}$. This follows from

$$(12) \qquad (-\triangle + 1)A = (-\triangle - \lambda^2)A + (\lambda^2+1)A = V + (1+\lambda^2)A$$

is bounded from L^4 to L^4. Meanwhile, for $|x| > 2R$ there is the uniform pointwise bound

$$|Af(x)| \lesssim \|Vf\|_1 |x|^{-1} \lesssim R^{\frac{9}{4}} \|V\|_\infty \|f\|_4 |x|^{-1}$$

Given $\varepsilon > 0$, we may choose $R_0 \sim R^9 \|V\|_\infty^4 \varepsilon^{-4}$ so that $\|\chi_{[|x|>R_0]} Af\|_4 < \varepsilon$ for all $\|f\|_4 \le 1$.

Let $\{f_j\}_{j=1}^{\infty} \subset L^4(\mathbb{R}^3)$ satisfy $f_j \rightharpoonup 0$ in L^4. Since $\sup_j \|Af_j\|_{W^{2,4}(\mathbb{R}^3)} < \infty$, Rellich's compactness theorem produces a subsequence f_{j_k} so that $Af_{j_k} \to 0$ in $L^4(|x| < R_0)$. Thus

$$\limsup_{k\to\infty} \|Af_{j_k}\|_4 \le (1 + C_\lambda)\varepsilon.$$

Sending $\varepsilon \to 0$ and passing to the diagonal subsequence finishes the proof. □

The following lemma establishes invertibility everywhere except on the imaginary axis.

Lemma 9. *Let $V \in L^p(\mathbb{R}^3) \cap L^{\frac{3}{2}}(\mathbb{R}^3)$, $\frac{3}{2} < p < 2$ and assume that V is real-valued. Then for any nonzero $\lambda \in \mathbb{H}$, the inverse $(I + R_0(\lambda^2)V)^{-1} : L^4(\mathbb{R}^3) \to L^4(\mathbb{R}^3)$ exists.*

Proof. By the previous lemma it suffices to show that

$$f \in L^4(\mathbb{R}^3), \quad f + R_0(\lambda^2)Vf = 0 \Longrightarrow f = 0.$$

Let f be as on the left-hand side and set $g = Vf$. Then $g \in L^r$, where $r = \frac{4p}{4+p} < \frac{4}{3}$. By Lemma 5, $f = -R_0(\lambda^2)g$ therefore belongs to $L^q \cap L^4$, where $\frac{1}{q} - \frac{1}{4} = \frac{3-2p}{3p} > 0$.

This bootstrapping procedure can be repeated until it is shown that $f \in L^{r'} \cap L^4$. In fact, one can continue to the point where $f \in L^\infty$, since $R_0(\lambda^2) : L^{\frac{3}{2}-\varepsilon} \cap L^{\frac{3}{2}+\varepsilon} \mapsto L^\infty$ is a bounded operator. What is important here is that f and g exist in spaces dual to each other.

Since V is real-valued, the duality pairing

$$\langle f, g\rangle = \langle f, Vf\rangle = -\langle R_0(\lambda^2)g, g\rangle$$

shows that $\langle R_0(\lambda^2 \pm i0)g, g\rangle$ is real-valued. If $\lambda^2 \notin \mathbb{R}$, then the condition

$$\Im\langle R_0(\lambda^2)g, g\rangle = \int_{\mathbb{R}^3} \frac{\Im(\lambda^2)}{\big(|\xi|^2 - \Re(\lambda^2)\big)^2 + \Im(\lambda^2)^2} |\hat{g}(\xi)|^2 \, d\xi = 0$$

requires that $\hat{g} = 0$ almost everywhere.

On the boundary $\lambda \in \mathbb{R}$, by the Stein-Tomas theorem

$$\Im\langle R_0((\lambda + i0)^2)g, g\rangle = \lim_{\varepsilon\to 0} \Im\langle R_0((\lambda + i\varepsilon)^2)g, g\rangle = c\lambda \int_{S^2} |\hat{g}(\lambda\omega)|^2 \, \sigma(d\omega)$$

with some constant $c \neq 0$. Hence, $\hat{g} = 0$ on $|\lambda|S^2$ in the L^2 sense. Since $g \in L^r(\mathbb{R}^3)$, one concludes from Proposition 7 above that $(1 + |x|)^{\delta-\frac{1}{2}} R_0(\lambda^2 \pm i0)g \in L^2(\mathbb{R}^3)$ for some $\delta > 0$. Hence also $(1 + |x|)^{\delta-\frac{1}{2}} f \in L^2(\mathbb{R}^3)$ for some $\delta > 0$. Since $(-\triangle + V - \lambda^2)f = 0$ in the distributional sense, and one checks easily from (12) (remembering that $f \in L^\infty \cap L^4$) that also $f \in W^{2,p}_{\text{loc}}(\mathbb{R}^3) \subset W^{1,2}_{\text{loc}}(\mathbb{R}^3)$, Theorem 2 implies that $f = 0$, as claimed. □

The following two lemmas show that the inverses in the previous lemma have uniformly bounded norms.

