STRICHARTZ ESTIMATES AND MAXIMAL OPERATORS FOR THE WAVE EQUATION IN $\mathbb{R}^3$

MARIUS BECEANU AND MICHAEL GOLDBERG

ABSTRACT. We prove sharp Strichartz-type estimates in three dimensions, including some which hold in reverse spacetime norms, for the wave equation with potential. These results are also tied to maximal operator estimates studied by Rogers–Villaroya, of which we prove a sharper version.

As a sample application, we use these results to prove the local well-posedness and the global well-posedness for small initial data of semilinear wave equations in $\mathbb{R}^3$ with quintic or higher monomial nonlinearities.

1. INTRODUCTION

1.1. Main result.

Consider the linear wave equation in $\mathbb{R}^3$

$$\partial_t^2 f - \Delta f + Vf = F, f(0) = f_0,\ \partial_t f(0) = f_1. \tag{1}$$

Under rather general conditions (i.e. if $H = -\Delta + V$ is selfadjoint on L^2), taking initial data in $L^2 \times \dot{H}^{-1}$ for example, the solution to this equation is given by the formula

$$f(t) = \cos(t\sqrt{H})f_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}} f_1 + \int_0^t \frac{\sin((t-s)\sqrt{H})}{\sqrt{H}} F(s)\, ds. \tag{2}$$

The expressions in (2) are independent of the branch chosen for $\sqrt{H}$, so are well defined even if H is not positive.

The homogeneous equation has a conserved energy, namely $E(t) = \|f\|_{\dot{H}^1}^2 + \|\partial_t f\|_2^2 + \int_{\mathbb{R}^3} V|f|^2\, dx$, which however does not preclude time-decay of solutions with respect to other norms. In the free ($V = 0$) case there are well known uniform decay bounds when the initial data f_0 and f_1 possess a sufficient degree of Sobolev regularity:

$$\|\cos(t\sqrt{-\Delta})f_0\|_\infty \lesssim |t|^{-1}\|f_0\|_{W^{2,1}(\mathbb{R}^3)}$$
$$\Big\|\frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}} f_1\Big\|_\infty \lesssim |t|^{-1}\|f_1\|_{W^{1,1}(\mathbb{R}^3)}$$

These operators act by pointwise multiplication of the Fourier transform, so their mapping properties on L^2 and H^s follow immediately by Placherel's identity. By combining them with the dispersive estimates above, one can extract the family of Strichartz inequalities that control each term in (2) with respect to certain norms $L^p_t L^q_x$. The full range of valid pairs of exponents is determined in [KeTa].

The first author received support from the Rutgers Research Council during the preparation of this work.

The second author received support from NSF grant DMS-1002515 during the preparation of this work.

We will prove dispersive and Strichartz estimates for a large class of short-range potentials $V(x)$, and also establish bounds of a similar nature in the reversed space-time norms

$$\|f\|_{L^p_x L^q_t} = \Big(\int_{R^d} \|f(x,t)\|_{L^q_t}^p \, dx \Big)^{1/p}.$$

Two applications of the reversed Strichartz inequalities are presented. For the linear evolution we prove an endpoint estimate for the maximal operator, and deduce almost-everywhere convergence to the initial data when (f_0, f_1) belong to the energy space. These results are new even when $V = 0$. We then state a concise global well-posedness result for small solutions of the energy-critical semilinear wave equation with a potential.

One substantial difference between the perturbed Hamiltonian $H = -\Delta + V$ and the Laplacian is the possible existence of eigenvalues. For the class of short-range potentials we consider, the essential spectrum of H is $[0, \infty)$ and the point spectrum may include a countable number of eigenvalues occupying a bounded subset of the real axis that is discrete away from zero. Embedded positive eigenvalues do not occur if $V \in L^{3/2}_{\mathrm{loc}}$ [IoJe]; when V is more singular we add this as an assumption. We further assume that zero is a regular point of the spectrum of H. Under these hypotheses H possesses pure absolutely continuous spectrum on $[0, \infty)$ and a finite number of negative eigenvalues.

If $E < 0$ is a negative eigenvalue, the associated eigenfunction responds to the wave equation propagators via scalar multiplication by $\cos(t\sqrt{E})$ or $E^{-1/2}\sin(t\sqrt{E})$, both of which grow exponentially due to $\sqrt{E}$ being purely imaginary. Dispersive estimates for H must include a projection onto the continuous spectrum in order to avoid this growth. Otherwise, further conditions on the potential are required to prevent the existence of eigenvalues entirely.

Results below refer to the Kato norm $\mathcal{K}$, introduced by Rodnianski–Schlag in [RoSc] and D'Ancona–Pierfelice in [DaPi]. In particular, potentials V are taken in the Kato norm closure of the set of bounded, compactly supported functions, which we denote by $\mathcal{K}_0$.

Definition 1. The Kato space is the Banach space of measures with the property that

$$\|V\|_{\mathcal{K}} := \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x)|}{|x-y|} \, dx < \infty. \tag{3}$$

$\mathcal{K}_0$ contains the Lorentz space $L^{3/2,1}$ by Young's inequality, as well as $\dot{W}^{1,1}$ by Lemma 7. Among compactly supported functions, K_0 coincides with the Kato class defined by the property $\sup_y \int_{|x-y|<r} |V(x)|/|x-y| \, dx \to 0$ as r decreases to zero (see [DaPi, Lemma 4.4]).

Note that the homogeneous wave equation (1) is left unchanged by the rescaling $f(x,t) \mapsto f(\alpha x, \alpha t)$, as long as the potential $V(x)$ changes to $\alpha^2 V(\alpha x)$. Quadratic decay at infinity is invariant under this rescaling, as is the Kato norm. At the level of operators, pointwise multiplication by $V \in \mathcal{K}_0$ is infinitesimally form-bounded relative to the Laplacian, hence there is a unique self-adjoint realization of $H = -\Delta + V$.

Several estimates in the following discussion use the complex interpolation spaces $\mathcal{K}^\theta := (L^1, \mathcal{K})_{[\theta]}$, for $0 \leq \theta \leq 1$, and their duals. Note that

$$\mathcal{K}^\theta := (L^1, \mathcal{K})_{[\theta]} = \{f \mid \sup_x \int_{\mathbb{R}^3} \frac{|f(y)|\, dy}{|x-y|^\theta} < \infty\}$$

and by Young's inequality

$$L^{3/(3-\theta),1} \subset \mathcal{K}^\theta. \tag{4}$$

The description of $(\mathcal{K}^\theta)^*$ is more involved:

$$(\mathcal{K}^\theta)^* := \{f \mid f(x) = \int g(x,y)\, d\mu(y),\ \int_{\mathbb{R}^3} \big(\sup_x |x-y|^\theta |g(x,y)|\big)\, d\mu(y) < \infty\}.$$

Also note that by duality, for $0 \leq \theta \leq 1$

$$(\mathcal{K}^\theta)^* \subset L^{3/\theta,\infty}. \tag{5}$$

Thus results expressed by means of $\mathcal{K}^\theta$ and $(\mathcal{K}^\theta)^*$ are sharper than those containing the scale of Lorentz spaces $L^{p,q}$.

In this paper we exhibit a new class of Strichartz inequalities in $\mathbb{R}^3$, which hold in reversed space-time norms of the form

$$\|f\|_{L^p_x L^q_t} = \Big(\int_{R^d} \|f(x,t)\|^p_{L^q_t}\, dx\Big)^{1/p}$$

and $\|f\|_{\mathcal{K}^\theta_x L^p_t}$, $\|f\|_{(\mathcal{K}^\theta)^*_x L^p_t}$ defined in the same manner. For pairs of exponents (p,q) with $q > p$ such inequalities will be stronger than the standard Strichartz bounds. In other cases, reversed space-time estimates will hold for pairs of coefficients for which the corresponding regular Strichartz estimates are false.

We allow for inhomogeneous terms F of the same types, which enables the familiar bootstrapping methods for semilinear equations to take place entirely in reversed space-time function spaces.

In the following discussion let $a \lesssim b$ denote $|a| \leq C|b|$ for various values of C.

The assumption that H has no eigenvalues or resonances at zero leads to a number of equivalences between Sobolev spaces based on applications of H in place of the Laplacian. Lemma 12 demonstrates that $\|\Delta f\|_1$ and $\|Hf\|_1$ are equivalent under these conditions, and the same applies to $\|\Delta f\|_{\mathcal{K}}$ and $\|Hf\|_{\mathcal{K}}$ (or $\|\cdot\|_{L^{3/2,1}}$, if $V \in L^{3/2,1}$).

It is also true that the positive quadratic form $\langle |H| f, f\rangle$ is equivalent to $\|f\|^2_{\dot H^1}$, though this fact will play a less prominent role in our analysis.

The main result of this paper is the following:

Theorem 1 (Reversed-norm dispersive estimates)**.** *Consider a real-valued potential $V \in \mathcal{K}_0$ on $\mathbb{R}^3$ such that $-\Delta + V$ has no eigenvalues on $[0,\infty)$ and no resonance at zero. Then*

$$\Big\| t \frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} f \Big\|_{L^\infty_x L^1_t} \lesssim \|f\|_1,$$

$$\Big\| \int_t^\infty \Big|\frac{\sin(s\sqrt{H})P_c}{\sqrt{H}} f\Big|\, ds \Big\|_{L^\infty_x L^1_t} \lesssim \|f\|_1,$$

$$\operatorname*{ess\,sup}_x \int_t^\infty \Big|\frac{\sin(s\sqrt{H})P_c}{\sqrt{H}} f\Big|\, ds \lesssim |t|^{-1}\|f\|_1$$

and also

$$\begin{aligned}\Big\|\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}}f\Big\|_{L^\infty_x L^1_t} &\lesssim \|f\|_{\mathcal{K}} \lesssim \|\nabla f\|_1,\\ \Big\|\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}}f\Big\|_{\mathcal{K}^*_x L^1_t} &\lesssim \|f\|_1.\end{aligned}$$

Assume that $f \in L^2$ *and* $\nabla f \in L^1$*. Then*

$$\begin{aligned}\|t\cos(t\sqrt{H})P_c f\|_{L^\infty_x L^1_t} &\lesssim \|\nabla f\|_1\\ \|\cos(t\sqrt{H})P_c f\|_{\mathcal{K}^*_x L^1_t} &\lesssim \|\nabla f\|_1\\ \Big\|\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}}f\Big\|_{\mathcal{K}^*_x L^\infty_t} &\lesssim \|\nabla f\|_1\\ \Big\|\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}}f\Big\|_{L^\infty_x} &\lesssim |t|^{-1}\|\nabla f\|_1.\end{aligned}$$

Assume that $f \in L^2$ *and* $\Delta f \in L^1$ *or* $\nabla f \in \mathcal{K}$*. Then*

$$\begin{aligned}\|\cos(t\sqrt{H})P_c f\|_{L^\infty_x L^1_t} &\lesssim \min(\|\Delta f\|_1, \|\nabla f\|_{\mathcal{K}}),\\ \|\cos(t\sqrt{H})P_c f\|_{L^\infty_x} &\lesssim |t|^{-1}\|\Delta f\|_1,\\ \Big\|\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}}f\Big\|_{L^\infty_t L^\infty_x} &\lesssim \min(\|\Delta f\|_1, \|\nabla f\|_{\mathcal{K}}).\end{aligned}$$

Furthermore,

$$\begin{aligned}\|\cos(t\sqrt{H})P_c f\|_{\mathcal{K}^*_x L^\infty_t} &\lesssim \|\Delta f\|_1\\ \|\cos(t\sqrt{H})P_c f\|_{L^\infty_{x,t}} &\lesssim \|\Delta f\|_{\mathcal{K}}.\end{aligned}$$

The resulting inhomogeneous estimates are, for $1 \leq p \leq \infty$*,*

$$\begin{aligned}\Big\|\int_{t'<t}\frac{\sin((t-t')\sqrt{H})P_c}{\sqrt{H}}F(t')\,dt'\Big\|_{\mathcal{K}^*_x L^p_t} &\lesssim \|F\|_{L^1_x L^p_t},\\ \Big\|\int_{t'<t}\frac{\sin((t-t')\sqrt{H})P_c}{\sqrt{H}}F(t')\,dt'\Big\|_{L^\infty_x L^p_t} &\lesssim \|F\|_{\mathcal{K}_x L^p_t}.\end{aligned}$$

Likewise,

$$\begin{aligned}\Big\|\int_{t'<t}\frac{\cos((t-t')\sqrt{H})P_c}{H}F(t')\,dt'\Big\|_{L^\infty_x L^p_t} &\lesssim \|F\|_{L^1_x L^p_t},\\ \Big\|\int_{t'<t}\frac{\cos((t-t')\sqrt{H})P_c}{H}F(t')\,dt'\Big\|_{\mathcal{K}^*_x L^\infty_t} &\lesssim \|F\|_{L^1_{x,t}},\\ \Big\|\int_{t'<t}\frac{\cos((t-t')\sqrt{H})P_c}{H}F(t')\,dt'\Big\|_{L^\infty_{x,t}} &\lesssim \|F\|_{\mathcal{K}_x L^1_t}.\end{aligned}$$

$\mathcal{K}^*$ can be replaced by $L^{3,\infty}$ (weak-L^3) wherever it appears in the statement of Theorem 1. Also note that $W^{1,1} \subset \mathcal{K}$ by Lemma 7.

By interpolation we obtain a wider family of inequalities:

Corollary 2. *Consider a real-valued potential $V \in \mathcal{K}_0$ on $\mathbb{R}^3$ such that $-\Delta + V$ has no eigenvalues on $[0,\infty)$ and no resonance at zero. Then, for $0 \le \theta \le 1$*

$$(6)\qquad \Big\| \int_{t'<t} \frac{\sin((t-t')\sqrt{H})P_c}{\sqrt{H}} F(t')\,dt' \Big\|_{(\mathcal{K}^{1-\theta})^*_x L^p_t} \lesssim \|F\|_{\mathcal{K}^\theta_x L^p_t}$$

$$\Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H} F(t')\,dt' \Big\|_{(\mathcal{K}^{1-\theta})^*_x L^\infty_t} \lesssim \|F\|_{\mathcal{K}^\theta_x L^1_t}.$$

More generally for any $0 \le \theta_1, \theta_2 \le 1$ and $1 \le p \le q \le \infty$ with $\theta_1 + \theta_2 \le 1$ and $\frac{1}{p} - \frac{1}{q} = \theta_1 + \theta_2$,

$$(7)\qquad \Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H} F(t')\,dt' \Big\|_{(\mathcal{K}^{\theta_2})^*_x L^q_t} \lesssim \|F\|_{\mathcal{K}^{\theta_1}_x L^p_t}.$$

Additionally, for θ_1, θ_2 as above and $\frac{1}{q} = 1 - \theta_1 - \theta_2$,

$$\Big\| \frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} f \Big\|_{(\mathcal{K}^{\theta_2})^*_x L^q_t} \lesssim \|\nabla f\|_{\mathcal{K}^{\theta_1}},$$

$$\big\| \cos(t\sqrt{H})P_c f \big\|_{(\mathcal{K}^{\theta_2})^*_x L^q_t} \lesssim \|\Delta f\|_{\mathcal{K}^{\theta_1}},$$

$$\Big\| \frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} f \Big\|_{(\mathcal{K}^{\theta_2})^*_x} \lesssim |t|^{\theta_1+\theta_2-1} \|\nabla f\|_{\mathcal{K}^{\theta_1}},$$

$$(8)\qquad \big\| \cos(t\sqrt{H})P_c f \big\|_{(\mathcal{K}^{\theta_2})^*_x} \lesssim |t|^{\theta_1+\theta_2-1} \|\Delta f\|_{K^{\theta_1}}.$$

The following Lorentz space inequalities are also valid in the range $1 < p, q < \infty$, $1 \le s \le \infty$:

$$(9)\qquad \Big\| \int_{t'<t} \frac{\sin((t-t')\sqrt{H})P_c}{\sqrt{H}} F(t')\,dt' \Big\|_{L^{q,s}_x L^r_t} \le C_{pq} \|F\|_{L^{p,s}_x L^r_t}$$

for $\frac{1}{p} - \frac{1}{q} = \frac{2}{3}$ and $1 \le r \le \infty$,

$$(10)\qquad \Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H} F(t')\,dt' \Big\|_{L^{q,s}_x L^r_t} \le C_{pq} \|F\|_{L^{p,s}_x L^{\tilde r}_t},$$

for $\frac{1}{p} - \frac{1}{q} = 1 - \frac{1}{3}(\frac{1}{\tilde r} - \frac{1}{r})$, $1 \le \tilde r < r \le \infty$, and

$$(11)\qquad \Big\| \frac{\cos(t\sqrt{H})P_c}{H} f \Big\|_{L^{q,s}} \le C_{pq} |t|^{-1/r} \|f\|_{L^{p,s}}$$

for $\frac{1}{p} - \frac{1}{q} = \frac{2}{3} + \frac{1}{3r}$, $1 < r \le \infty$, $1 < p, q < \infty$.

Under the stronger assumption $V \in L^{3/2,1}(\mathbb{R}^3)$ and the same (p,q,r) as in (11), these inequalities hold as well:

$$\|\cos(t\sqrt{H})P_c f\|_{L^{q,s}_x L^r_t} \le C_{pq} \|\Delta f\|_{L^{p,s}},$$

$$\|\cos(t\sqrt{H})P_c f\|_{L^{q,s}} \le C_{pq} |t|^{-1/r} \|\Delta f\|_{L^{p,s}}.$$

The constant C_{pq} satisfies the bound $C_{pq} \lesssim \frac{(p'+q)^2}{p'q}$, where $p' := \frac{p}{p-1}$. In particular, C_{pq} is uniformly bounded over all pairs where $p' = q$.

Finally, we state our results concerning homogeneous Strichartz estimates. Several of the endpoint bounds are valid, and one of them doubles as a statement about maximal operators:

Theorem 3 (Strichartz and reversed-norm Strichartz estimates)**.** *Consider a real-valued potential $V \in \mathcal{K}_0$ on $\mathbb{R}^3$ such that $-\Delta + V$ has no eigenvalues on $[0, \infty)$ and no resonance at zero. Then for all $0 \leq s < \frac{3}{2}$ and $\frac{1}{r} + \frac{3}{q} = \frac{3}{2} - s$ in the range $0 \leq \frac{1}{r} < \frac{s}{2}$,*

$$\Big\| \frac{e^{it\sqrt{H}} P_c}{H^{s/2}} f \Big\|_{L^r_t L^q_x} \lesssim \|f\|_2. \tag{12}$$

The initial-value problem for the wave equation (1) *satisfies*

$$\Big\| \cos(t\sqrt{H}) P_c f_0 + \frac{\sin(t\sqrt{H}) P_c}{\sqrt{H}} f_1 \Big\|_{L^r_t L^q_x} \lesssim \|f_0\|_{\dot H^s} + \|f_1\|_{\dot H^{s-1}}, \tag{13}$$

for all $s \in [0, 1]$, and also for $1 < s < \frac{3}{2}$ under the additional assumption $V \in L^{3/2,1}$. The case $r = \infty$ is included for all $0 \leq s < 3/2$ and the case $q = \infty$ for $1 < s < \frac{3}{2}$.

For $\frac{1}{2} < s < \frac{3}{2}$, the following reversed-norm Strichartz inequalities are also valid: For all θ, r with $\max(0, 1-s) \leq \theta \leq \min(\frac{1}{2}, \frac{3}{2} - s)$ and $\theta + \frac{1}{r} = \frac{3}{2} - s$,

$$\Big\| \frac{e^{it\sqrt{H}} P_c}{H^{s/2}} f \Big\|_{(\mathcal{K}^\theta)^*_x L^{r,2}_t} \lesssim \|f\|_{L^2}, \tag{14}$$

which includes the endpoint case $L^\infty_x L^{\frac{2}{3-2s},2}_t$ for $1 \leq s < \frac{3}{2}$ and $\theta = 0$. Furthermore

$$\Big\| \frac{e^{it\sqrt{H}} P_c}{H^{s/2}} f \Big\|_{L^{q,2}_x L^r_t} \lesssim \|f\|_{L^2} \tag{15}$$

for all pairs (q, r) with $\max(6, \frac{6}{3-2s}) \leq q \leq \frac{3}{\max(1-s,0)}$, $q \neq \infty$, and $\frac{3}{q} + \frac{1}{r} = \frac{3}{2} - s$. This includes the endpoint $L^{\frac{6}{3-2s},2}_x L^\infty_t$ when $1 \leq s < \frac{3}{2}$.

Solutions to the homogeneous wave equation (1) *satisfy*

$$\Big\| \cos(t\sqrt{H}) P_c f_0 + \frac{\sin(t\sqrt{H}) P_c}{\sqrt{H}} f_1 \Big\|_{(\mathcal{K}^\theta)^*_x L^r_t \cap L^{q,2}_x L^r_t} \lesssim \|f_0\|_{\dot H^s} + \|f_1\|_{\dot H^{s-1}} \tag{16}$$

for $s \in [0, 1]$ and also for $1 < s < \frac{3}{2}$ under the additional assumption $V \in L^{3/2,1}$.

In several non-endpoint estimates, L^p spaces can be strengthened to $L^{p,2}$, as shown in the proof.

Remark 1. There is a Kato-class version of (13) analogous to (14). These estimates are described in terms of Schechter's spaces $M_{3-2s\theta,s+1}$ (as defined in [Sim]), which form a complex interpolation family between $\mathcal{K}^\theta$ and $L^2(\mathbb{R}^3)$.

Remark 2. Theorem 3 is presented as a consequence of Corollary 2; it is not a complete list of valid Strichartz and reversed-norm Strichartz inequalities. A self-contained proof of the "sharp-admissible" case $r = \frac{2}{s}$, $s < 1$ in (13) likely demands several extra steps (e.g. Littlewood-Paley decomposition) to make L^∞ safe for complex interpolation.

On the other hand, the main technical step (Theorem 14) makes it possible to transfer reversed-norm estimates from the Laplacian to H regardless of their initial proof. This allows us to confirm Strichartz inequalities over half of the sharp-admissible range.

Theorem 4. *Consider a real-valued potential $V \in L^{3/2,1}(\mathbb{R}^3)$ such that $-\Delta + V$ has no eigenvalues on $[0,\infty)$ and no resonances at zero. then for all $0 \leq s \leq \frac{1}{2}$,*

$$\Big\| \frac{e^{it\sqrt{H}} P_c}{H^{s/2}} \Big\|_{L_t^{2/s} L_x^{2/(1-s)}} \lesssim \|f\|_2 \qquad \text{and} \tag{17}$$

$$\Big\| \cos(t\sqrt{H}) P_c f_0 + \frac{\sin(t\sqrt{H}) P_c}{\sqrt{H}} f_1 \Big\|_{L_t^{2/s} L_x^{2/(1-s)}} \lesssim \|f_0\|_{\dot H^s} + \|f_1\|_{\dot H^{s-1}}. \tag{18}$$

The $L_x^{\frac{6}{3-2s},2} L_t^\infty$ endpoint in (16) is a bound on the maximal function for the wave propagator, with the following consequence:

Theorem 5. *Consider a real-valued potential $V \in \mathcal{K}_0$ on $\mathbb{R}^3$ such that $-\Delta + V$ has no eigenvalues on $[0,\infty)$ and no resonance at zero. Then for $f_0 \in \dot H^1$ and $f_1 \in L^2$*

$$\lim_{t \to 0} \Big(\cos(t\sqrt{H}) f_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}} f_1 \Big)(x) = f_0(x) \tag{19}$$

at almost every $x \in \mathbb{R}^3$.

More generally, such results hold for $\dot H^s$, $1 \leq s < 3/2$.

1.2. **History of the problem.** Strichartz estimates are a fundamental tool in the study of the wave equation. Other inequalities employed in its study include local energy decay, Morawetz estimates [Mor], weighted Keel–Smith–Sogge-type estimates [KSS], and Killing-field-based, i.e. Klainerman–Sobolev [Kla], estimates. These techniques are not mutually exclusive and are often used together or even in combination.

Strichartz-type estimates only hold for the dispersive part of the evolution, corresponding to the projection on the continuous spectrum. The optimal rate of decay requires the absence of zero eigenvalues and resonances, i.e. of nonzero solutions of the equation $H\Psi = 0$. Eigenvalues and resonances at zero, even when specifically excluded by means of a spectral projection, nevertheless lead to a lower rate of decay for the dispersive part of the evolution.

In the case of Schrödinger's equation in one dimension, Kato-type smoothing estimates hold in a similar norm:

$$\| |\nabla|^{1/2} e^{it\Delta} f \|_{L_x^\infty L_t^2} \lesssim \|f\|_2.$$

Such estimates have also been obtained previously by Kenig–Ponce–Vega [KPV1], [KPV2] in one dimension for the Korteweg-deVries and Benjamin–Ono equations. Reversed-norm estimates are new in higher dimensions for the wave equation; see [LiZh] for a related result for Schrödinger's equation with radial data.

Strichartz estimates for the wave equation have a long history, going back to the works of Strichartz [Str] and Segal [Seg] in $\mathbb{R}^2$. Previously-known Strichartz inequalities for the free wave equation (without potential) belong to Georgiev–Karadzhov–Visciglia [GKV], Harmse [Har], Kapitanski [Kap], Ginebre–Velo [GiVe1] [GiVe2], Oberlin [Obe], Lindblad–Sogge [LiSo], and Nakamura–Ozawa [NaOz]. In other settings we cite the work of Mockenhaupt–Seeger–Sogge [MSS].

Keel–Tao [KeTa] also obtained sharp Strichartz estimates for the free wave equation in $\mathbb{R}^4$ and higher dimensions — and everything except the endpoint in $\mathbb{R}^3$. These estimates were further extended by Foschi [Fos], in the inhomogeneous case.

In this paper we seek to prove similar estimates for the cosine and sine evolutions $\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}}$ and $\cos(t\sqrt{H})P_c$ related to the perturbed Hamiltonian $H = -\Delta + V$.

Many such results were initially proved only for the free wave equation. However, the boundedness of wave operators shown by Yajima in [Yaj] implies that results in the free case can be extended to the case of a perturbed Hamiltonian $H = -\Delta + V$, if the potential has sufficient decay: $|V(x)| \lesssim \langle x\rangle^{-6-\epsilon}$ in $\mathbb{R}^3$. Still, Yajima's method does not apply to the results of Theorem 1, because the L^p boundedness of the wave operators does not address the issue of reversed spacetime norms.

Strichartz estimates for the wave equation in $\mathbb{R}^d$ with critical potentials were obtained by Burq–Planchon–Stalker–Tahvildar-Zadeh [BPST], [BPST2] and by D'Ancona–Pierfelice [DaPi]. The former results apply to potentials an with inverse-square pointwise bound, $|V(x)| \lesssim |x|^{-2}$, the primary examples of which lie outside $\mathcal{K}_0$. Strichartz estimates are shown to fail in this larger class if there is no control over the negative part of the potential; some additional smoothness in the radial direction (where $x = r\omega$) is also assumed. The dispersive and Strichartz estimates in [DaPi] apply to to all $V \in \mathcal{K}_0$ whose negative part satisfies $\|V^-\|_{\mathcal{K}} < 2\pi$. In this paper the size bound on V^- is replaced with the weaker hypothesis that the operator $-\Delta + V$ has no eigenvalues or resonances along its essential spectrum.

The current paper's results are also comparable to those of Rogers–Villarroya. In [RoVi], these authors studied maximal operators for the wave equation in $\mathbb{R}^d$, defined by

$$M_1 f(x) := \sup_{t\in[0,1]} |(e^{it\sqrt{\Delta}} f)(x)| \text{ and } Mf(x) := \sup_{t\in\mathbb{R}} |(e^{it\sqrt{-\Delta}} f)(x)|.$$

In particular they showed estimates of the form

$$\|Mf\|_{L^q} \lesssim \|f\|_{H^s},$$

where $q \geq \frac{2(d+1)}{d-1}$ and $s > d(1/2-1/q)$. The estimate (15) implies that for $1 \leq s < \frac{3}{2}$

$$\|\sup_t |(e^{it\sqrt{H}} P_c f)(x)|\|_{L_x^{\frac{6}{3-2s},2}} \lesssim \|f\|_{\dot{H}^s_x}.$$

This corresponds to an endpoint not treated in [RoVi]. We also prove this inequality for the general case of $H = -\Delta + V$, in addition to the free wave equation.

Some of the results obtained here for $\mathcal{K}_0$ potentials can be extended to more general Kato-class potentials, including singular measures; see, in this direction, the results of [Gol].

Many papers are dedicated to Strichartz estimates for Schrödinger's equation, whose proof is largely similar to that of Strichartz estimates for the wave equation. Indeed, the key step of using an operator-valued Wiener L^1-inversion theorem was originally applied to Schrödinger's equation by the authors in [Bec], [BeGo], and [Gol]. The abstract Wiener theorem presented in Section 3 is borrowed largely intact from these works, however the wave equation encourages it to appear in several new disguises including a weighted-L^1 version (Proposition 15) not previously considered.

1.3. **Nonlinear applications.**

Proposition 6. *For $V \in L^{3/2,1}$, consider the energy-critical equation*

$$\partial_t^2 f - \Delta f + Vf = F \pm f^5, (f(0), \partial_t f(0)) = (f_0, f_1).$$

Assume that $H = -\Delta + V$ has neither eigenvalues, nor resonances. Then for sufficiently small $(f_0, f_1) \in \dot H^1 \times L^2$ and $F \in L^{6/5,2}_x L^\infty_t$, there exists a unique solution $f \in L^{6,2}_x L^\infty_t$.

When $(f_0, f_1) \in \dot H^1 \times L^2$ and $F \in L^{6/5,2}_x L^\infty_t \cap L^{3/2,1}_x L^2_t$, the solution f is in $L^{6,2}_x L^\infty_t \cap L^\infty_x L^2_t$.

By the conservation of energy, in both cases $(f, \partial_t f) \in L^\infty_t \dot H^1_x \times L^\infty_t L^2_x$.

Similar results hold for $\dot H^s$-critical wave equations with $1 \leq s < 3/2$.

Proof. By Strichartz estimates (Theorem 3) we obtain
$$\|\cos(t\sqrt{H})P_c f\|_{L^\infty_x L^2_t \cap L^{6,2}_x L^\infty_t} \lesssim \|f\|_{\dot H^1}.$$
Likewise,
$$\Big\|\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} f(x)\Big\|_{L^\infty_x L^2_t \cap L^{6,2}_x L^\infty_t} \lesssim \|f\|_2$$
and by Theorem 1 and Corollary 2
$$\Big\|\int_{s<t} \frac{\sin((t-s)\sqrt{H})P_c}{\sqrt{H}} F(x,s)\,ds\Big\|_{L^\infty_x L^2_t} \lesssim \|F\|_{\mathcal{K}_x L^2_t} \lesssim \|F\|_{L^{3/2,1}_x L^2_t},$$
$$\Big\|\int_{s<t} \frac{\sin((t-s)\sqrt{H})P_c}{\sqrt{H}} F(x,s)\,ds\Big\|_{L^{6,2}_x L^\infty_t} \lesssim \|F\|_{L^{6/5,2}_x L^\infty_t}.$$
We retrieve the solution by means of a fixed-point scheme, based on the Strichartz inequalities above. Note that
$$\|f^5\|_{L^{6/5,2}_x L^\infty_t} \leq \|f\|^5_{L^{6,2}_x L^\infty_t}$$
and
$$\|f^5\|_{L^{3/2,1}_x L^2_t} \leq \|f\|^4_{L^{6,2}_x L^\infty_t} \|f\|_{L^\infty_x L^2_t}.$$
□

2. Definitions and basic estimates

The global Kato norm, which defines our admissible class of potentials $V(x)$, is highly compatible with Sobolev spaces based on $L^1(\mathbb{R}^3)$ or $L^\infty(\mathbb{R}^3)$. Let $\dot W^{1,1}$ indicate the completion of compactly supported test functions under the norm $\|f\|_{\dot W^{1,1}} := \|\nabla f\|_1$.