Lemma 10. *Let $V \in L^p(\mathbb{R}^3)$, $\frac{3}{2} \le p \le 2$. The map $\lambda \mapsto R_0(\lambda^2)V$ is continuous from the domain $\mathbb{H} \setminus \{0\} \subset \mathbb{C}$ to the space of bounded operators on $L^4(\mathbb{R}^3)$.*

Proof. First suppose V is bounded and has compact support in the ball $\{|x| < R\}$. The convolution kernel associated to $R_0(\lambda^2) - R_0(\zeta^2)$ has the bounds

$$|K(x)| \lesssim \begin{cases} |\lambda - \zeta|, & \text{if } |x| < |\lambda - \zeta|^{-1} \\ |x|^{-1}, & \text{if } |x| \geq |\lambda - \zeta|^{-1} \end{cases}$$

Then for any pair $\lambda, \zeta \in \mathbb{H}$, $|\lambda - \zeta| \leq \frac{1}{2R}$, we have

$$|(R_0(\lambda^2) - R_0(\zeta^2))Vf(x)| \lesssim \begin{cases} |\lambda - \zeta| \|Vf\|_1, & \text{if } |x| < |\lambda - \zeta|^{-1} \\ |x|^{-1} \|Vf\|_1, & \text{if } |x| \geq |\lambda - \zeta|^{-1} \end{cases}$$

Thus $\|(R_0(\lambda^2) - R_0(\zeta)^2)Vf\|_4 \lesssim |\lambda - \zeta|^{1/4} R^{9/4} \|V\|_\infty \|f\|_4$.

Approximate V by compactly supported $\tilde{V} \in L^\infty$ so that $\|V - \tilde{V}\|_p < \varepsilon$. By the above calculation, Lemma 5, and the simple identity

$$(R_0(\lambda^2) - R_0(\zeta)^2)V = R_0(\lambda^2)(V - \tilde{V}) + (R_0(\lambda^2) - R_0(\zeta^2))\tilde{V} - R_0(\zeta^2)(V - \tilde{V}),$$

we see that $\limsup_{\zeta \to \lambda} \|(R_0(\lambda^2) - R_0(\zeta)^2)V\|_{4 \to 4} \lesssim |\lambda|^{(3-2p)/p} \varepsilon$. ∎

Lemma 11. *Let V be as in the previous lemma and suppose $\lambda_0 > 0$. Then*

$$\sup_{|\Re(\lambda)| \geq \lambda_0} \left\| (I + R_0(\lambda^2)V)^{-1} \right\|_{4 \to 4} < \infty. \tag{13}$$

Proof. In view of Lemma 5, there is some finite $\lambda_1 \in \mathbb{R}$ so that $\|R_0(\lambda^2)V\|_{4\to4} < \frac{1}{2}$ provided $|\lambda| > \lambda_1$. It therefore suffices to prove (13) on the compact set $\{\lambda \in \mathbb{C} : \lambda_0 \leq |\lambda| \leq \lambda_1, |\Re(\lambda)| \geq \lambda_0\}$. The previous two lemmas, however, show that $(I + R_0(\lambda^2)V)^{-1}$ is a continuous function of λ on this set, hence it is uniformly bounded from above. ∎

It is now a simple matter to prove Theorem 1.

Proof of Theorem 1. By the resolvent identity, for any $\varepsilon \neq 0$,

$$R_V(\lambda^2 + i\varepsilon) = R_0(\lambda^2 + i\varepsilon) - R_0(\lambda^2 + i\varepsilon)VR_V(\lambda^2 + i\varepsilon).$$

By Lemma 11 one therefore has

$$R_V(\lambda^2 + i\varepsilon) = (I + R_0(\lambda^2 + i\varepsilon)V)^{-1} R_0(\lambda^2 + i\varepsilon)$$

and the right-hand side is uniformly bounded for $\lambda \geq \lambda_0 \geq 0$ as well as $0 < \varepsilon \leq 1$ in the L^4 operator norm. In fact, the last factor contributes a decaying factor of $\lambda^{-\frac{1}{2}}$ as L^4 operator norm in view of Lemma 4. ∎

Proof of Proposition 3. There is only one point in the argument where the condition $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ is used, namely the step in Lemma 9 where we wish to make use of Theorem 2. It otherwise suffices to assume that $V \in L^p(\mathbb{R}^3)$, $\frac{3}{2} < p < 2$.

For the second claim, one observes the following consequence of Lemma 5: If $V \in L^p$, $\frac{3}{2} < p \leq 2$, and $r > 4$, then $R_0(\lambda^2 \pm 10)V : L^4 \cap L^r \mapsto L^4 \cap L^s$, where $\frac{1}{s} = \max(\frac{1}{r} + \frac{1}{p} - \frac{2}{3}, 0)$. The same is true for any $V \in L^q, p \leq q \leq 2$. This allows the bootstrapping procedure on f to continue normally, and furthermore $g = Vf$ is still an element of $L^{\frac{4}{3}-\varepsilon}$, as desired. Therefore, the only matter of concern is whether the conclusion of Theorem 2 will hold for such a potential V. ∎

4 Further estimates on the free resolvent

Returning to Proposition 7, we note that a sharper estimate can be made at the endpoint $p = 1$.