Lemma 7. *If $f \in \dot W^{1,1}$ then $f \in \mathcal{K}_0$ and $\|f\|_{\mathcal{K}} \leq \|f\|_{\dot W^{1,1}}$.*

Proof. Suppose $f \in C^1_c(\mathbb{R}^3)$. Fix any point y, and then write the integral for the Kato norm in polar coordinates $x = y + r\omega$.
$$\int_{\mathbb{R}^3} \frac{|f(x)|}{|x-y|}\,dx = \int_0^\infty \int_{S^2} |f(y+r\omega)|\,d\omega\, r\,dr =$$
$$= -\int_0^\infty \int_{S^2} \partial_r |f(y+r\omega)|\,d\omega \frac{r^2}{2}\,dr \lesssim \|f\|_{\dot W^{1,1}}.$$
since $\big|\partial_r |f(y+r\omega)|\big| = |\partial_r f(y+r\omega)|$ almost everywhere. Take the supremum over $y \in \mathbb{R}^3$ to obtain the same bound for $\|f\|_{\mathcal{K}}$, and limits to extend the result to all $f \in \dot W^{1,1}$. □

The next lemma shows that functions with two derivatives in $\mathcal{K}$ and some decay at infinity are in fact bounded.

Lemma 8. *Assume that $D^2 f \in \mathcal{K}$ and f can be approximated by smooth functions of compact support. Then $\|f\|_{L^\infty} \lesssim \|D^2 f\|_{\mathcal{K}}$.*

Proof. Note that for bounded functions f of compact support, for every x

$$\lim_{r\to\infty} \int_{S^2} f(x + r\omega)\, d\omega = 0.$$

Then

$$\begin{aligned} f(x) &= -\frac{1}{4\pi} \int_{S^2} \int_0^\infty \frac{\partial f}{\partial_r}(x + r\omega)\, dr\, d\omega \\ &= \frac{1}{4\pi} \int_{S^2} \int_0^\infty \int_r^\infty \frac{\partial^2 f}{\partial_r^2}(x + s\omega)\, ds\, dr\, d\omega \\ &= \frac{1}{4\pi} \int_{S^2} \int_0^\infty s \frac{\partial^2 f}{\partial_r^2}(x + s\omega)\, ds\, d\omega \\ &\lesssim \int_{\mathbb{R}^3} \frac{|D^2 f(y)|}{|x-y|}\, dy \lesssim \|D^2 f\|_{\mathcal{K}}. \end{aligned}$$

By approximation we infer the result in the general case. □

Finally, we show the following inclusion between Kato-type spaces:

Lemma 9. *If f is approximable by bounded compactly supported functions*

$$\sup_x \int_{\mathbb{R}} \frac{|f(y)|\, dy}{|x-y|^2} \lesssim \|\nabla f\|_{\mathcal{K}}.$$

Proof. Take f bounded and compactly supported. Then for every x

$$\lim_{r\to\infty} \int_{S^2} |f(x + r\omega)| r\, d\omega = 0.$$

Consequently

$$\int_{\mathbb{R}} \int_{S^2} |f(x + r\omega)|\, d\omega\, dr = -\int_{\mathbb{R}} \int_{S^2} \partial_r |f(x + r\omega)| r\, d\omega\, dr \lesssim \|\nabla f\|_{\mathcal{K}}$$

because $|\partial_r |f(x + r\omega)|| = |\partial_r f(x + r\omega)|$ almost everywhere.

The conclusion then follows by approximation. □

A straightforward approximation argument shows that every potential $V \in \mathcal{K}_0$ satisfies the local Kato condition

$$\lim_{\epsilon\to 0} \sup_{y\in\mathbb{R}^3} \int_{|x-y|<\epsilon} \frac{|V(x)|}{|x-y|}\, dx = 0 \tag{20}$$

and the distal Kato condition

$$\lim_{R\to\infty} \sup_{y\in\mathbb{R}^3} \int_{|x-y|>R} \frac{|V(x)|}{|x-y|}\, dx = 0. \tag{21}$$

These properties imply that $H = -\Delta + V$ is essentially self-adjoint with spectrum bounded below by $-M$ for some $M < \infty$ [Sim]. With the assumption that zero is neither an eigenvalue nor a resonance, there are at most finitely many negative eigenvalues λ_j with corresponding orthogonal spectral projections P_j. We denote by $P_c := I - \sum_j P_j$ the projection on the absolutely continuous spectrum.

For each $z \in \mathbb{C} \setminus \mathbb{R}^+$ define $R_0(z) := (-\Delta - z)^{-1}$. The operators $R_0(z)$ are all bounded on $L^2(\mathbb{R}^3)$ and act explicitly by convolution with the kernel

$$R_0(z)(x, y) = \frac{e^{i\sqrt{z}|x|}}{4\pi|x|},$$

where $\sqrt{z}$ is taken to have positive imaginary part. On the boundary $\lambda \in \mathbb{R}^+$ the resolvents $R_0(\lambda \pm i0)$ are defined as $\lim_{\varepsilon\downarrow 0} R_0(\lambda \pm i\varepsilon)$. These operators are not L^2-bounded but instead satisfy a uniform mapping estimate from $L^{6/5,2}$ to $L^{6,2}$.

The perturbed resolvent $R_V(z) := (H - z)^{-1}$ does not have a closed-form formula for its integral kernel, however it can be expressed in terms of $R_0(z)$ via the identity

$$R_V(z) = (I + R_0(z)V)^{-1}R_0(z) = R_0(z)(I + VR_0(z))^{-1}. \tag{22}$$

Boundary values $R_V^{\pm}(\lambda)$ along the positive real axis are once again defined by continuation, that is $R_V^{\pm}(\lambda) = \lim_{\varepsilon\downarrow 0} R_V(\lambda \pm i\varepsilon)$. The two continuations do not coincide; the difference between them is (up to a constant factor) the absolutely continuous spectral measure of H.

Returning to the formal solution (2) of the wave equation with a potential, we wish to analyze the component of the solution orthogonal to eigenfunctions of H, that is

$$P_c f(t) = \cos(t\sqrt{H})P_c f_0 + \frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} g_0 + \int_0^t \frac{\sin((t-s)\sqrt{H})P_c}{\sqrt{H}} F(s)\, ds,$$

where $\cos(t\sqrt{H})P_c$ and $\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}}$ are defined by means of operator calculus:

$$\cos(t\sqrt{H})P_c f := \frac{1}{2\pi i}\int_0^\infty \cos(t\sqrt{\lambda})[R_V^+(\lambda) - R_V^-(\lambda)]f\, d\lambda,$$
$$\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} f := \frac{1}{2\pi i}\int_0^\infty \frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}[R_V^+(\lambda) - R_V^-(\lambda)]f\, d\lambda.$$

This definition makes $\cos(t\sqrt{H})P_c$ and $\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}}$ L^2_x-bounded for every t. So long as zero is a regular point of the spectrum, it is possible to effect a change of variables $\lambda \mapsto \lambda^2$ leading to the identities

$$\cos(t\sqrt{H})P_c f = \frac{1}{\pi i}\int_{-\infty}^\infty \lambda \cos(t\lambda) R_V^+(\lambda^2) f\, d\lambda \tag{23}$$

$$\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} f = \frac{1}{\pi i}\int_{-\infty}^\infty \sin(t\lambda) R_V^+(\lambda^2) f\, d\lambda. \tag{24}$$

To be precise, $R_V^+(\lambda^2)$ denotes $\lim_{\varepsilon\downarrow 0} R_V((\lambda + i\varepsilon)^2)$, which coincides with $R_V^{\pm}(\lambda^2)$ according to the sign of λ. Presented this way, we see that the linear propagators for the perturbed wave equation are closely tied to the Fourier transform (in λ) of the resolvent $R_V^+(\lambda^2) = R_0^+(\lambda^2)(I + VR_0^+(\lambda^2))^{-1}$. Indeed, the principal effort behind Theorem 1 is to establish useful bounds on the Fourier transform of the perturbative factor $(I + VR_0^+(\lambda^2))^{-1}$. We obtain these bounds as an application of the abstract Wiener inversion theorem in Section 3.

There is no material change to these estimates if one considers the resolvent continuation from below (i.e. $R_0^-(\lambda^2)$) or applies inverse Fourier transforms instead, as has been done in previous studies. Keeping the notation in [BeGo], let

$$\hat{T}^{\pm}(\lambda, x, y) := VR_0^{\pm}(\lambda^2)(x, y) = V(x)\frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}. \tag{25}$$

Its inverse Fourier transform in the λ variable has the explicit form

$$\begin{aligned} T^{\pm}(\rho, x, y) &= (4\pi|x-y|)^{-1}V(x)\delta_{\mp|x-y|}(\rho) \\ &= (\mp 4\pi\rho)^{-1}V(x)\delta_{\mp|x-y|}(\rho), \end{aligned} \tag{26}$$

which is the restriction of $V(x)/(4\pi|x-y|)$ to the light cone $|x-y| = \pm\rho$.

We will return to these operators in Section 4 with the goal of understanding properties of $(I + \hat{T}^{\pm}(\lambda))^{-1}$. In Section 3, where the techniques for constructing an operator inverse are developed, the family of operators $\hat{T}(\lambda)$ will be of a more general character.

3. An Abstract Wiener Theorem

Let X be a Banach lattice of complex-valued functions over a measure space $(\mathcal{X}, \mu)$. The main lattice properties are that $f \mapsto |f|$ is an isometry in X, and if $f \in X$ and $|g| \leq f$, then $g \in X$ with lesser norm. Relevant examples include $L^p(\mu)$, $1 \leq p \leq \infty$, $\mathcal{K}$ and $\mathcal{K}^*$ as defined in Section 1, and their interpolants. The space of finite complex-valued measures on $\mathbb{R}$, which we denote by $\mathbb{M}$, is also a Banach lattice.

The compound space $X_x\mathbb{M}_\rho$ then consists of measures ν on $\mathbb{R} \times \mathcal{X}$ for which $M(x) = \|\nu(\,\cdot\,, x)\|_{\mathbb{M}}$ is finite μ-almost everywhere and belongs to X.

Definition 2. Let $\mathcal{U}_X$ be the set of bounded operators from $X_x\mathbb{M}_\rho$ to itself defined formally by the integral

$$(TF)(\rho, x) := \int_{-\infty}^{\infty}\int_{\mathcal{X}} T(\rho - \sigma, x, y)F(\sigma, y)\,\mu(dy)\,d\sigma, \tag{27}$$

and possessing the property that $|T(\rho - \sigma, x, y)|$ also produces a bounded integral operator on $X_x\mathbb{M}_\rho$. In the general case, $T(\rho, x, y)$ may be a measure on $\mathbb{R} \times \mathcal{X} \times \mathcal{X}$ and its associated linear operator is defined weakly by pairing with a second function in $X_x^*B(\mathbb{R})_\rho$.

Setting the norm $\|T\|_{\mathcal{U}_X} := \||T|\|_{\mathcal{B}(X_x\mathbb{M}_\rho)}$ gives $\mathcal{U}_X$ the structure of a Banach lattice. In addition, $\mathcal{U}_X$ is a Banach algebra under the natural composition of operators on $X_x\mathbb{M}_\rho$.

Given an element $T \in \mathcal{U}_X$, let $M(T)$ be the marginal distribution of $|T|$ on $\mathcal{X} \times \mathcal{X}$, which can be written formally as

$$M(T)(x, y) = \int_{-\infty}^{\infty} |T(\rho, x, y)|\,d\rho. \tag{28}$$

It follows from the construction that $\|T\|_{\mathcal{B}(X_x\mathbb{M}_\rho)} \leq \|M(T)\|_{\mathcal{B}(X)} = \|T\|_{\mathcal{U}_X}$.

Kernels $T \in \mathcal{U}_X$ also define bounded operators from X to $X_x\mathbb{M}_\rho$ by

$$(Tf)(\rho, x) = \int_{-\infty}^{\infty}\int_{\mathcal{X}} T(\rho, x, y)f(y)\,\mu(dy). \tag{29}$$

For many of the statements in Theorem 1, it will be significant that $T \in \mathcal{U}_X$ acts as a bounded operator on $X_xL^p_\rho$ as well for any $1 \leq p \leq \infty$. This is a consequence of (27) and (28) and the fact that convolution with a finite measure preserves $L^p(\mathbb{R})$.

Finally, each element $T \in \mathcal{U}_X$ has an adjoint operator $T^*(\rho, x, y) = \overline{T(-\rho, y, x)}$ which belongs to $\mathcal{U}_{X^*}$ because $M(T^*) = M(T)^*$.

We take the Fourier transform of $T \in \mathcal{U}_X$, $\hat{T}(\lambda) \in \mathcal{B}(X)$, to be the operator with kernel

$$\hat{T}(\lambda, x, y) = \int_{-\infty}^{\infty} e^{-i\rho\lambda} T(\rho, x, y)\, d\rho, \tag{30}$$

if $T(\rho, x, y)$ is a function, or more generally the marginal distribution of $e^{-i\rho\lambda}T$. It satisfies the standard product relation $(ST)^\wedge(\lambda) = \hat{S}(\lambda)\hat{T}(\lambda)$. For each $\lambda \in \mathbb{R}$, $\hat{T}(\lambda)$ is dominated pointwise by $M(T)$, hence $\|\hat{T}(\lambda)\|_{\mathcal{B}(X)} \leq \|T\|_{\mathcal{U}_X}$.

There is an identity element $\mathbf{1} \in \mathcal{U}_X$ whose integral kernel $\mathbf{1}(\rho, x, y)$ is the lifting of $\delta_x(y)$ onto the "centered" diagonal $\{\rho = 0\} \times \{x = y\} \subset \mathbb{R} \times \mathcal{X} \times \mathcal{X}$ via the natural identification with points in X. One can verify that $M(\mathbf{1})$ is the identity operator on X, and that the Fourier transform of $\mathbf{1}$ is $\hat{\mathbf{1}}(\lambda) = I_X$ for each $\lambda \in \mathbb{R}$.

Proposition 10. *Suppose $T \in \mathcal{U}_X$ is such that*

C1 *For some $N > 0$, $\lim_{\delta \to 0} \|T^N(\rho, x, y) - T^N(\rho - \delta, x, y)\|_{\mathcal{U}_X} = 0$.*

C2 $\lim_{R \to \infty} \|\chi_{|\rho| \geq R} T\|_{\mathcal{U}_X} = 0$.

If $I + \hat{T}(\lambda)$ is an invertible element of $\mathcal{B}(X)$ for every $\lambda \in \mathbb{R}$, then $\mathbf{1} + T$ is invertible in $\mathcal{U}_X$.

The argument used in [BeGo], used for a slightly different algebra based on $L^1_\rho X_x$ instead of $X_x \mathbb{M}_\rho$, can be repeated without substantial modification. We reproduce it below for the reader's convenience.

Proof. It suffices to show that $(I + \hat{T}(\lambda))^{-1}$ is the Fourier transform of an element $S \in \mathcal{U}_X$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a standard cutoff function. For any $L \in \mathbb{R}$, the large-λ restriction $(1 - \eta(\lambda/L))\hat{T}(\lambda)$ is the Fourier transform of

$$S_L(\rho) = \big(T - L\check{\eta}(L\,\cdot\,) * T\big)(\rho) = \int_{\mathbb{R}} L\check{\eta}(L\sigma)[T(\rho) - T(\rho - \sigma)]\, d\sigma.$$

Dependence on (x, y) in the kernels $T(\rho, x, y)$ is suppressed in the expression above. If condition C1 is satisfied with $N = 1$, then the $\mathcal{U}_X$ norm of the right-hand side vanishes as $L \to \infty$. Thus there is a fixed number L so that $\sum_{k=0}^{\infty}(-S_L)^k$ is a convergent series in $\mathcal{U}_X$. Its Fourier transform is

$$\sum_{k=0}^{\infty}(-1)^k\Big(\big(1 - \eta(\lambda/L)\big)\hat{T}(\lambda)\Big)^k = \Big(I + \big(1 - \eta(\lambda/L)\big)\hat{T}(\lambda)\Big)^{-1}$$

which agrees with $(I + \hat{T}(\lambda))^{-1}$ for all $\lambda > 2L$. Apply the decomposition

$$\begin{aligned}(I + \hat{T}(\lambda))^{-1} &= \eta(\lambda/2L)(I + \hat{T}(\lambda))^{-1} + \big(1 - \eta(\lambda/2L)\big)(I + \hat{T}(\lambda))^{-1} \\ &= \eta(\lambda/2L)(I + \hat{T}(\lambda))^{-1} + (1 - \eta(\lambda/2L))\big(I + (1 - \eta(\lambda/L))\hat{T}(\lambda)\big)^{-1}\end{aligned}$$

If instead T^N satisfies C1 then one observes that

$$(1 - \eta(\lambda/2L))\big(I + \hat{T}(\lambda)\big)^{-1} = (1 - \eta(\lambda/2L))\big(I - (-\hat{T}(\lambda))^N\big)^{-1}\sum_{k=0}^{N-1}(-1)^k\hat{T}^k(\lambda).$$

We then construct a local inverse for $I + \hat{T}(\lambda)$ in the neighborhood of any $\lambda_0 \in \mathbb{R}$. For simplicity, consider the representative case $\lambda_0 = 0$, and let $A_0 = I + \hat{T}(0)$, which is bounded pointwise by $I + M(T)$ as an integral operator in $\mathcal{B}(X)$. One can write $\eta(\lambda/\epsilon)(I + \hat{T}(\lambda) - A_0)$ as the Fourier transform of

$$\begin{aligned} S_\epsilon(\rho) &= \epsilon\check{\eta}(\epsilon\,\cdot\,) * T(\rho) - \epsilon\check{\eta}(\epsilon\rho)(I - A_0) \\ &= \int_{\mathbb{R}} \epsilon\big(\check{\eta}(\epsilon(\rho-\sigma)) - \check{\eta}(\epsilon\rho)\big)T(\sigma)\,d\sigma. \end{aligned}$$

Here we are again suppressing the (x, y) dependence to highlight the convolution in the ρ variable. The second equality uses the fact that $I - A_0(x, y) = -\hat{T}(0, x, y) = -\int_{\mathbb{R}} T(\rho, x, y)\,d\rho$.

By the mean value theorem, $\int_{\mathbb{R}} \epsilon|\check{\eta}(\epsilon(\rho-\sigma)) - \check{\eta}(\epsilon\rho)|\,d\rho \lesssim \min(\epsilon|\sigma|, 1)$. As a consequence, for any $R > 0$

$$M(S_\varepsilon) \lesssim \varepsilon R M(T) + M(\chi_{|\rho|>R}T),$$

which immediately implies that $\|S_\varepsilon\|_{\mathcal{U}_X} \lesssim \varepsilon R\|T\|_{\mathcal{U}_X} + \|\chi_{|\rho|>R}T\|_{\mathcal{U}_X}$. By Assumption C2 there is a choice of R and ε to make this norm arbitrarily small.

For any smooth function ϕ supported in $[-\frac{\epsilon}{2}, \frac{\epsilon}{2}]$, there exists a series

$$\begin{aligned} \phi(\lambda)(I + \hat{T}(\lambda))^{-1} &= \phi(\lambda)\big(A_0 + \eta(\lambda/\epsilon)(1 + \hat{T}(\lambda) - A_0)\big)^{-1} \\ &= \phi(\lambda)A_0^{-1}\big(I + \hat{S}_\epsilon(\lambda)A_0^{-1}\big)^{-1} = \phi(\lambda)A_0^{-1}\sum_{k=0}^{\infty}(-1)^k\big(\hat{S}_\epsilon(\lambda)A_0^{-1}\big)^k. \end{aligned}$$

When ϵ is chosen sufficiently small, the inverse Fourier transform of this series converges in $\mathcal{U}_X$.

For λ in the compact interval $[-2L, 2L]$, there is a nonzero lower bound on the length ε required for convergence of the power series.

Choose a finite covering of the compact set $[-2L, 2L]$ and a subordinated partition of the unity $(\phi_j)_j$ with $\sum_j \phi_j = \eta(\lambda/2L)$, such that for each j the local inverse $\phi_j\big((\lambda - \lambda_j)/\epsilon\big)\big((I + \widehat{T}(\lambda)\big)^{-1} \in \mathcal{U}_X$ is given by an explicit series as above. Thus $\eta(\lambda/2L)\big((I + \widehat{T}(\lambda)\big)^{-1}$ is the sum of finitely many elements of $\mathcal{U}_X$.

□

Remark 3. Condition C2 implies that $\hat{T}(\lambda)$ is a norm-continuous function from $\mathbb{R}$ to $\mathcal{B}(X)$. Condition C1 implies that $\lim_{|\lambda|\to\infty} \|\hat{T}^N(\lambda)\|_{\mathcal{B}(X)} = 0$.

4. Proof of the main result

Recall that $\hat{T}^{\pm}(\lambda) = VR_0^{\pm}(\lambda^2)$ as defined in (25), hence $T^{\pm}(\rho, x, y)$ is represented by the distribution kernel $V(x)/(4\pi|x-y|)$ supported on the surface $\{|x-y| \pm \rho = 0\}$. The proof of Theorem 1 is based on applying Proposition 10 to $T^{\pm}$ in the space $\mathcal{U}_{L^1(\mathbb{R}^3)}$. With the help of duality and algebraic relations, this result will extend to $\mathcal{U}_{\mathcal{K}}$ and the family of interpolation spaces spanning them.

The pointwise invertibility of $I + \hat{T}^{\pm}(\lambda)$ in $\mathcal{B}(L^1)$ at each λ follows by Fredholm's alternative from the absence of resonances or eigenvalues, once we show that $\hat{T}^{\pm}(\lambda)$ is a compact operator in $\mathcal{B}(L^1)$.

Lemma 11. *$\hat{T}^{\pm}(\lambda)$ is a compact operator in $\mathcal{B}(L^1)$ for all $\lambda \in \mathbb{R}$.*

Proof. By an approximation argument (since $V \in \mathcal{K}_0$), it suffices to treat the case when V is smooth and compactly supported. Then all functions $\hat{T}^{\pm}(\lambda)f =$

$VR_0^{\pm}(\lambda^2)f$ are supported within $\operatorname{supp} V$. In addition,

$$(1-\Delta)VR_0^{\pm}(\lambda^2)f = Vf + (1+\lambda^2)VR_0^{\pm}(\lambda^2)f - 2\nabla V\cdot\nabla R_0^{\pm}(\lambda^2)f - (\Delta V)R_0^{\pm}(\lambda^2)f.$$

Under the assumption that $V \in C_c^{\infty}(\mathbb{R}^3)$ each of the above terms belongs to $L^1(\mathbb{R}^3)$ with a norm bound of $C(1+\lambda^2)\|f\|_1$.

For λ fixed, $\hat{T}^{\pm}(\lambda)$ maps L^1 to $(1-\Delta)^{-1}L^1$ with fixed support inside $\operatorname{supp} V$. Hence it is a compact operator on L^1 itself. □

Now the only obstacle to invertibility of $I + \hat{T}^{\pm}(\lambda)$ is the presence of a nonzero solution to $\phi + VR_0^{\pm}(\lambda)\phi = 0$, $\phi \in L^1$. At the same time, $R_0^{\pm}(\lambda)\phi$ would be a distributional solution to $(-\Delta + V - \lambda)f = 0$. We show below that such a scenario is not possible under the spectral conditions of Theorem 1.

Lemma 12. *Assume that $V \in \mathcal{K}_0$, and suppose that for some $\lambda \in [0,\infty)$ there is a nonzero $\phi \in L^1$ satisfying $\phi + VR_0^{\pm}(\lambda)\phi = 0$. Then in fact $\phi \in L^1 \cap \mathcal{K}$, and $R_0^{\pm}(\lambda)\phi \in \mathcal{K}^* \cap L^{\infty}$ belongs to $\langle x\rangle^{\sigma}L^2$ for each $\sigma > \frac{1}{2}$.*

When $\lambda > 0$ the stronger conclusion $R_0^{\pm}(\lambda)\phi \in L^2$ is also valid.

Consequently, under the conditions of Theorem 1 the operators $I + R_0^{\pm}(\lambda)V$ and $I + VR_0^{\pm}(\lambda)$ are invertible for all $\lambda \in [0,\infty)$ and

$$(I + R_0^{\pm}(\lambda)V)^{-1} \in \mathcal{B}(L^{\infty}) \cap \mathcal{B}(\mathcal{K}^*),\ \ (I + VR_0^{\pm}(\lambda))^{-1} \in \mathcal{B}(L^1) \cap \mathcal{B}(\mathcal{K}).$$

If $V \in L^{3/2,1}$, then $\mathcal{K}$ and $\mathcal{K}^*$ may be replaced respectively by $L^{3/2,1}$ and $L^{3,\infty}$ in the above conclusion.

Proof. Choose an approximation $V_\varepsilon \in C_c^b(\mathbb{R}^3)$ so that $\|V - V_\varepsilon\|_{\mathcal{K}} < 4\pi$. Then $V_\varepsilon R_0^{\pm}(\lambda)\phi$ belongs to both L^1 and $L^{3,\infty}$, which also includes $\mathcal{K}$. Thanks to the identity

$$\phi + (V - V_\varepsilon)R_0^{\pm}(\lambda)\phi = -V_\varepsilon R_0^{\pm}(\lambda)\phi$$

and the smallness of $V - V_\varepsilon$, we can rewrite

$$\phi = -(I + (V - V_\varepsilon)R_0^{\pm}(\lambda))^{-1}V_\varepsilon R_0^{\pm}(\lambda)\phi$$

as an element of $L^1 \cap \mathcal{K}$. Dominating the resolvent kernel by $(4\pi|x-y|)^{-1}$ leads to the conclusion $R_0^{\pm}(\lambda)\phi \in \mathcal{K}^* \cap L^{\infty} \cap \langle x\rangle^{\sigma}L^2$ for each $\sigma > \frac{1}{2}$.

When $\lambda > 0$, one can use the fact that

$$\operatorname{Im}(\langle R_0^{\pm}(\lambda)\phi, \phi\rangle) = -\operatorname{Im}(\langle R_0^{\pm}(\lambda)\phi, VR_0^{\pm}(\lambda)\phi\rangle) = 0$$

to conclude that the Fourier transform of ϕ vanishes on the sphere of radius $\sqrt{\lambda}$ in frequency space. Then Corollary 4.2 of [GoSc] asserts that $R_0^{\pm}(\lambda)\phi \in L^2$.

The spectral assumptions in Theorem 1 rule out solutions of this kind. Direct appliction of the Fredholm alternative then shows that $(I + VR_0^{\pm}(\lambda))^{-1}$ exists in $\mathcal{B}(L^1)$, and by duality $(I + R_0^{\pm}(\lambda)V)^{-1} \in \mathcal{B}(L^{\infty})$. The identity

$$(I + VR_0^{\pm}(\lambda))^{-1} = I - V(I + VR_0^{\pm}(\lambda))^{-1}R_0^{\pm}(\lambda)$$

defines an inverse for $I + VR_0^{\pm}(\lambda)$ in $\mathcal{B}(\mathcal{K})$ (also in $\mathcal{B}(L^{3/2,1})$ if $V \in L^{3/2,1}$) and the dual statement defines an inverse for $I + R_0^{\pm}(\lambda)V$ in $\mathcal{B}(\mathcal{K}^*)$. □

The case $\lambda = 0$ generates several direct equivalences between $-\Delta$ and H.

Lemma 13. *Let $V \in \mathcal{K}_0$, and suppose that $H = -\Delta + V$ does not have an eigenvalue or resonance at zero. Then $H\Delta^{-1}$ acts as an isomorphism on both $L^1(\mathbb{R}^3)$ and $\mathcal{K}$ (and also on $L^{3/2,1}$ if $V \in L^{3/2,1}$).*

Moreover, H is an invertible linear map from $\dot H^1(\mathbb{R}^3)$ to its dual, and the positive quadratic form $\langle |H|^s f, f\rangle$ is equivalent to $\|f\|^2_{\dot H^s} = \langle (-\Delta)^s f, f\rangle$ for $|s| \leq 1$.

Finally, assuming that $V \in L^{3/2,1}$, then $\||H|^{s/2} f\|_{L^2} \sim \|f\|_{\dot H^s}$ whenever $|s| < 3/2$.

Proof. The statements about $H\Delta^{-1} = -(I + VR_0(0))$ acting on L^1 and on $\mathcal{K}$ are a restatement of Lemma 12 with $\lambda = 0$. It is well known that $V \in \mathcal{K}$ is form-bounded with respect to the Laplacian (see for example [Sim], p. 459), hence $\Delta^{-1}H$ is a bounded map on $\dot H^1(\mathbb{R}^3)$. Its invertibility (assuming zero is a regular point of the spectrum of H) follows from a similar compactness and Fredholm alternative argument. We refer to [Gol] for the details in the general case where V is a locally finite measure.

Note that $H - |H|$ is a finite linear combination of projections onto the point spectrum of H, and the same is true of $H^{-1} - |H|^{-1}$ provided zero is not an eigenvalue or resonance. Each eigenfunction is exponentially decaying and belongs to $\dot H^1(\mathbb{R}^3) \cap \dot H^{-1}(\mathbb{R}^3)$, so the projections are bounded on any $H^s, |s| \leq 1$. Then $|H|$ is also an isomorphism between $\dot H^1(\mathbb{R}^3)$ and its dual space. Its Hermitian square root $\sqrt{|H|} : \dot H^1 \to L^2$ is another isomorphism, meaning $\langle |H| f, f\rangle = \|\sqrt{|H|} f\|_2^2 \sim \|f\|^2_{\dot H^1}$.

Since the quadratic forms $\langle |H| f, f\rangle$ and $\langle -\Delta f, f\rangle$ are equivalent — each is form-bounded by a multiple of the other — the same holds for $\langle |H|^s f, f\rangle$ and $\langle (-\Delta)^s f, f\rangle$ for $0 \leq s \leq 1$. Thus $\||H|^{s/2} f\|_{L^2} \sim \|f\|_{\dot H^s}$ for $0 \leq s \leq 1$ and then by duality the same is true when $-1 \leq s \leq 0$.

Next, assume that $V \in L^{3/2,1}$. Since the eigenstates of H are in $\dot H^1$, by bootstrapping in the eigenstate equation $f = R_0(E)Vf$, $E < 0$, we first obtain that $f \in L^\infty$, then that $(-\Delta + 1)f \in L^{3/2,1}$. Consequently $f \in \dot H^s$ for any $s \in (-3/2, 3/2)$ — due to exponential decay for the negative range.

Further note that now $H = -\Delta + V$ is a bounded operator from $\dot H^s$ to $\dot H^{s-2}$ for any $s \in (1/2, 3/2)$, hence $(-\Delta)^{-1}H$ is a bounded map on $\dot H^s$. Its invertibility is a consequence of Lemma 12. Thus $|H|$ is an isomorphism between $\dot H^s$ and $\dot H^{s-2}$. Since $\dot H^s$ and $\dot H^{s-2}$ are dual with respect to the $\dot H^{s-1}$ dot product, we obtain that $\sqrt{|H|} \in \mathcal{B}(\dot H^s, \dot H^{s-1})$ is an isomorphism. Since $|H|^{(s-1)/2}$ is an isomorphism from $\dot H^{s-1}$ to L^2, we obtain that $|H|^{s/2}$ is an isomorphism from $\dot H^s$ to L^2 for any $s \in [0, 3/2)$. The conclusion extends by duality to $s \in (-3/2, 0]$. □

It is now an exercise to show that $T^{\pm}(\rho, x, y)$ falls within the framework of Proposition 10. Without loss of generality, the result is stated in terms of T^- alone.

Theorem 14. *Let $V \in \mathcal{K}_0$ be a scalar potential in $\mathbb{R}^3$ satisfying the assumptions of Theorem 1. Then*

$$\begin{aligned} \|T^-(\rho)\|_{\mathcal{U}_X} &\leq \frac{\|V\|_{\mathcal{K}}}{4\pi} \\ \text{and} \quad \left\|(\mathbf{1} + T^-)^{-1}\right\|_{\mathcal{U}_X} &< \infty, \end{aligned}$$

where X may be any one of the function spaces $\mathcal{K}^\theta, 0 \leq \theta \leq 1$. In particular this includes $\mathcal{K} = \mathcal{K}^1$ and $L^1(\mathbb{R}^3) = \mathcal{K}^0$.

If $V \in L^{3/2,1}$ then one may also choose X to be $L^{3/2,1}$ or any $L^{p,q}$, $1 < p < \frac{3}{2}$, $1 \leq q \leq \infty$ by replacing the right side of the first inequality with $C\|V\|_{3/2,1}$.

Before attempting a proof that spans the gamut of admissible spaces X, it will be convenient to introduce one additional bit of abstract notation. Let X and Y be two Banach lattices of functions over $(\mathcal{X}, \mu)$ and $(\mathcal{Y}, \nu)$ respectively.