Proposition 12. *Let f be a function in $L^1(\mathbb{R}^3)$ such that $\hat{f} = 0$ on the unit sphere S^2. Then*

$$\sup_{\varepsilon>0} \big\| R_0(1 \pm i\varepsilon) f \big\|_2 \leq \frac{1}{\sqrt{8\pi}} \|f\|_1 \tag{14}$$

Proof. Define the trace function

$$G(\lambda) = \lambda^{-2} \big\| \hat{f}|_{\lambda S^2} \big\|_2^2 = 4\pi \iint_{\mathbb{R}^6} f(x) \frac{\sin(\lambda|x-y|)}{\lambda|x-y|} \bar{f}(y)\, dx\, dy \tag{15}$$

By inspection,

$$G(\lambda) = 2\pi \iiint_{\mathbb{R}\times\mathbb{R}^6} \frac{f(x)\bar{f}(y)}{|x-y|} \chi_{|x-y|}(\tau) e^{i\lambda\tau}\, d\tau dx dy \tag{16}$$

where $\chi_{|x-y|}$ denotes the characteristic function of the interval $\{|\tau| \leq |x-y|\}$. The integrand on the right-hand side is in $L^1(\mathbb{R}^7)$, so Fubini's Theorem implies that G is the inverse Fourier transform of an L^1 function.

Using the Plancherel identity (in 3 dimensions), and noting that G is an even function,

$$\|R_0(1 \pm i\varepsilon) f\|_2^2 = \frac{1}{2(2\pi)^3} \int_{-\infty}^{\infty} G(\lambda) \frac{\lambda^2}{|\lambda^2 - (1+i\varepsilon)|^2} \tag{17}$$

For any $\varepsilon > 0$, the multiplier $M_\varepsilon(\lambda) = \frac{\lambda^2}{|\lambda^2-(1+i\varepsilon)|^2}$ is integrable, hence it has Fourier transform $\hat{M}_\varepsilon \in L^\infty(d\tau)$. By Parseval's formula, this time in one dimension,

$$\|R_0(1 \pm i\varepsilon) f\|_2^2 = \frac{1}{2(2\pi)^4} \int_{\mathbb{R}} \hat{G}(\tau) \hat{M}_\varepsilon(-\tau)\, d\tau \tag{18}$$

An explicit formula for $\hat{M}_\varepsilon(\tau)$ can be obtained via residue integrals:

$$\hat{M}_\varepsilon(\tau) = \frac{\pi}{2\varepsilon} \big(\sqrt{1+i\varepsilon}\, e^{i|\tau|\sqrt{1+i\varepsilon}} + \sqrt{1-i\varepsilon}\, e^{-i|\tau|\sqrt{1-i\varepsilon}} \big) \tag{19}$$

This, along with (16), can be immediately substituted back into equation (18).

$$\begin{aligned} \|R_0(1 \pm i\varepsilon) f\|_2^2 &= \frac{1}{8\pi\varepsilon} \iint_{\mathbb{R}^6} \int_0^{|x-y|} \frac{f(x)\bar{f}(y)}{|x-y|} \big(\sqrt{1+i\varepsilon}\, e^{i\tau\sqrt{1+i\varepsilon}} + \sqrt{1-i\varepsilon}\, e^{-i\tau\sqrt{1-i\varepsilon}} \big)\, d\tau\, dx\, dy \\ &= \frac{1}{8\pi i\varepsilon} \iint_{\mathbb{R}^6} \frac{f(x)\bar{f}(y)}{|x-y|} \big(e^{i|x-y|\sqrt{1+i\varepsilon}} - e^{-i|x-y|\sqrt{1-i\varepsilon}} \big) dx\, dy \end{aligned}$$

Boundedness of $\hat{M}_\varepsilon$ enables us to continue applying Fubini's theorem to the multiple integral. We have also simplified the expression by noting that $\hat{M}_\varepsilon$ is an even function. Recall definition (15) and subtract $\frac{1}{16\pi^2\varepsilon}G(1)$ from both sides of the equation.

$$
\begin{aligned}
&\|R_0(1\pm i\varepsilon)f\|_2^2 - \frac{1}{16\pi^2\varepsilon}G(1) \\
&\quad = \frac{1}{8\pi i\varepsilon}\iint_{\mathbb{R}^6}\frac{f(x)\bar{f}(y)}{|x-y|}\Big(\big(e^{i|x-y|\sqrt{1+i\varepsilon}}-e^{i|x-y|}\big)-\big(e^{-i|x-y|\sqrt{1-i\varepsilon}}-e^{-i|x-y|}\big)\Big)\,dx\,dy \qquad (20)\\
&\quad = \frac{1}{8\pi i\varepsilon}\iint_{\mathbb{R}^6}\frac{f(x)\bar{f}(y)}{|x-y|}K(|x-y|)\,dx\,dy
\end{aligned}
$$

where $|K(|x-y|)|\leq \varepsilon|x-y|$. This leads to the conclusion

$$\big|\|R_0(1\pm i\varepsilon)f\|_2^2 - \frac{1}{16\pi^2\varepsilon}G(1)\big| \leq \frac{1}{8\pi}\|f\|_1^2$$