Definition 3. Let $\mathcal{U}_{X,Y}$ be the set of bounded operators from $Y_y\mathbb{M}_\rho$ to $X_x\mathbb{M}_\rho$ of the form

$$(Tf)(\rho, x) = \int_{\mathcal{Y}} \int_{-\infty}^{\infty} f(\rho - \sigma, y) T(\sigma, x, y)\, d\sigma\, \nu(dy),$$

where $T(\rho, x, y)$ has the property that $T(\rho, x, y)\, d\rho \in \mathbb{M}_\rho$ for a.e. x and y and

$$M(T)(x, y) = \int_{-\infty}^{\infty} |T(\rho, x, y)|\, d\rho$$

is the integral kernel for a bounded operator from Y to X. $\mathcal{U}_{X,Y}$ is a Banach space under the norm $\|T\|_{\mathcal{U}_{X,Y}} = \|M(T)\|_{\mathcal{B}(Y,X)}$

Though in general $\mathcal{U}_{X,Y}$ cannot be an algebra under the composition of operators, it has the more general property that for any three Banach lattices X, Y, and Z

$$\|ST\|_{\mathcal{U}_{X,Z}} \leq \|S\|_{\mathcal{U}_{X,Y}} \|T\|_{\mathcal{U}_{Y,Z}}.$$

This structure is an algebroid.

The Fourier transform defined by (30) fulfills $\|\hat{T}(\lambda)\|_{\mathcal{B}(Y,X)} \leq \|T\|_{\mathcal{U}_{X,Y}}$ and $(ST)^\wedge(\lambda) = \hat{S}(\lambda)\hat{T}(\lambda)$. Kernels $T \in \mathcal{U}_{X,Y}$ define, through (29), bounded operators from Y to $X_x\mathbb{M}_\rho$.

Proof of Theorem 14. In our previously introduced notation, T^- has the kernel

$$T^-(\rho, x, y) = (4\pi\rho)^{-1} V(x) \delta_{|x-y|}(\rho).$$

This is a finite measure in ρ, for all x and y, and

$$M(T^-)(x, y) = \int_{-\infty}^{\infty} |T^-(\rho, x, y)|\, d\rho = \frac{|V(x)|}{4\pi|x-y|}.$$

$M(T^-)$ belongs to $\mathcal{B}(L^1)$ exactly when $V \in \mathcal{K}$. Indeed,

$$\begin{aligned} \Big\| \int_{\mathbb{R}^3} \frac{|V(x)|}{4\pi|x-y|} f(y)\, dy \Big\|_{L^1_x} &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x)|}{4\pi|x-y|} |f(y)|\, dy\, dx \\ &\leq \sup_y \int_{\mathbb{R}^3} \frac{|V(x)|}{4\pi|x-y|}\, dx \int_{\mathbb{R}^3} |f(y)|\, dy \leq \frac{\|V\|_{\mathcal{K}}}{4\pi} \|f\|_1 \end{aligned}$$

with equality being nearly achieved if f is concentrated on points $y \in \mathbb{R}^3$ that nearly optimize the supremum in the Kato norm. These computations show that $T^- \in \mathcal{U}_{L^1}$ and $\|T^-\|_{\mathcal{U}_{L^1}} \leq \|V\|_{\mathcal{K}}/4\pi$.

To work in other spaces, consider $\check{R}^-(\rho, x, y) = (4\pi\rho)^{-1}\delta_{|x-y|}(\rho)$. Its Fourier transform is the family of free resolvents $R_0^-(\lambda^2)$. An elementary calculation similar to the one above shows that $\check{R}^- \in \mathcal{U}_{\mathcal{K}^*, L^1} \cap \mathcal{U}_{L^\infty, \mathcal{K}}$ with norm $1/4\pi$. Multiplication by V maps L^∞ to $\mathcal{K}$, and maps $\mathcal{K}^*$ to L^1, so $T^- = V\check{R}^-$ belongs to $\mathcal{U}_{\mathcal{K}}$.

Suppose the second conclusion of the theorem is valid for $X = L^1(\mathbb{R}^3)$. Then the resolvent identity $(I + \hat{T}^-(\lambda))^{-1} = I - V(I + \hat{T}^+(\lambda)^*)^{-1} R_0^-(\lambda)$ leads to a bound

$$\|(\mathbf{1} + T^-)^{-1}\|_{\mathcal{U}_{\mathcal{K}}} \leq 1 + \|V\|_{\mathcal{K}} \big\| \big((\mathbf{1} + T^+)^*\big)^{-1} \big\|_{\mathcal{U}_{L^\infty}} \|\check{R}^-\|_{\mathcal{U}_{L^\infty}, \mathcal{K}} < \infty$$

The remaining cases $X = \mathcal{K}^\theta$ follow by interpolation. If $V \in L^{3/2,1}$ the argument can be repeated for Lorentz spaces by embedding $L^{3/2,1} \subset \mathcal{K}$ and $\mathcal{K}^* \subset L^{3,\infty}$.

The case $X = L^1(\mathbb{R}^3)$ was treated in [BeGo, Theorem 2] as an application of Proposition 10, and is summarized below.

We have determined that $T^- \in \mathcal{U}_{L^1}$, with $\|T^-\|_{\mathcal{U}_{L^1}} \leq \|V\|_{\mathcal{K}}/4\pi$. Lemma 12 shows that $I + \hat{T}^-(\lambda)$ is an invertible element of $\mathcal{B}(L^1)$ for each $\lambda \in \mathbb{R}$ provided that hypotheses of Theorem 1 are satisfied.

When applied to T^- specifically, condition (C2) of Proposition 10 is equivalent to the statement

$$\lim_{R\to\infty} \Big\| \chi_{|x-y|>R} \frac{|V(x)|}{4\pi|x-y|} \Big\|_{\mathcal{B}(L^1)} = 0,$$

which in turn reduces to the distal Kato property (21).

It suffices to verify condition (C1) for bounded and compactly supported potentials. Norm-continuity of the mapping $V \in \mathcal{K} \mapsto T^- \in \mathcal{U}_{L^1}$ allows the extension of property (C1) to all potentials $V \in \mathcal{K}_0$.

If $V \in C_c^b(\mathbb{R}^3)$, then $\hat{T}^-(\lambda) = VR_0^-(\lambda^2)$ maps $L^1(\mathbb{R}^3)$ to $L^{4/3}(\mathbb{R}^3)$ and vice versa. By Theorem 2.3 of [KRS] and scaling, it is also true that

$$\|\hat{T}^-(\lambda)f\|_{\frac{4}{3}} \lesssim |\lambda|^{-1/2}\|V\|_2\|f\|_{\frac{4}{3}}.$$

Then $\|\hat{T}^-(\lambda)^{10}\|_{\mathcal{B}(L^1)} \lesssim (1+\lambda^2)^{-2}$, which is more than sufficient to imply that $\|\partial_\rho^k (T^-)^{10}(\rho, x, y)\|_{\mathcal{B}(L^1)}$ is uniformly bounded over all $\rho \in \mathbb{R}$ and $k = 0, 1$.

The compact-approximation property (C2) is preserved by products in $\mathcal{U}_{L^1}$, therefore it suffices to verify that $\eta(\rho)(T^-)^{10}$ satisfies condition (C1) for all compactly supported functions η. However $\|\partial_\rho\, \eta(T^-)^{10}(\rho, x, y)\|_{\mathcal{B}(L^1)}$ will be uniformly bounded, and vanishes away from the support of η. It follows that

$$\begin{aligned} &\big\|\eta(T^-)^{10}(\rho, x, y) - \eta(T^-)^{10}(\rho - \delta, x, y)\big\|_{\mathcal{U}_{L^1}} \\ \leq\ &\int_{\mathbb{R}} \big\| |\eta(T^-)^{10}(\rho, x, y) - \eta(T^-)^{10}(\rho - \delta, x, y)| \big\|_{\mathcal{B}(L^1)}\, d\rho \leq C\delta \end{aligned}$$

which converges to zero as $\delta \to 0$. The constant depends on parameters such as the size and support of V and of η but is independent of δ. □

A kernel's membership in $\mathcal{U}_X$ ensures only integrability in the ρ variable. Several statements within Theorem 1 require an additional weighted integrability condition $\rho T(\rho, x, y) \in \mathcal{U}_{\mathcal{K},L^1}$. In this setting, we have the following extension of Theorem 14:

Proposition 15. *Assume that $V \in \mathcal{K}_0$ satisfies the conditions of Theorem 1. Then*

$$\|\rho T^-\|_{\mathcal{U}_{\mathcal{K},L^1}} \leq \frac{\|V\|_{\mathcal{K}}}{4\pi}$$
$$\textit{and} \quad \big\|\rho(\mathbf{1} + T^-)^{-1}\big\|_{\mathcal{U}_{\mathcal{K},L^1}} < \infty.$$

Proof. $\rho T^-(\rho)$ has the kernel

$$\rho T^-(\rho)(x, y, \rho) = (4\pi)^{-1}V(x)\delta_{|x-y|}(\rho).$$

Therefore

$$M(\rho T^-(\rho))(x, y) = (4\pi)^{-1}|V(x)|$$

is the kernel of a bounded operator from L^1 to $\mathcal{K}$ as long as $V \in \mathcal{K}$. This implies that $\|\rho T^-\|_{\mathcal{U}_{L^1,\mathcal{K}}} \leq \frac{\|V\|_{\mathcal{K}}}{4\pi}$.

The second inequality is obtained from Theorem 14, because of the fact that

$$\rho\big(\mathbf{1} + T\big)^{-1} = -(\mathbf{1} + T)^{-1}(\rho)\big(\rho T(\rho)\big)(\mathbf{1} + T)^{-1}(\rho). \tag{31}$$

By taking the Fourier transform, (31) is equivalent to

$$\partial_\lambda (I + \hat{T}(\lambda))^{-1} = -(I + \hat{T}(\lambda))^{-1} \partial_\lambda \hat{T}(\lambda)(I + \hat{T}(\lambda))^{-1}. \tag{32}$$

To prove (32), one divides the formula

$$(I + \hat{T}(\lambda_1))^{-1} - (I + \hat{T}(\lambda_2))^{-1} = (I + \hat{T}(\lambda_1))^{-1}(\hat{T}(\lambda_2) - \hat{T}(\lambda_1))(I + \hat{T}(\lambda_2))^{-1}$$

by $\lambda_1 - \lambda_2$ and passes to the strong limit. □

It is also useful to obtain bounds on $\rho^\theta T(\rho, x, y)$ for $0 \le \theta \le 1$:

Corollary 16. *Assume that* $V \in \mathcal{K}_0$ *satisfies the conditions of Theorem 1. Then, for* $0 \le \theta_1 \le \theta_2 \le 1$,

$$\begin{aligned} \|\rho^{\theta_2 - \theta_1} T^-\|_{\mathcal{U}_{\mathcal{K}_{\theta_2}, \mathcal{K}_{\theta_1}}} &\le \frac{\|V\|_{\mathcal{K}}}{4\pi} \\ \textit{and} \quad \big\||\rho|^{\theta_2-\theta_1}(\mathbf{1} + T^-)^{-1}\big\|_{\mathcal{U}_{\mathcal{K}_{\theta_2}, \mathcal{K}_{\theta_1}}} &< \infty. \end{aligned}$$

In fact, all these measures have positive support in ρ, so the absolute value is superfluous.

Proof. The statements follow by complex interpolation between the three cases addressed in Theorem 14 and Proposition 15. □

Finally, we can prove the main result of this paper.

Proof of Theorem 1. Start with the functional calculus formula in (24),

$$\begin{aligned} \frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} f &= \frac{1}{\pi i} \int_{-\infty}^{\infty} \sin(t\lambda) R_V^+(\lambda^2) f \, d\lambda \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{it\lambda} - e^{-it\lambda}) R_V^+(\lambda^2) f \, d\lambda \\ &= -\frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{it\lambda} - e^{-it\lambda})(I + \hat{T}^-(\lambda)^*)^{-1} R_0^+(\lambda^2) f \, d\lambda \end{aligned}$$

This is a symmetrization of the (inverse) Fourier transform of $R_V^+(\lambda^2)$. Most of the desired estimates do not rely on cancellation so it suffices to bound the Fourier transform in L_t^p by itself.

In the event that $V = 0$, the factor $(I + \hat{T}^-(\lambda)^*)^{-1}$ reduces to the identity operator and one is left to calculate the inverse Fourier transform of $R_0^+(\lambda^2)$. Thanks to the explicit free resolvent kernel $R_0^+(\lambda^2)(x, y) = (4\pi|x - y|)^{-1} e^{i\lambda|x-y|}$, the result is a measure supported on the light cone $|x - y| = |t|$. In the notation of Section 3,

$$\check{R}_0^+(\rho, x, y) = (4\pi\rho)^{-1} \delta_0(\rho + |x - y|),$$

which is precisely the backward propagator of the free wave equation. The symmetrized version contains both the forward and backward propagators.

Direct inspection of of the operator kernel shows that

$$\begin{aligned} \|\rho \check{R}_0^+\|_{\mathcal{U}_{L^\infty, L^1}} &= \frac{1}{2\pi} \\ \|\check{R}_0^+\|_{\mathcal{U}_{L^\infty, \mathcal{K}}} &= \frac{1}{2\pi}, \ \|\check{R}_0^+\|_{\mathcal{U}_{\mathcal{K}^*, L^1}} = \frac{1}{2\pi}, \end{aligned}$$

where we have taken $\check{R}_0^+$ to indicate the inverse Fourier transform of $R_0^+(\lambda^2)$ in a convenient abuse of notation.

Let $S := (\mathbf{1}+T^-)^{-1}$ be defined in $\mathcal{U}_X$ (for various function spaces X) according to Theorem 14. Then $S^* \in \mathcal{U}_{X^*}$ satisfies $(S^*)^\wedge(\lambda) = (I+\hat{T}^-(\lambda)^*)^{-1}$. Observe that $\rho\int_{\mathbb{R}} e^{i\rho\lambda}\hat{S}^*(\lambda)R_0^+(\lambda^2)\,d\lambda = (\rho S^*)\check{R}_0^+ + S^*(\rho\check{R}_0^+)$, where the products on the right-hand side are taken in the algebroid structure of $\mathcal{U}_{X,Y}$.

$$\begin{aligned}\|\rho\check{R}_V^+\|_{\mathcal{U}_{L^\infty,L^1}} &\le \|\rho S^*\|_{\mathcal{U}_{L^\infty,\mathcal{K}^*}}\|\check{R}_0^+\|_{\mathcal{U}_{\mathcal{K}^*,L^1}}\\ &\quad+\|S^*\|_{\mathcal{U}_{L^\infty}}\|\rho\check{R}_0^+\|_{\mathcal{U}_{L^\infty,L^1}}<\infty\end{aligned}$$

by the collected results in Theorem 14 and Proposition 15. It immediately follows that $\big\|t\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_cf\big\|_{L^\infty_x L^1_t} \lesssim \|f\|_1$. The subsequent two inequalities in Theorem 1 express the fact that the tail integral of a function in $t^{-1}L^1_t$ belongs to both L^1_t and $t^{-1}L^\infty_t$. After substituting Hf in place of f, one obtains that

$$\Big\|\int_t^\infty \sqrt{H}\sin(s\sqrt{H})P_cf\,ds\Big\|_{L^\infty_x} \lesssim t^{-1}\|Hf\|_1$$

for all $t>0$. Typically $\|Hf\|_1 = \|(I+V(-\Delta)^{-1})\Delta f\|_1 \lesssim \|\Delta f\|_1$. Lemma 13 asserts that the two norms are even equivalent if H has no resonance or eigenvalue at zero.

The same tail integral bounds also hold for $\cos(t\sqrt{H})P_cf$ once it is established that $\cos(t\sqrt{H})P_c = \int_t^\infty \sqrt{H}\sin(s\sqrt{H})P_c\,ds$ in an appropriate sense. Note that the difference

$$Af := \cos(t\sqrt{H})P_cf - \int_t^\infty \sqrt{H}\sin(s\sqrt{H})P_cf\,ds$$

is independent of t and is a bounded linear operator from $L^2\cap H^{-1}L^1$ to L^2+L^∞. By Lemma 13 it is permissible to replace $H^{-1}L^1$ with the equivalent space $(-\Delta)^{-1}L^1$. However $\cos(t\sqrt{H})P_c$ converges weakly to zero in $\mathcal{B}(L^2)$ as $t\to\infty$, and the norm of $\int_t^\infty \frac{\sin(s\sqrt{H})}{\sqrt{H}}P_c\,ds$ in $\mathcal{B}(L^1,L^\infty)$ is dominated by t^{-1}. That forces $\langle Af,g\rangle = 0$ for any pair of test functions f,g, which means $A=0$.