If f satisfies the hypothesis $\hat{f}|_{S^2}=0$, then $G(1)=0$. □

Corollary 13. *Let f be a function in $L^1(\mathbb{R}^3)$ such that $\hat{f}=0$ on the unit sphere S^2. Then*

$$\|R_0(1\pm i0)f\|_2 \leq \frac{1}{\sqrt{8\pi}}\|f\|_1 \qquad (21)$$

Proof. This follows immediately from (17) and monotone convergence. □

The condition $\hat{f}=0$ is crucial in Proposition 7. Indeed, recall that for $f\in L^p(\mathbb{R}^3)$ real-valued with $1\leq p\leq \frac{4}{3}$ one has

$$\Im R_0(1+i0)f = c\,(\widehat{\sigma_{S^2}}*f)$$

for some constant c. This follows by writing $R_0(1+i\varepsilon)$ as a sum of its real and imaginary parts, as well as from the fact that the operation of restriction $f\mapsto \hat{f}(r\cdot)$ is continuous in $r>0$ as a map $L^p(\mathbb{R}^3)\to L^2(S^2)$. However, it is clear that for any $\delta>0$

$$\|(1+|x|)^{\delta-\frac{1}{2}}[\widehat{\sigma_{S^2}}*f]\|_2=\infty \qquad (22)$$

even for smooth bump-functions f since the function inside the norm decays like $(1+|x|)^{\delta-\frac{3}{2}}$ which just fails to be $L^2(\mathbb{R}^3)$. The following simple lemma shows, on the other hand, that $\delta<0$ does lead to a finite norm in (22).

Lemma 14. *For any $R\geq 1$ one has*

$$\Big\|\chi_{[|x|<R]}[\widehat{\sigma_{S^2}}*f]\Big\|_2 \lesssim \sqrt{R}\,\|f\|_{\frac{4}{3}}$$

for all $f\in L^{\frac{4}{3}}(\mathbb{R}^3)$.

Proof. Let ϕ be a smooth cut-off function with $\hat{\phi}$ compactly supported. Then by Plancherel, and Cauchy-Schwartz,

$$
\begin{aligned}
&\Big\|\chi\Big(\frac{\cdot}{R}\Big)[\widehat{\sigma_{S^2}}*f]\Big\|_2^2 = R^6\int_{\mathbb{R}^3}\Big|\int_{S^2}\hat{\chi}(R(\xi-\eta))\hat{f}(\eta)\,\sigma_{S^2}(d\eta)\Big|^2\,d\xi\\
&\lesssim R^6\int_{\mathbb{R}^3}\int_{S^2}|\hat{\chi}(R(\xi-\eta'))|\,d\eta'\int_{S^2}|\hat{\chi}(R(\xi-\eta))||\hat{f}(\eta)|^2\,\sigma_{S^2}(d\eta)\,d\xi\\
&\lesssim R\,\|\hat{f}\|_{L^2(S^2)}^2 \lesssim R\,\|f\|_{\frac{4}{3}}^2,
\end{aligned}
$$

as claimed. □

The previous lemma suggests that one should also have the bound

$$\sup_{\varepsilon>0} \left\| \chi_{[|x|<R]} R_0(1 \pm i\varepsilon) f \right\|_2 \lesssim \sqrt{R}\, \|f\|_{\frac{4}{3}}. \tag{23}$$

While this bounds remains open[1], it is easy to show that

$$\sup_{\varepsilon>0} \left\| \chi_{[|x|<R]} R_0(1 \pm i\varepsilon) f \right\|_2 \lesssim R^{\frac{3}{4}}\, \|f\|_{\frac{4}{3}}. \tag{24}$$

Indeed, denoting the operator on the left-hand side by T for a fixed $\varepsilon > 0$, observe that by Lemma 4

$$T^*T = R_0(1 - i\varepsilon)\chi_{[|x|<R]} R_0(1 + i\varepsilon)$$

satisfies $\|T^*T\|_{\frac{4}{3}\to 4} \lesssim \|\chi_{[|x|<R]}\|_2 \lesssim R^{\frac{3}{2}}$, which is the same as (24). One would of course expect that the Knapp example determines the power of R in (23). Note that in the case of a Knapp example of dimensions $\delta \times \sqrt{\delta} \times \sqrt{\delta}$ where $\delta = R^{-1}$, one does not encounter the L^2 norm of $\|\chi_{[|x|<R]}\|_2$ in the previous T^*T argument, but rather the L^2 norm of a $R \times \sqrt{R} \times \sqrt{R}$-tube, which gives the conjectured R. Conversely, in what follows we show how to approach (23) by a decomposition into Knapp examples. This leads to an improvement over the simple bound (24) by $\frac{1}{8}$. Such arguments originate in the analysis of Bochner Riesz multipliers as well as the restriction theory of the Fourier transform. We will use a square function bound from Bourgain [Bou1]. For the convenience of the reader, we reproduce the details.

The following lemma is a discrete version of the Stein-Tomas theorem, see Lemma 6.2 in [Bou1]. It can also be proved by expanding the L^4-norm on the left-hand side explicitly and then using the usual geometric arguments based on counting overlap. However, the following approach does not depend on any special arithmetic properties of the Stein-Tomas exponent and therefore generalizes to other dimensions as well.

Lemma 15. *Let $R \geq 1$ and let $\{\xi_\alpha\}_\alpha \subset S^2$ be a collection of $R^{-\frac{1}{2}}$-separated points and $\{a_\alpha\}_\alpha$ a sequence of arbitrary numbers. Then*

$$\Big\| \sum_\alpha e^{ix\cdot\xi_\alpha}\, a_\alpha \Big\|_{L^4(Q)} \lesssim R^{\frac{1}{2}} \Big(\sum_\alpha |a_\alpha|^2 \Big)^{\frac{1}{2}} \tag{25}$$

for any cube Q of side-length $R^{\frac{1}{2}}$.

Proof. Fix $R \geq 1$, some cube Q and a smooth cut-off function χ_Q adapted to Q. We assume that $\mathrm{supp}(\widehat{\chi_Q})$ is contained inside a $CR^{-\frac{1}{2}}$-cube with some big constant C. Then

$$\begin{aligned}
\Big\| \sum_\alpha e^{ix\cdot\xi_\alpha} a_\alpha \Big\|_{L^4(Q)} &\lesssim \Big\| \Big[\sum_\alpha a_\alpha \widehat{\chi_Q}(\cdot - \xi_\alpha) \Big]^\vee \Big\|_4 \\
&\lesssim \int_{1-R^{-\frac{1}{2}}}^{1+R^{-\frac{1}{2}}} \Big\| \int_{S^2} e^{ir\omega\cdot x} \sum_\alpha a_\alpha \widehat{\chi_Q}(r\omega - \xi_\alpha)\, \sigma(d\omega) \Big\|_{L^4_x} dr \\
&\lesssim R^{-\frac{1}{2}} R^{\frac{3}{2}} \Big(\sum_\alpha R^{-1} |a_\alpha|^2 \Big)^{\frac{1}{2}} = R^{\frac{1}{2}} \Big(\sum_\alpha |a_\alpha|^2 \Big)^{\frac{1}{2}},
\end{aligned}$$

as claimed. □

[1] Note added in proof: This conjecture is solved in a forthcoming paper by Ionescu and the second author [IonSch], which also contains other improvements on the results obtained here.

In what follows, we associate with every $R \geq 1$ a decomposition $\{S_\alpha^{(R)}\}_\alpha$ of the shell $\mathcal{S}_R := \{\xi \in \mathbb{R}^3 : ||\xi| - 1| < R^{-1}\}$ into Knapp caps $S_\alpha^{(R)}$ of size about $R^{-\frac12} \times R^{-\frac12} \times R^{-1}$. We furthermore assume bounded overlap, i.e., that

$$\sup_{R \geq 1} \Big\| \sum_\alpha \chi_{S_\alpha^{(R)}} \Big\|_\infty < \infty.$$

Lemma 16. *Let $g \in L^4(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ be so that $\hat{g} \subset \mathcal{S}_R$. Write $g = \sum_\alpha g_\alpha$ where each $\widehat{g_\alpha}$ is supported in $S_\alpha^{(R)}$. Then*

$$\|g\|_4 \lesssim R^{\frac18} \Big\| \Big(\sum_\alpha |g_\alpha|^2 \Big)^{\frac12} \Big\|_4. \tag{26}$$

Dually, one has for any $f \in L^{\frac43}(\mathbb{R}^3)$ that

$$\Big\| \Big(\sum_\alpha |f_\alpha|^2 \Big)^{\frac12} \Big\|_{\frac43} \lesssim R^{\frac18} \, \|f\|_{\frac43} \tag{27}$$

where $\widehat{\sum_\alpha f_\alpha} = \hat{f} \chi_{\mathcal{S}_R}$ and each f_α is Fourier supported in $S_\alpha^{(R)}$.

Proof. Firstly, fix $R \geq 1$, a cube Q of size $\sqrt{R}$, the cap decomposition $\{S_\alpha^{(R)}\}_\alpha$, as well as some $\xi_\alpha \in S_\alpha^{(R)}$ for each α. Secondly, fix some smooth cut-off χ_0 so that $\chi_0(\sqrt{R}\, y)$ is adapted to the cube $Q_0 := [-c_0\sqrt{R}, c_0\sqrt{R}]^3$ where $c_0 > 0$ is small. We require that $\chi_0(\sqrt{R}\,(\xi - \xi_\alpha)) = 1$ for all $\xi \in S_\alpha^{(R)}$ and each α. Set

$$\mu(\xi) := \chi_0(\sqrt{R}\,\xi) \left(\fint_{Q_0} e^{iy\cdot\xi} \, dy \right)^{-1}$$