The immediate consequence is that $\|t\cos(t\sqrt{H})P_cf\|_{L^\infty_x}$ and $\|\cos(t\sqrt{H})P_cf\|_{L^\infty_x L^1_t}$ are both controlled by $\|\Delta f\|_1$, which is comparable to $\|Hf\|_1$. A second integration in the t direction shows that $\big\|\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_cf\big\|_{L^\infty_{x,t}} \lesssim \|\Delta f\|_1$.

Estimates for $\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_cf$ in unweighted L^1_t are straightforward, requiring only the composition

$$\begin{aligned}\|\check{R}_V^+\|_{\mathcal{U}_{L^\infty,\mathcal{K}}} &\le \|S^*\|_{\mathcal{U}_{L^\infty}}\|\check{R}_0^+\|_{\mathcal{U}_{L^\infty,\mathcal{K}}}\\ \|\check{R}_V^+\|_{\mathcal{U}_{\mathcal{K}^*,L^1}} &\le \|S^*\|_{\mathcal{U}_{\mathcal{K}^*}}\|\check{R}_0^+\|_{\mathcal{U}_{\mathcal{K}^*,L^1}}.\end{aligned}$$

It follows that $\big\|\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c\big\|_{L^\infty_x L^1_t} \lesssim \|f\|_{\mathcal{K}}$ and $\big\|\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c\big\|_{\mathcal{K}^*_x L^1_t} \lesssim \|f\|_1$. Integrating in from $t=\infty$ as above proves the corresponding estimates for $\cos(t\sqrt{H})P_c$ in L^∞_t, with norm controlled by $\|\Delta f\|_{\mathcal{K}}$ and $\|\Delta f\|_1$ respectively. Lemma 13 is used to show the equivalence of Δf and Hf in the Kato norm.

Initial conditions $f(x)$ belonging to a Sobolev space such as $\dot{W}^{1,1}(\mathbb{R}^3)$ require extra care because the Banach lattice structure is absent. Indeed, $I+\hat{T}^-(\lambda)$ need not be a bounded operator here, nor possess a bounded inverse.

We consider the action of $\cos(t\sqrt{H})P_c$ on functions with one weak derivative. Start with the spectral representation (23) to derive

$$\begin{aligned}\cos(t\sqrt{H})P_c f &= \frac{1}{\pi i}\int_{-\infty}^{\infty} \lambda\cos(t\lambda)R_V^+(\lambda^2)f\,d\lambda \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}(e^{it\lambda}+e^{-it\lambda})\lambda R_V^+(\lambda^2)f\,d\lambda \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}(e^{it\lambda}+e^{-it\lambda})(I+\hat{T}^-(\lambda)^*)^{-1}\lambda R_0^+(\lambda^2)f\,d\lambda\end{aligned}$$

As before it suffices to bound the inverse Fourier transform of $\lambda R_V^+(\lambda^2)f$ alone rather than the symmetrized version.

In the free case $(\lambda R_0^+(\lambda^2)f)^\vee(\rho)$ provides a solution to the wave equation in $\mathbb{R}_\rho^- \times \mathbb{R}_x^3$ with boundary conditions $u(x,0)=f$, $u_\rho(x,0)=0$. The explicit formula is

$$\frac{1}{2\pi i}(\lambda R_0^+(\lambda^2)f)^\vee(\rho,x) = \int_{S^2} f(x+\rho\omega)+\rho\partial_\rho f(x+\rho\omega)\,d\omega$$

This is an even function of ρ, so for the purposes of estimating its norm in L^1_ρ we may integrate over $\mathbb{R}^+$ instead of $\mathbb{R}^-$. If f is bounded and has compact support then

$$\begin{aligned}\big\|(\lambda R_0^+(\lambda^2)f)^\vee(\,\cdot\,,x)\big\|_{L^1_\rho} &\le C\int_0^\infty\int_{S^2}|f(x+\rho\omega)|+|\rho\,\partial_\rho f(x+\rho\omega)|\,d\omega d\rho \\ &\lesssim (|f|*|x|^{-2})(x)+(|\nabla f|*|x|^{-1})(x) \\ &\lesssim (|\nabla f|*|x|^{-1})(x),\end{aligned}$$

because $|f(x)|\lesssim |\nabla f|*|x|^{-2}$ pointwise almost everywhere. Consequently

$$\big\|(\lambda R_0^+(\lambda^2)f)^\vee\big\|_{\mathcal{K}^*_x L^1_\rho}\lesssim\|\nabla f\|_1 \quad\text{and}\quad \big\|(\lambda R_0^+(\lambda^2)f)^\vee\big\|_{L^\infty_x L^1_\rho}\lesssim\|\nabla f\|_{\mathcal{K}}.$$

The fact that $S^*\in\mathcal{U}_{\mathcal{K}^*}\cap\mathcal{U}_{L^\infty}$, together with the prior estimates for $(\lambda R_0^+(\lambda^2)f)^\vee$, leads to immediate bounds of the form

$$\begin{aligned}\|\cos(t\sqrt{H})P_c f\|_{\mathcal{K}^*_x L^1_t}&\lesssim\|\nabla f\|_1 \\ \|\cos(t\sqrt{H})P_c f\|_{L^\infty_x L^1_t}&\lesssim\|\nabla f\|_{\mathcal{K}}.\end{aligned}$$

Integrating this pair of inequalities with respect to t produces the further bounds $\big\|\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c f\big\|_{\mathcal{K}^*_x L^\infty_t}\lesssim\|\nabla f\|_1$ and $\big\|\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c f\big\|_{L^\infty_{x,t}}\lesssim\|\nabla f\|_{\mathcal{K}}$.

Time-weighted bounds in the free case are also derived from the explicit form of the propagator kernel.

$$\begin{aligned}\big\|\rho(\lambda R_0^+(\lambda^2)f)^\vee(\,\cdot\,,x)\big\|_{L^1_\rho} &\le C\int_0^\infty\int_{S^2}|\rho f(x+\rho\omega)|+|\rho^2\,\partial_\rho f(x+\rho\omega)|\,d\omega d\rho \\ &\lesssim |f|*|x|^{-1}+\|\nabla f\|_1 \\ &\lesssim \|\nabla f\|_1.\end{aligned}$$

which means that $\big\|\rho(\lambda R_0^+(\lambda^2)f)^\vee\big\|_{L^\infty_x L^1_\rho}\lesssim\|\nabla f\|_1$. Lemma 7 was used in the last line to control the size of $|f|*|x|^{-1}$.

Proposition 15 provides the additional information that $\rho S^* \in \mathcal{U}_{L^\infty,\mathcal{K}^*}$. Imitating the weighted L^1 arguments for the sine propagator,

$$\begin{aligned}\|\rho(\lambda R_V^+)^\vee f\|_{L^\infty_x L^1_t} &\leq \|\rho S^*\|_{\mathcal{U}_{L^\infty,\mathcal{K}^*}} \|(\lambda R_0^+)^\vee f\|_{\mathcal{K}^*_x L^1_\rho} + \|S^*\|_{\mathcal{U}_{L^\infty}} \|\rho(\lambda R_0^+)^\vee f\|_{L^\infty_x L^1_\rho} \\ &\lesssim \|\nabla f\|_1\end{aligned}$$

The same bound is true of $\rho(\lambda R_V^-)^\vee f$, and their sum shows that

$$\|t\cos(t\sqrt{H})P_c f\|_{L^\infty_x L^1_t} \lesssim \|\nabla f\|_1.$$

Once again the tail integral of a function in $t^{-1}L^1_t$ belongs both to L^1_t as well as $t^{-1}L^\infty_t$. This provides mapping bounds for $\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c$ once it is established that the operator

$$Bf := \frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c f + \int_t^\infty \cos(t\sqrt{H})P_c f\, ds$$

is trivial. Here B is bounded from $L^2 \cap \dot{W}^{1,1}$ to $\dot{H}^1 + L^\infty$ (recalling that $\dot{H}^1 \cong H^{-1/2}L^2$), and for any pair of smooth test functions, f, g one can show that $\lim_{t\to\infty}\langle Bf, g\rangle = 0$. Since Bf is in fact independent of t it follows that $B = 0$.

The first consequence, that $\big\|\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c f\big\|_{L^\infty_x L^1_t} \lesssim \|\nabla f\|_1$, was proved earlier in the discussion with a stronger bound in terms of $\|f\|_{\mathcal{K}}$ in place of $\|\nabla f\|_1$. The second consequence is the dispersive estimate

$$\Big\|\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c f\Big\|_{L^\infty} \lesssim |t|^{-1}\|\nabla f\|_1. \tag{33}$$

The inhomogeneous estimates presented in Theorem 1 are elementary extensions of the dispersive bounds proved above. They reduce to the following statements about the propagator kernels for $\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c$ and $\frac{\cos(t\sqrt{H})}{H}P_c$, denoted by $K_1(t,x,y)$ and $K_2(t,x,y)$ respectively:

$$\begin{aligned}&\|K_1(\,\cdot\,,x,y)\|_{\mathcal{M}} \text{ defines bounded maps taking } L^1 \to \mathcal{K}^* \text{ and } \mathcal{K} \to L^\infty.\\ &\|K_2(\,\cdot\,,x,y)\|_\infty \text{ defines bounded maps taking } L^1 \to \mathcal{K}^* \text{ and } \mathcal{K} \to L^\infty.\\ &\|K_2(\,\cdot\,,x,y)\|_1 \lesssim 1 \text{ at almost every } x \text{ and } y.\end{aligned} \tag{34}$$

Each of these is a well known fact when $H = -\Delta$, and is readily transferred to $H = -\Delta + V$ by composition with the operator $S^* \in \mathcal{U}_{\mathcal{K}^*} \cap \mathcal{U}_{L^\infty}$. □

Proof of Corollary 2. The inhomogeneous propagator estimates involving $\mathcal{K}^\theta$ and its dual all follow from complex interpolation of the kernel bounds in (34), recalling that $\mathcal{K}^0 = L^1(\mathbb{R}^3)$. The fact that K_2 belongs to $L^1_t \cap L^\infty_t$ allows for a second parameter of interpolation.

Each of the homogeneous estimates that follow is an interpolation between three bounds stated in Theorem 1. For example one is given that $\frac{\sin(t\sqrt{H})}{\sqrt{H}}P_c f$ is controlled in both the $\mathcal{K}^*_x L^\infty_t$ and the $L^\infty_x L^1_t$ norms by $\|\nabla f\|_1$, and controlled in the $L^\infty_{x,t}$ norm by $\|\nabla f\|_{\mathcal{K}}$. The time-decay estimates as stated in Corollary 2 are in fact slightly weaker than what naturally arises via this method. A more precise statement is

$$\begin{aligned}\Big\|t^{1-\theta_1-\theta_2}\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}}f\Big\|_{(\mathcal{K}^{\theta_2})^*_x L^\infty_t} &\lesssim \|\nabla f\|_{\mathcal{K}^{\theta_1}}\\ \big\|t^{1-\theta_1-\theta_2}\cos(t\sqrt{H})P_c f\big\|_{(\mathcal{K}^{\theta_2})^*_x L^\infty_t} &\lesssim \|\Delta f\|_{\mathcal{K}^{\theta_1}}\end{aligned}$$

The first two Lorentz-space inequalities are also derived from (34), this time using real interpolation. Recall that $L^{3/2,1} \subset \mathcal{K}$ and $\mathcal{K}^* \subset L^{3,\infty}$. Then $\|K_1(\,\cdot\,, x, y)\|_{\mathcal{M}}$ defines a linear map from L^1 to $L^{3,\infty}$, and $L^{3/2,1}$ to L^∞. By Hunt's interpolation theorem [Hun] the same map is also bounded from $L^{p,s}$ to $L^{q,s}$ so long as $\frac{1}{p} - \frac{1}{q} = \frac{2}{3}$ and $1 < p, q < \infty$. For the cosine propagator K_2, first use complex interpolation as above to show that $\|K_2(\,\cdot\,, x, y)\|_p$ maps L^1 to $(\mathcal{K}^{1-1/p})^* \subset L^{3p/(p-1),\infty}$ and $L^{3p/(2p+1),1} \subset \mathcal{K}^{1-1/p}$ to L^∞. Hunt's theorem establishes mapping bounds in the intermediate Lorentz spaces and the convolution with L^p_t is handled by Young's inequality.

The dispersive estimate (11) with polynomial time-decay is derived from (8). By Lemma 13 and complex interpolation, Δf and Hf have equivalent norms in $\mathcal{K}^\theta$ for all $0 \le \theta \le 1$. Once H has been brought back to the left side of the inequality, apply real interpolation between the endpoint cases $(\theta_1, \theta_2) = (0, \frac{r-1}{r})$ and $(\theta_1, \theta_2) = (\frac{r-1}{r}, 0)$.

The final two bounds are derived in the same manner. An additional assumption that $V \in L^{3/2,1}$ is needed to ensure that Δf and Hf are comparable in the function space $L^{3/2,1}$ as well as in L^1. The norm equivalence holds in all intermediate Lorentz spaces $L^{p,s}$, $1 < p < \frac{3}{2}$ as well. □

Proof of Theorem 3. Split the operator $\frac{e^{it\sqrt{H}}}{\sqrt{H}} P_c = \frac{\cos(t\sqrt{H})}{\sqrt{H}} P_c + i \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c$ into its real and imaginary components. Since the continuous spectrum of H lies on $[0, \infty)$, both parts are well defined and (up to the factor of i) self-adjoint. Both the regular and reversed-norm Strichartz inequalities are proved by applying TT^* argument to each component separately. We begin with the reversed-norm estimates (14)–(16), first in the energy- critical case.

Consider the operator $Tf = \dfrac{\sin(t\sqrt{H})P_c}{\sqrt{H}} f$, with t ranging over the entire real line. Then $T^*F = \displaystyle\int_{\mathbb{R}} \frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} F(t)\, dt$. Consequently

$$
\begin{aligned}
(TT^*F)(t) &= \int_{\mathbb{R}} \frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} \frac{\sin(s\sqrt{H})P_c}{\sqrt{H}} F(s)\, ds \\
&= \frac{1}{2} \int_{\mathbb{R}} \Big(\frac{\cos((t-s)\sqrt{H})P_c}{H} - \frac{\cos((t+s)\sqrt{H})P_c}{H} \Big) F(s)\, ds.
\end{aligned}
$$

By the combined estimates (7) and (10), TT^* is a bounded operator from $\mathcal{K}^\theta_x L^{r'}_t$ (in particular $L^1_x L^2_t$) to its dual and from $L^{q',2}_x L^{r'}_t$ to its dual. Since we are interested in mappings between dual spaces, the constant $C_{q'q}$ in (10) is uniformly bounded. That establishes (14) and (15) for the imaginary part of $\frac{e^{it\sqrt{H}}}{\sqrt{H}} P_c$.

The proof for the real part is nearly identical. Let $Tf = \frac{\cos(t\sqrt{H})}{\sqrt{H}} P_c f$. Then the corresponding TT^* operator is defined by

$$
\int_{\mathbb{R}} \frac{\cos(t\sqrt{H})P_c}{\sqrt{H}} \frac{\cos(s\sqrt{H})P_c}{\sqrt{H}}\, ds = \frac{1}{2} \int_{\mathbb{R}} \frac{\cos((t-s)\sqrt{H})P_c}{H} + \frac{\cos((t+s)\sqrt{H})P_c}{H}\, ds,
$$

which is once again controlled by (7) and (10) as above.

The reversed-norm Strichartz estimate for the wave equation with initial data $f_0 \in \dot H^1$ follows readily from the decomposition

$$\|\cos(t\sqrt{H})f_0\|_X \leq \Big\|\frac{\cos(t\sqrt{H})P_c}{\sqrt{H}}\Big\|_{\mathcal{B}(L^2,X)} \|(HP_c)^{1/2}\|_{\mathcal{B}(\dot H^1,L^2)} \|f_0\|_{\dot H^1}$$

with X standing in for any of the target spaces $(\mathcal{K}^\theta)^*_x L^r_t \cap L^{q,2}_x L^r_t$. Since $V \in \mathcal{K}_0$ is form-bounded with respect to the Laplacian, H is a bounded map from $\dot H^1$ into $\dot H^{-1}$. The spectral projection P_c removes a finite number of eigenfunctions, each of which belong to $\dot H^1 \cap \dot H^{-1}$, hence HP_c defines a positive bounded quadratic form on $\dot H^1$. Its square root then belongs to $\mathcal{B}(\dot H^1, L^2)$.