and $\mu_\alpha(\xi) = \mu(\xi - \xi_\alpha)$ for every α. Note that this is again a smooth cut-off function adapted to a ball of size $R^{-\frac12}$ (together with the natural derivative bounds with constants uniform in R). In particular, $\sup_\alpha \|\widehat{\mu_\alpha}\|_1 = \|\hat{\mu}\|_1 < \infty$ uniformly in R. Then

$$\begin{aligned}
\|g\|_{L^4(Q)}^4 &= \Big\| \sum_\alpha \int e^{ix\cdot\xi} \, \widehat{g_\alpha}(\xi) \, d\xi \Big\|_{L^4(Q)}^4 \\
&\leq \int_Q \fint_{Q_0} \Big| \sum_\alpha e^{-iy\cdot\xi_\alpha} \int e^{i(x+y)\cdot\xi} \, \widehat{g_\alpha}(\xi)\mu_\alpha(\xi) \, d\xi \Big|^4 \, dy \, dx && (28) \\
&\lesssim \fint_{2Q} \int_{Q_0} \Big| \sum_\alpha e^{-iy\cdot\xi_\alpha} \int e^{ix\cdot\xi} \widehat{g_\alpha}(\xi)\mu_\alpha(\xi) \, d\xi \Big|^4 \, dy \, dx && (29) \\
&\lesssim \fint_{2Q} R^2 \left(\sum_\alpha \Big| \int e^{ix\cdot\xi} \, \widehat{g_\alpha}(\xi)\mu_\alpha(\xi) \, d\xi \Big|^2 \right)^2 \, dx && (30) \\
&\lesssim R^{\frac12} \Big\| \Big(\sum_\alpha |h_\alpha|^2 \Big)^{\frac12} \Big\|_{L^4(2Q)}^4, && (31)
\end{aligned}$$

where $h_\alpha = g_\alpha * k_\alpha$ and $\widehat{k_\alpha} = \mu_\alpha$. Here (28) follows from Jensen's inequality as well as the definition of μ_α, (29) follows by changing variables (and $2Q$ is the cube with the same center but twice the size), whereas (30) is a consequence of Lemma 15. Summing (31) over a partition $\{Q\}$ of $\mathbb{R}^3$ consisting of congruent cubes one obtains that

$$\|g\|_4 \lesssim R^{\frac{1}{8}} \Big\| \Big(\sum_\alpha |h_\alpha|^2 \Big)^{\frac{1}{2}} \Big\|_4 \lesssim R^{\frac{1}{8}} \Big\| \Big(\sum_\alpha |g_\alpha|^2 * \sup_\alpha |k_\alpha| \Big)^{\frac{1}{2}} \Big\|_4 \lesssim R^{\frac{1}{8}} \Big\| \sum_\alpha |g_\alpha|^2 * |\check{\mu}| \Big\|_2^{\frac{1}{2}}$$
$$\lesssim R^{\frac{1}{8}} \Big\| \Big(\sum_\alpha |g_\alpha|^2 \Big)^{\frac{1}{2}} \Big\|_4$$

since $\|\check{\mu}\|_1 \lesssim 1$. This proves (26) and (27) follows by duality. Indeed,

$$\|\{f_\alpha\}\|_{L^{\frac{4}{3}}(\ell^2)} = \sup_{\|\{g_\alpha\}\|_{L^4(\ell^2)} \le 1} \Big| \int \sum_\alpha f_\alpha \, \overline{g_\alpha} \Big| \lesssim \sup_{\|g\|_4 \le C R^{\frac{1}{8}},\, \mathrm{supp}(\hat{g}) \subset \mathcal{S}_R} |\langle f, g \rangle| \tag{32}$$

$$\lesssim \sup_{\|g\|_4 \le C R^{\frac{1}{8}}} |\langle f, g \rangle| \lesssim R^{\frac{1}{8}} \|f\|_{\frac{4}{3}}. \tag{33}$$

In line (32), the supremum is taken over all g as in (26), whereas in (33), we drop the condition $\mathrm{supp}(\hat{g}) \subset \mathcal{S}_R$. $\square$

The powers of R in (25), as well as in (26) and (27) are optimal. In the case of (25), this can be seen by taking all $a_\alpha = 1$ and similarly for the square function.
Recall that Agmon's limiting absorption principle states that

$$\sup_{\varepsilon > 0} \big\| (1 + |x|)^{-\frac{1}{2}-} R_0(1 \pm i\varepsilon) f \big\|_2 \lesssim \big\| (1 + |x|)^{\frac{1}{2}+} f \big\|_2. \tag{34}$$

This in particular implies that

$$\sup_{\varepsilon > 0} \Big\| \chi_{[|x|<R]} R_0(1 \pm i\varepsilon) \chi_{[|x|<R]} f \Big\|_2 \lesssim R^{1+} \, \|f\|_2. \tag{35}$$

The following lemma is an improved version of (35), see the decay in $|v|$. The proof is self-contained. In particular, it does not rely on (34).

Lemma 17. *For any $v \in \mathbb{R}^3$ one has*

$$\sup_{\varepsilon > 0} \Big\| \chi_{[|x|<R]} R_0(1 \pm i\varepsilon) \chi_{[|x-v|<R]} f \Big\|_2 \lesssim R(1 + |v|/R)^{-1} \, \|f\|_2. \tag{36}$$

for all $R \ge 1$. The constants are uniform in R and v.

Proof. This is basically a simple consequence of Hörmander's variable coefficient Plancherel theorem, see [Hör], and especially Wolff's notes [Wol], page 55. We start with the case $|v| \gg R$. Consider the operator

$$T_{R,v} f(x) = \int_{\mathbb{R}^3} \psi(x) \frac{e^{iR|x-y|}}{|x-y|} \psi(y - v/R) f(y) \, dy$$

where ψ is a smooth bump function at zero. This operator is of the form

$$T_{R,v} f(x) = \int_{\mathbb{R}^3} e^{iR\Phi(x,y)} a_v(x, y) f(y) \, dy$$

where Φ is smooth on the support of $a_v(x,y)$, and $\mathrm{rank}[\partial^2_{xy}\Phi(x,y)] = 2$. Moreover, $a_v(x,y)$ is smooth, the size of its support is uniformly bounded in v, and $\|\partial^\beta a_v\|_\infty \lesssim (1+|v|/R)^{-1}$. Hence, Hörmander's variable coefficient Plancherel theorem implies that

$$\|T_{R,v}\|_{2\to 2} \lesssim R^{-1}(1+|v|/R)^{-1}.$$

One now checks that (36) follows from this by means of a change of variables.
If $|v| \lesssim R$, one can argue similarly, but needs to introduce a Whitney decomposition away from the singularity $x = y$. Firstly, note that with ψ as above,

$$\left\| \int_{\mathbb{R}^3} \psi(x) \frac{e^{iR|x-y|}}{|x-y|} \chi_{[|x-y|\le R^{-\frac12}]} \psi(y) f(y)\, dy \right\|_2 \lesssim \int_{[|x|<R^{-\frac12}]} \frac{dx}{|x|}\, \|f\|_2 \lesssim R^{-1}\, \|f\|_2.$$

Next, let $\rho \in C_0^\infty(\mathbb{R})$ be a smooth function so that $\rho(t) = 1$ if $1 < t < 2$ and $\rho(t) = 0$ if $t > 2$ or $t < \frac12$. Then we claim that

$$\Big\| \int_{\mathbb{R}^3} \psi(x) \frac{e^{iR|x-y|}}{|x-y|} \rho(2^{-j}|x-y|) \psi(y) f(y)\, dy \Big\|_2 \lesssim 2^j\, R^{-1}\, \|f\|_2 \tag{37}$$

for all $R^{-\frac12} < 2^j \le 1$. To this end introduce a further decomposition $1 = \sum_\ell \omega\big(2^{-j}(x - x_\ell)\big)$ where the sum runs over a lattice of points $\{x_\ell\}_\ell$ in $\mathbb{R}^3$ that are 2^j-spaced, and ω is some smooth cut-off which is adapted to the unit cube. Exploiting orthogonality, (37) follows from the following estimate

$$\Big\| \int_{\mathbb{R}^3} \omega\big(2^{-j}(x-x_\ell)\big) \frac{e^{iR|x-y|}}{|x-y|} \rho(2^{-j}|x-y|) \omega\big(2^{-j}(y-x_k)\big)\ f(y)\, dy \Big\|_2 \lesssim 2^j\, R^{-1}\, \|f\|_2.$$

This, however, is again reduced the Hörmander's bound by means of an obvious rescaling. □

We are now ready to formulate our estimate which lies between the conjecture (23) and the simple bound (24).

Proposition 18. *Let $f \in L^{\frac43}(\mathbb{R}^3)$. Then there is the bound*

$$\sup_{\varepsilon>0} \big\| \chi_{[|x|<R]} R_0(1 \pm i\varepsilon) f \big\|_2 \lesssim R^{\frac58}\, \|f\|_{\frac43}. \tag{38}$$

for all $R \ge 1$.

Proof. If $\mathrm{supp}(\hat f) \subset \big\{\xi \in \mathbb{R}^3 \ :\ ||\xi| - 1| > \frac12\big\}$, then one has

$$\sup_{\varepsilon>0} \big\| \chi_{[|x|<R]} R_0(1 \pm i\varepsilon) f \big\|_2 \lesssim \|(-\triangle + 1)^{-1} f\|_2 \lesssim \|f\|_{\frac43}.$$

Hence we can assume that $\mathrm{supp}(\hat f) \subset \big\{\xi \in \mathbb{R}^3 \ :\ ||\xi| - 1| \le \frac12\big\}$ and we write $f = \sum_k f_k$, where

$$\mathrm{supp}(\hat f_k) \subset \big\{\xi \in \mathbb{R}^3 \ :\ ||\xi| - 1| \asymp 2^k R^{-1}\big\}$$

for $k \ge 1$ and

$$\mathrm{supp}(\hat f_0) \subset \big\{\xi \in \mathbb{R}^3 \ :\ ||\xi| - 1| \le R^{-1}\big\}.$$