The same proof based on the TT^* method generalizes to the case of initial data in $\dot H^s \times \dot H^{s-1}$, for $\frac12 < s < \frac32$. For $0 < s < \frac32$, $s \neq 1/2$, the propagator $\frac{\cos(t\sqrt{-\Delta})}{(-\Delta)^s}\,|_{t\geq 0}$ has a radially symmetric kernel of the form

$$\begin{aligned} K_s(r=|x-y|,t) &= \partial_t \frac{C_s}{t}\int_0^\pi \frac{t^2\sin\phi}{(r^2-2rt\cos\phi+t^2)^{\frac32-s}}\,d\phi \\ &= \partial_t \frac{C_s}{r}\left(\frac{1}{|r-t|^{1-2s}} - \frac{1}{(r+t)^{1-2s}}\right) \\ &= \frac{C_s}{r}\left(\frac{\operatorname{sgn}(t-r)}{|r-t|^{2-2s}} - \frac{1}{(r+t)^{2-2s}}\right) \end{aligned}$$

and a similar expression is retrieved in the case $s = \frac12$ by means of a computation that involves logarithms. For $\frac12 < s \leq 1$ we obtain that

$$\|K_s(r,t)\|_{L^1_t} \lesssim_s \frac{1}{r^{2-2s}} \text{ and } \|K_s(r,t)\|_{L^{1/(2-2s),\infty}_t} \lesssim_s \frac{1}{r}.$$

Furthermore, for $1 < s < \frac32$ there is a similar set of bounds $\|K_s(r,t)\|_{L^{1/(3-2s),\infty}_t} \lesssim_s 1$ and $\|K_s(r,t)\|_{L^\infty_t} \lesssim_s \frac{1}{r^{3-2s}}$, and more generally, for $0 \leq \theta \leq 3-2s$, $\|t^\theta K_s(r,t)\|_{L^\infty_t} \lesssim_s \frac{1}{r^{3-2s-\theta}}$.

Thus for $\frac12 < s \leq 1$ $\|K_s(r=|x-y|,t)\|_{L^1_t}$ is a bounded map from L^1 to $(\mathcal{K}^{2-2s})^*$ and from $\mathcal{K}^{2-2s}$ to L^∞, while

$$\|K_s(r=|x-y|,t)\|_{L^{1/(2-2s),\infty}_t} \in \mathcal{B}(L^1,\mathcal{K}^*) \cap \mathcal{B}(\mathcal{K},L^\infty).$$

For $1 < s < \frac32$, $0 \leq \theta \leq 3-2s$,

$$\begin{aligned} \|K_s(r=|x-y|,t)\|_{L^{1/\theta,\infty}_t} &\lesssim \|t^\theta K_s(r=|x-y|,t)\|_{L^\infty_t} \\ &\in \mathcal{B}(L^1,(\mathcal{K}^{3-2s-\theta})^*) \cap \mathcal{B}(\mathcal{K}^{3-2s-\theta},L^\infty). \end{aligned}$$

Since by Corollary 16 $\rho^{\theta_2-\theta_1}S^* \in \mathcal{U}_{(\mathcal{K}^{\theta_1})^*,(\mathcal{K}^{\theta_2})^*}$, these bounds carry over to the kernel of $\frac{\cos(t\sqrt{H})P_c}{H^s}\,|_{t\geq0}$. Thus for $\frac12 < s < 1$ ($s=1$ is the energy-critical case treated above), $0 \leq \theta \leq 2-2s$, $\frac1p - \frac1q = 2s-1$, $1 < p \leq q < \infty$,

$$\Big\|\int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H^s} F(t')\,dt'\Big\|_{(\mathcal{K}^{2-2s-\theta})^*_x L^{p,\sigma}_t} \lesssim \|F\|_{\mathcal{K}^\theta_x L^{p,\sigma}_t}, \tag{35}$$

$$\Big\|\int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H^s} F(t')\,dt'\Big\|_{(\mathcal{K}^{1-\theta})^*_x L^{q,\sigma}_t} \lesssim \|F\|_{\mathcal{K}^\theta_x L^{p,\sigma}_t}, \tag{36}$$

and for $1 < s < \frac{3}{2}$, $0 \le \theta_1 + \theta_2 \le 3 - 2s$, $\frac{1}{p} - \frac{1}{q} = \theta_1 + \theta_2 + 2s - 2$, $1 < p \le q < \infty$,

$$\Big\| \int_{t'<t} (t-t')^{3-2s-\theta_1-\theta_2} \frac{\cos((t-t')\sqrt{H})P_c}{H^s} F(t')\, dt' \Big\|_{(\mathcal{K}^{\theta_2})^*_x L^\infty_t} \lesssim \|F\|_{\mathcal{K}^{\theta_1}_x L^1_t}, \tag{37}$$

$$\Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H^s} F(t')\, dt' \Big\|_{(\mathcal{K}^{\theta_2})^*_x L^{q,\sigma}_t} \lesssim \|F\|_{\mathcal{K}^{\theta_1}_x L^{p,\sigma}_t}. \tag{38}$$

Interpolating between (35) and (36) we obtain that, for $\frac{1}{2} < s < 1$, $2 - 2s \le \theta_1 + \theta_2 \le 1$ and $\frac{1}{p} - \frac{1}{q} = \theta_1 + \theta_2 + 2s - 2$, $1 < p \le q < \infty$,

$$\Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H^s} F(t')\, dt' \Big\|_{(\mathcal{K}^{\theta_2})^*_x L^{q,\sigma}_t} \lesssim \|F\|_{\mathcal{K}^{\theta_1}_x L^{p,\sigma}_t}.$$

A Lorentz space version of the above inequality is, for $\frac{1}{2} < s < 1$, $0 \le \frac{1}{r} - \frac{1}{\tilde r} \le 2s-1$, $\frac{1}{r} - \frac{1}{\tilde r} = 1 - \frac{3}{p} + \frac{3}{q} + 2s$, $1 < \tilde r \le r < \infty$, $1 < p \le q < \infty$,

$$\Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H^s} F(t')\, dt' \Big\|_{L^{q,\sigma}_x L^{r,\tilde\sigma}_t} \lesssim \|F\|_{L^{p,\sigma}_x L^{\tilde r,\tilde\sigma}_t}.$$

Likewise, a Lorentz space version of (38) is that for $1 < s < \frac{3}{2}$, $2s - 2 \le \frac{1}{r} - \frac{1}{\tilde r} \le 1$, $\frac{1}{r} - \frac{1}{\tilde r} = 1 - \frac{3}{p} + \frac{3}{q} + 2s$, $1 < \tilde r \le r < \infty$, $1 < p \le q < \infty$

$$\Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H^s} F(t')\, dt' \Big\|_{L^{q,\sigma}_x L^{r,\tilde\sigma}_t} \lesssim \|F\|_{L^{p,\sigma}_x L^{\tilde r,\tilde\sigma}_t}$$

and for $1 < s < \frac{3}{2}$, $\frac{3}{p} - \frac{3}{q} = 2s$, $1 < p \le q < \infty$,

$$\Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H^s} F(t')\, dt' \Big\|_{L^{q,\sigma}_x L^\infty_t} \lesssim \|F\|_{L^{p,\sigma}_x L^1_t}.$$

Identifying dual spaces in the previous inequalities, by the TT^* method we arrive at the following estimates for $\frac{\cos(t\sqrt{H})P_c}{H^{s/2}}$ and $\frac{\sin(t\sqrt{H})P_c}{H^{s/2}}$: when $\frac{1}{2} < s < 1$, $1 - s \le \theta \le \frac{1}{2}$, $\theta + \frac{1}{r} = \frac{3}{2} - s$ and when $6 \le q \le \frac{3}{1-s}$, $\frac{3}{q} + \frac{1}{\tilde r} = \frac{3}{2} - s$

$$\Big\| \frac{e^{it\sqrt{H}} P_c}{H^{s/2}} f \Big\|_{(\mathcal{K}^\theta)^*_x L^{r,2}_t \cap L^{q,2}_x L^{\tilde r,2}_t} \lesssim \|f\|_{L^2}.$$

For $1 < s < 3/2$, one obtains that when $0 \le \theta \le \frac{3}{2} - s$ (note this includes L^∞ when $\theta = 0$), $\theta + \frac{1}{r} = \frac{3}{2} - s$, $r < \infty$ and when $\frac{6}{3-2s} < q < \infty$, $\frac{3}{q} + \frac{1}{\tilde r} = \frac{3}{2} - s$, $\tilde r < \infty$,

$$\Big\| \frac{e^{it\sqrt{H}} P_c}{H^{s/2}} f \Big\|_{(\mathcal{K}^{\frac{3}{2}-s})^*_x L^\infty_t \cap L^{\frac{6}{3-2s},2}_x L^\infty_t \cap (\mathcal{K}^\theta)^*_x L^{r,2}_t \cap L^{q,2}_x L^{\tilde r,2}_t} \lesssim \|f\|_{L^2}.$$

The regular Strichartz inequalities (12) can be proved in the manner of Ginibre-Velo [GiVe2]. In the energy-critical case $s = 1$, thanks to the dispersive bound (11) and to Young's inequality, we obtain that when $\frac{1}{r} - \frac{1}{\tilde r} = \theta_1 + \theta_2$, $1 < r < \tilde r < \infty$,

$$\Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H} F(t')\, dt' \Big\|_{L^{\tilde r,\tilde\sigma}_t (\mathcal{K}^{\theta_2})^*_x} \lesssim \|F\|_{L^{r,\tilde\sigma}_t \mathcal{K}^{\theta_1}_x}$$

and $\frac{1}{r} - \frac{1}{\tilde r} = 3 - \frac{3}{p} + \frac{3}{q}$, $1 < p \le q < \infty$, $1 < r < \tilde r < \infty$

$$\Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H})P_c}{H} F(t')\, dt' \Big\|_{L^{\tilde r,\tilde\sigma}_t L^{q,\sigma}_x} \lesssim \|F\|_{L^{r,\tilde\sigma}_t L^{p,\sigma}_x}.$$

From (37) we infer that for $1 < s < \frac{3}{2}$, $\frac{3}{p} - \frac{3}{q} = \frac{1}{r} + 2s$, and $1 < p \leq q < \infty$,

$$\Big\| t^{1/r} \frac{\cos(t\sqrt{H}) P_c}{H^s} f \Big\|_{L^{q,\sigma}_x L^\infty_t} \lesssim \|f\|_{L^{p,\sigma}_x}.$$

Thus for $1 < s < \frac{3}{2}$, $\frac{1}{r} - \frac{1}{\tilde r} = 1 - \frac{3}{p} + \frac{3}{q} + 2s$, $1 < p \leq q < \infty$, $1 < r < \tilde r < \infty$,

$$\Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H}) P_c}{H^s} F(t')\, dt' \Big\|_{L^{\tilde r, \tilde \sigma}_t L^{q,\sigma}_x} \lesssim \|F\|_{L^{r,\tilde\sigma}_t L^{p,\sigma}_x}$$

and for $1 < s < \frac{3}{2}$, $\frac{1}{r} - \frac{1}{\tilde r} = \theta_1 + \theta_2 + 2s - 2$, $1 < r < \tilde r < \infty$,

$$\Big\| \int_{t'<t} \frac{\cos((t-t')\sqrt{H}) P_c}{H^s} F(t')\, dt' \Big\|_{L^{\tilde r, \tilde \sigma}_t (\mathcal{K}^{\theta_2})^*_x} \lesssim \|F\|_{L^{r,\tilde\sigma}_t \mathcal{K}^{\theta_1}_x}.$$

Identifying dual spaces, by a TT^* argument we obtain that in the energy- critical case, for $0 < \theta < \frac{1}{2}$, $\theta + \frac{1}{r} = \frac{1}{2}$ and for $6 < q < \infty$, $\frac{3}{q} + \frac{1}{\tilde r} = \frac{1}{2}$,

$$\Big\| \frac{e^{it\sqrt{H}} P_c}{H^{s/2}} f \Big\|_{L^{r,2}_t (\mathcal{K}^\theta)^*_x \cap L^{\tilde r,2}_t L^{q,2}_x} \lesssim \|f\|_2.$$

In addition, the L^2 conservation law also makes this expression bounded in $L^\infty_t L^{6,2}_x$.

When $1 < s < 3/2$, the same TT^* argument shows that for $0 \leq \theta < \frac{3}{2} - s$, $\theta + \frac{1}{r} = \frac{3}{2} - s$ and $\frac{6}{3-2s} < q < \infty$, $\frac{3}{q} + \frac{1}{\tilde r} = \frac{3}{2} - s$

$$\Big\| \frac{e^{it\sqrt{H}} P_c}{H^{s/2}} f \Big\|_{L^{r,2}_t (\mathcal{K}^\theta)^*_x \cap L^{\tilde r,2}_t L^{q,2}_x} \lesssim \|f\|_2.$$

By L^2 conservation, this expression is also bounded in $L^\infty_t L^{\frac{6}{3-2s},2}_x$. Setting $\theta = 0$ we also obtain a $L^{\frac{2}{3-2s},2}_t L^\infty_x$ bound.

For the remaining case $0 \leq s \leq 1$, separate the real and imaginary parts of $H^{-1/2} e^{it\sqrt{H}} P_c$ as above. When $s = 1$, a TT^* construction leads to the operator defined by convolution in time with $\frac{\cos(t\sqrt{H})}{H} P_c$. Thanks to the dispersive bound (11) and Young's inequality, it is a bounded linear map from $L^{r'}_t L^{q',2}_x$ to its dual so long as (r, q) is Strichartz-admissible with $r > 2$.

When $s = 0$ the functional calculus of H provides a conservation law in $L^2(\mathbb{R}^3)$. Moreover, every imaginary power of (HP_c) is a partial isometry on L^2. Thus (12) holds whenever $\mathrm{Re}(s) = 1$ and $\mathrm{Re}(s) = 0$, uniformly with respect to the imaginary part of s. Complex interpolation fills in the intermediate cases $0 < s < 1$.

Estimates (13) for the solution of an initial-value problem are equivalent to (12), in view of Lemma 13. □

Proof of Theorem 4. It suffices to prove the second inequality (18). Then (17) follows by the equivalence of homogeneous and perturbed Sobolev spaces set forth in Lemma 13.

The main estimates are another consequence of the formulas

$$\cos(t\sqrt{H}) P_c f_0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (e^{it\lambda} + e^{-it\lambda})(I + \hat{T}^-(\lambda)^*)^{-1} \lambda R_0^+(\lambda^2) f_0\, d\lambda$$

$$\frac{\sin(t\sqrt{H}) P_c}{\sqrt{H}} f_1 = -\frac{1}{2\pi} \int_{-\infty}^{\infty} (e^{it\lambda} - e^{-it\lambda})(I + \hat{T}^-(\lambda)^*)^{-1} R_0^+(\lambda^2) f_1\, d\lambda$$

that appeared in the proof of Theorem 1. Returning to the notation there,

$$\cos(t\sqrt{H})P_c f_0(t,x) = -i\big(S^*(\lambda R_0^+)^\vee f_0)(t,x) + (S^*(\lambda R_0^+)^\vee f_0)(-t,x)\big) \tag{39}$$

$$\frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} f_1(t,x) = -\big((S^*\check{R}_0^+ f_1)(t,x) - (S^*\check{R}_0^+ f_1)(-t,x)\big). \tag{40}$$

Here $\check{R}_0^+(t,x,y)$ is the sine propagator of the free wave equation restricted to $t \le 0$ and $(\lambda R_0^+)^\vee(t,x,y)$ is the free cosine propagator.