Now $R_0(1+i\varepsilon)f_0 = R_0(1+i\varepsilon)\sum_\alpha f_\alpha$ with $\sum_\alpha f_\alpha$ as in Lemma 16. Consider any of the pieces f_α. Then one can write $f_\alpha = \sum_j f_{\alpha,j}$ where each $f_{\alpha,j}$ lives on a tube of dimensions $R \times \sqrt{R} \times \sqrt{R}$ and the tubes corresponding to different j's are disjoint. Moreover, the bound

$$\|f_{\alpha,j}\|_\infty \lesssim \|f_{\alpha,j}\|_p \, R^{-\frac{2}{p}}$$

holds for every $1 \le p \le \infty$. Therefore,

$$\begin{aligned}
\|f_\alpha\|_2^2 &\lesssim \sum_j \|f_{\alpha,j}\|_2^2 \lesssim \sum_j \|f_{\alpha,j}\|_\infty^2 R^2 \\
&\lesssim \sum_j \|f_{\alpha,j}\|_{\frac{4}{3}}^2 R^{-1} \lesssim \Bigg(\sum_j \|f_{\alpha,j}\|_{\frac{4}{3}}^{\frac{4}{3}}\Bigg)^{\frac{3}{2}} R^{-1} \\
&\lesssim \|f_\alpha\|_{\frac{4}{3}}^2 \, R^{-1}. \qquad (39)
\end{aligned}$$

We will apply this bound not to f_α, but to $\psi((\cdot - v)/R) f_\alpha$ for a smooth cut-off ψ and arbitrary v. This is justified since the Fourier support of $\psi((\cdot - v)/R) f_\alpha$ still lies in the cap $S_\alpha^{(R)}$. In combination with (36), (39) yields

$$\begin{aligned}
&\sup_{\varepsilon>0} \big\|\chi_{[|x|<R]} R_0(1 \pm i\varepsilon) f_\alpha\big\|_2 \lesssim \sum_{v \in R\mathbb{Z}^3} \sup_{\varepsilon>0} \big\|\chi_{[|x|<R]} R_0(1\pm i\varepsilon)\chi_{[|x-v|<R]} f_\alpha\big\|_2 \\
&\lesssim R \sum_{v\in R\mathbb{Z}^3} (1+|v|/R)^{-1} \|\psi((\cdot - v)/R) f_\alpha\|_2 \\
&\lesssim R \sum_{v\in R\mathbb{Z}^3} (1+|v|/R)^{-1} \, R^{-\frac{1}{2}} \, \|\psi((\cdot - v)/R) f_\alpha\|_{\frac{4}{3}} \\
&\lesssim R^{\frac{1}{2}} \Big(\sum_{j\in\mathbb{Z}^3} (1+|j|)^{-4}\Big)^{\frac{1}{4}} \Big(\sum_{j\in\mathbb{Z}^3} \|\psi((\cdot - Rj)/R) f_\alpha\|_{\frac{4}{3}}^{\frac{4}{3}}\Big)^{\frac{3}{4}} \lesssim R^{\frac{1}{2}} \|f_\alpha\|_{\frac{4}{3}}. \qquad (40)
\end{aligned}$$

It remains to sum up (40) exploiting orthogonality as provided by Lemma 16:

$$\begin{aligned}
\sup_{\varepsilon>0} \Big\|\chi_{[|x|<R]} R_0(1\pm i\varepsilon) f_0\Big\|_2^2 &\le \sum_\alpha \sup_{\varepsilon>0} \Big\|\psi\big(\tfrac{\cdot}{R}\big) R_0(1\pm i\varepsilon) f_\alpha\Big\|_2^2 \\
&\lesssim R \sum_\alpha \|f_\alpha\|_{\frac{4}{3}}^2 \lesssim R^{\frac{5}{4}} \|f\|_{\frac{4}{3}}^2.
\end{aligned}$$

The final bound uses (27). It is easier to deal with f_k where $k \ge 1$. Fix such a k and let $f_k = \sum_\beta f_\beta$ be the corresponding decomposition into Knapp caps. Then, by (39),

$$\|f_\beta\|_2 \lesssim \|f_\beta\|_{\frac{4}{3}} (2^k \, R^{-1})^{\frac{1}{2}}$$

and instead of (40) one now has

$$\sup_{\varepsilon>0} \big\|\chi_{[|x|<R]} R_0(1\pm i\varepsilon) f_\beta\big\|_2 \lesssim (2^k R^{-1})^{-1} \|f_\beta\|_2 \lesssim (2^{-k} R)^{\frac{1}{2}} \|f_\beta\|_{\frac{4}{3}}.$$

Invoking (27) finally allows one to conclude that

$$\sup_{\varepsilon>0} \left\| \chi_{[|x|<R]} R_0(1 \pm i\varepsilon) f_k \right\|_2 \lesssim (2^{-k}R)^{\frac{5}{4}} \|f\|_{\frac{4}{3}}.$$

Summing over k finishes the proof. □

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