It is well known from [KeTa] that the free evolution satisfies the Strichartz estimates $\|\check{R}_0^+ f_1\|_{L^4_{x,t}} \lesssim \|f_1\|_{\dot{H}^{-1/2}}$ and $\|(\lambda R_0^+)^\vee f_0\|_{L^4_{x,t}} \lesssim \|f_0\|_{\dot{H}^{1/2}}$. Because V is assumed to belong to $L^{3/2,1}(\mathbb{R}^3)$, Theorem 14 indicates that $S = (I - T^-)^{-1} \in \mathcal{U}_{L^{4/3}}$. Then $S^* \in \mathcal{U}_{L^4}$ is a bounded operator on $L^4_{x,t}$. This establishes the bound

$$\Big\| \cos(t\sqrt{H})P_c f_0 + \frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} f_1 \Big\|_{L^4_t L^4_x} \lesssim \|f_0\|_{\dot{H}^{1/2}} + \|f_1\|_{\dot{H}^{-1/2}}$$

The other endpoint of the range,

$$\| \cos(t\sqrt{H})P_c f_0 + \frac{\sin(t\sqrt{H})P_c}{\sqrt{H}} f_1 \|_{L^\infty_t L^2_x} \lesssim \|f_0\|_{\dot{H}^1} + \|f_1\|_2,$$

follows directly from the spectral theorem applied to H and from Lemma 13. Complex interpolation proves (18) in the intermediate cases $0 < s < \frac{1}{2}$.

□

Proof of Theorem 5. It suffices to show that $f(t,x) = \cos(t\sqrt{H})P_c f_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f_1$ can be approximated in $L^6_x L^\infty_t$ by functions that are uniformly continuous with respect to t. Under the assumptions of Theorem 1 there are a finite number of eigenvalues, and each eigenfunction ψ_j belongs to $\dot{H}^1(\mathbb{R}^3) \cap \dot{H}^{-1}(\mathbb{R}^3)$. Then P_c acts boundedly on both $f_0 \in \dot{H}^1$ and $f_1 \in L^2$, and the coefficients of ψ_j in the full (i.e. unprojected) evolution are a linear combination of $\cosh(tE_j)$ and $\sinh(tE_j)$, which does not influence the short-time continuity of solutions.

By Lemma 13 the cross-sections $f(t,\,\cdot\,)$ are well defined elements of $\dot{H}^1$ and converge in norm to $f(0,\,\cdot\,) = P_c f_0$ as t approaches zero.

We look at functions of the form $\tilde{f} = \cos(t\sqrt{\tilde{H}})\tilde{P}_c \tilde{f}_0 + \frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c \tilde{f}_1$, where $\tilde{f}_0, \tilde{f}_1 \in C_c^\infty(\mathbb{R}^3)$ are smooth approximations of the initial data and $\tilde{H} = -\Delta + \tilde{V}$ has a smooth compactly supported approximation of the potential. The projection $\tilde{P}_c$ is taken according to the spectral measure of $\tilde{H}$.

Both the continuity of $\tilde{f}$ and its approximation properties are derived from formulas (39) and (40). Recall once more that $\check{R}_0^+(t,x,y)$ is the backwards sine propagator of the free wave equation. Choosing $\tilde{f}_1 \in C_c^\infty(\mathbb{R}^3)$ insures that $\check{R}_0^+\tilde{f}_1$ is Lipschitz continuous where it crosses the plane $\{t = 0\}$ and is globally bounded with bounded derivative. The action of $S^* \in \mathcal{U}_{L^\infty}$ preserves $L^\infty_{x,t}$ norms, but thanks to its convolution structure in the t variable it also preserves Lipschitz continuity in the t direction. Precisely,

$$\begin{aligned} |S^*g(t+\delta,x) - S^*g(t,x)| &= |S^*(g(\,\cdot\,+\delta,x) - g(\,\cdot\,,x))| \\ &\le \|S^*\|_{\mathcal{U}_{L^\infty}} \|g(\,\cdot\,+\delta,x) - g(\,\cdot\,,x)\|_{L^\infty_{x,t}}. \end{aligned}$$

It follows that $\frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c \tilde{f}_1$ is uniformly Lipschitz. The density of smooth test functions in L^2 combined with (16) shows that it can be used to approximate $\frac{\sin(t\sqrt{H})}{\sqrt{H}} P_c f_1$ in the space $L^6_x L^\infty_t$.

Proving continuity of the cosine evolution is slightly more complicated. Within the formula

$$\cos(t\sqrt{\tilde{H}})\tilde{P}_c \tilde{f}_0(t,x) = -i\big((\tilde{S}^*(\lambda R_0^+)^\vee \tilde{f}_0)(t,x) + (\tilde{S}^*(\lambda R_0^+)^\vee \tilde{f}_0)(-t,x)\big)$$

one encounters the obstruction that $(\lambda R_0^+)^\vee \tilde{f}_0$ has a jump discontinuity across the plane $\{t = 0\}$ of size $\tilde{f}_0(x)$, even though it is smooth elsewhere. Imitating the argument above with $\tilde{S}^* \in \mathcal{U}_{L^\infty}$ only shows that $\tilde{S}^*(\lambda R_0^+)^\vee \tilde{f}_0$ has bounded variation in the t direction.

Recall that the Fourier transform of $\tilde{S}^*$ is $(I + R_0^+(\lambda^2)\tilde{V})^{-1}$. Then

$$i\cos(t\sqrt{\tilde{H}})\tilde{P}_c \tilde{f}_0\Big|_{t<0} = \tilde{S}^*(\lambda R_0^+)^\vee \tilde{f}_0 = (\lambda R_0^+)^\vee \tilde{f}_0 - \tilde{S}^* \check{R}_0^+ \tilde{V}(\lambda R_0^+)^\vee \tilde{f}_0.$$

This is more or less a restatement of Duhamel's formula over the half-space $\{t < 0\}$, since $\check{R}_0^+$ and $(\lambda R_0^+)^\vee$ are the backward fundamental solutions of the free wave equation. On the assumption that $\tilde{V} \in C_c^\infty(\mathbb{R}^3)$, we have that $\tilde{V}(\lambda R_0^+)^\vee \tilde{f}_0$ is bounded with bounded derivatives when $t < 0$, and consequently $\check{R}_0^+(\lambda R_0^+)^\vee \tilde{f}_0$ is uniformly Lipschitz.

From here the previous argument applies to show that $\cos(t\sqrt{\tilde{H}})\tilde{P}_c \tilde{f}_0$ is uniformly Lipschitz when $t < 0$. It is an even function of t, thus the same Lipschitz constant is valid over the entire range $t \in (-\infty, \infty)$.

By (16), $\tilde{f}_0$ can be chosen so that $\cos(t\sqrt{\tilde{H}})\tilde{P}_c \tilde{f}_0$ is a close approximation to $\cos(t\sqrt{\tilde{H}})\tilde{P}_c f_0$. The remaining task is to show that the spectral multiplier $\cos(t(-\Delta + V))$ varies continuously with $V \in \mathcal{K}_0$ in the absence of nonnegative eigenvalues and resonances.

This is easiest to accomplish when $V \in L^{3/2,1}$. In that case both $(T^-)^*$ and S^* belong to $\mathcal{U}_{L^{6,2}}$ by Theorem 14. Furthermore $\tilde{S}^* = (\mathbf{1} + (\tilde{T}^-)^*)^{-1}$ has a comparable norm in $\mathcal{U}_{L^{6,2}}$ provided $\|V - \tilde{V}\|_{L^{3/2,1}}$ is sufficiently small. Then the difference between S^* and $\tilde{S}^*$ is controlled by

$$\|S^* - \tilde{S}^*\|_{\mathcal{U}_{L^6}} = \|S^*(T^- - \tilde{T}^-)\tilde{S}^*\|_{\mathcal{U}_{L^6}} \lesssim \|S^*\|^2_{\mathcal{U}_{L^6}} \|V - \tilde{V}\|_{L^{3/2,1}}.$$

Since $\mathcal{U}_{L^6} \subset \mathcal{B}(L^6_x L^\infty_t)$ it follows that

$$i\cos(t\sqrt{\tilde{H}})\tilde{P}_c f_0 - i\cos(t\sqrt{H})P_c f_0\Big|_{t<0} = (S^* - \tilde{S}^*)(\lambda R_0^+)^\vee f_0$$

has $L^6_x L^\infty_t$ norm controlled by $\|S^*\|^2_{\mathcal{U}_{L^6}} \|V - \tilde{V}\|_{L^{3/2,1}} \|f_0\|_{\dot{H}^1}$.

In the more general case where $V \in \mathcal{K}_0$, bounds in $\mathcal{U}_{L^6}$ are not directly available. The difference between the the evolution of H and $\tilde{H}$ is instead estimated by TT^* arguments and interpolation, similar to the proof of Theorem 1. Set

$$Tf_0 = (S^* - \tilde{S}^*)\big(\cos(t\sqrt{-\Delta})f_0\big|_{t<0}\big),$$

with $f_0 \in \dot{H}^1(\mathbb{R}^3)$, which makes

$$TT^* = \frac{1}{2}(S^* - \tilde{S}^*)\Big(\frac{\cos((t+s)\sqrt{-\Delta}) - \cos((t-s)\sqrt{-\Delta})}{-\Delta}\Big|_{s,t<0}\Big)(S - \tilde{S}).$$

The central operator maps $L^1_{x,t}$ to $\mathcal{K}^*_x L^\infty_t$. Meanwhile $\|S-\tilde S\|_{\mathcal{U}_{L^1}}$ is controlled by $\|S\|^2_{\mathcal{U}_{L^1}}\|V-\tilde V\|_{\mathcal{K}}$ as above, and similarly $\|S^*-\tilde S^*\|_{\mathcal{U}_{\mathcal{K}^*}} \lesssim \|S^*\|^2_{\mathcal{U}_{\mathcal{K}^*}}\|V-\tilde V\|_{\mathcal{K}}$. Thus the norm of $TT^* : L^1_{x,t} \to \mathcal{K}^*_x L^\infty_t$ is less than a constant times $\|V-\tilde V\|^2_{\mathcal{K}}$.

Taking adjoints, the norm of $TT^* : \mathcal{K}_x L^1_t \to L^\infty_{x,t}$ has the same bound. Real interpolation provides the desired mapping estimate between $L^{6/5}_x L^1_t$ and $L^6_x L^\infty_t$, with the conclusion that

$$\|\cos(t\sqrt{H})P_c f_0 - \cos(t\sqrt{\tilde H})\tilde P_c f_0\|_{L^6_x L^\infty_t} \lesssim \|S\|_{\mathcal{U}_{L^1}}\|S\|_{\mathcal{U}_{\mathcal{K}}}\|V-\tilde V\|_{\mathcal{K}}\|f_0\|_{\dot H^1}.$$

□

References

[Bec] M. Beceanu, *New estimates for a time-dependent Schrödinger equation*, Duke Math. J. 159 (2011), no. 3, pp. 417–477.

[BeGo] M. Beceanu, M. Goldberg, *Schrödinger dispersive estimates for a scaling-critical class of potentials*, Comm. Math. Phys. 314 (2012), no. 2, 471–481.

[BPST] N. Burq, F. Planchon, J. Stalker, S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential*, Journal of Functional Analysis 203 (2003), pp. 519– 549.

[BPST2] N. Burq, F. Planchon, J. Stalker, S. Tahvildar-Zadeh, *Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay*, Indiana Univ. Math. J. 53, (6), 2004, pp. 1665–1680.

[DaPi] P. D'Ancona, V. Pierfelice *On the wave equation with a large rough potential*, J. Funct. Anal. 227 (2005), no. 1, pp. 30–77.

[Fos] D. Foschi, *Inhomogeneous Strichartz estimates*, J. Hyperbolic Differ. Equ. 2 (1) (2005), pp. 1–24.

[GiVe1] J. Ginibre, G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys., 123 (1989), pp. 535–573.

[GiVe2] J. Ginibre, G. Velo, *Generalized Strichartz inequalities for the wave equation*, J. Func. Anal., 133 (1995), pp. 50– 68.

[GKV] V. Georgiev, G. Karadzhov, N. Visciglia, *Endpoint Strichartz estimates for the wave equation in the critical case*, Pubbl. CRM Ennio Giorgi, Scuola Norm. Sup., Pisa 1 (2004), 225–233.

[Gol] M. Goldberg, *Dispersive estimates for Schrödinger operators with measure-valued potentials in* $\mathbb{R}^3$, to appear in Indiana Univ. Math. J.

[GoSc] M. Goldberg, W. Schlag, *A limiting absorption principle for the three-dimensional Schrödinger equation with* L^p *potentials*, Intl. Math. Res. Not. 2004:75 (2004), pp. 4049-4071.

[Har] J. Harmse, *On Lebesgue space estimates for the wave equation*, Indiana Univ. Math. J. 39 (1990), no. 1, pp. 229–248.

[Hun] R.A. Hunt, *An extension of the Marcinkiewicz interpolation theorem to Lorentz spaces*, Bull. Amer. Math. Soc. 70 (1964), pp. 803–807.

[IoJe] A. Ionescu, D. Jerison, *On the absence of positive eigenvalues of Schrödinger operators with rough potentials*, Geom. and Funct. Anal. 13 (2003), no. 5, pp. 1029–1081.

[Kap] L. Kapitanski, *Some Generalizations of the Strichartz-Brenner Inequality*, Leningrad Math. J., 1 (1990), pp. 693–726.

[KSS] M. Keel, H. Smith, C. Sogge, *On global existence for nonlinear wave equations outside of convex obstacles*, Amer. J. Math. 122 (2000), no. 4, pp. 805–842.

[KeTa] M. Keel, T. Tao, *Endpoint Strichartz estimates*, Amer. Math. J. 120 (1998), pp. 955–980.

[KPV1] C.E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. 40 (1991) pp. 33–69.

[KPV2] C.E. Kenig, G. Ponce and L. Vega, *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc. 4 (1991), pp. 323–347.

[KRS] C.E. Kenig, A. Ruiz, C.D. Sogge, *Uniform Sobolev inequalities and unique continuation for second-order constant coefficient differential operators*, Duke Math. J. 55 (1987), pp. 329–347.

[Kla] S. Klainerman, *On "almost global" solutions to quasilinear wave equations in three space dimensions*, Comm. Pure Appl. Math. 36 (1983), no. 3, pp. 325–344.

[LiZh] D. Li, X. Zhang, Smoothing estimates of the radial Schr"odinger propagator in dimensions $n \geq 2$, Acta Mathematica Scientia 2010,30B(6), pp. 2103–2109.

[LiSo] H. Lindblad, C. D. Sogge, *On Existence and Scattering with Minimal Regularity for Semilinear Wave Equations*, Jour. Func. Anal., 130 (1995), pp. 357–426.

[MSS] G. Mockenhaupt, A. Seeger, C. D. Sogge, *Local Smoothing of Fourier Integrals and Carleson-Sjölin Estimates*, J. Amer. Math. Soc., 6 (1993), pp. 65–130.

[Mor] C. Morawetz, Time decay for the nonlinear Klein-Gordon equation. Proc. Roy. Soc. A 306, pp. 291–296, 1968.

[NaOz] M. Nakamura, T. Ozawa, *The Cauchy problem for nonlinear Klein–Gordon equations in the Sobolev spaces*, Publ. Res. Inst. Math. Sci. 37 (3) (2001), pp. 255–293.

[Obe] D. M. Oberlin, *Convolution estimates for some distributions with singularities on the light cone*, Duke Mathematical Journal 59 (1989), no. 3, pp. 747–757.

[RoSc] I. Rodnianski, W. Schlag, *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*, Invent. Math. 155 (2004), no. 3, pp. 451–513.

[RoVi] K. M. Rogers, P. Villarroya, *Sharp estimates for maximal operators associated to the wave equation*, Arkiv för Matematik, Volume 46 (2008), no. 1, pp. 143–151.

[Seg] I. E. Segal, *Space-time Decay for Solutions of Wave Equations*, Adv. Math., 22 (1976), pp. 305–311.

[Sim] B. Simon, *Schrödinger semigroups*, Bull. Amer. Math. Soc. 7 (1982), pp. 447–526.

[Ste] E. Stein, *Harmonic Analysis*, Princeton University Press, Princeton, 1994.

[Str] R. Strichartz, *A priori estimates for the wave equation and some applications*, Journal of Functional Analysis, 5 (1970), pp. 218–235.

[Yaj] K. Yajima, *The $W^{k,p}$-continuity of wave operators for Schrödinger operators*, Proc. Japan Acad., 69, Ser. A (1993), pp. 94–99.

School of Mathematics, Institute for Advanced Study, Princeton, NJ, 08540
E-mail address: `mbeceanu@math.rutgers.edu`

Department of Mathematics, University of Cincinnati, Cincinnati, OH 45221-0025
E-mail address: `Michael.Goldberg@uc.edu`