SCHRÖDINGER DISPERSIVE ESTIMATES FOR A SCALING-CRITICAL CLASS OF POTENTIALS

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ABSTRACT. We prove a dispersive estimate for the evolution of Schrödinger operators $H = -\Delta + V(x)$ in $\mathbb{R}^3$. The potential should belong to the closure of $C_b^c(\mathbb{R}^3)$ with respsect to the global Kato norm. Some additional spectral conditions are imposed, namely that no resonances or eigenfunctions of H exist anywhere within the interval $[0, \infty)$. The proof is an application of a new version of Wiener's L^1-inversion theorem.

1. INTRODUCTION

Solutions to the linear Schrödinger equation are governed by a number of dispersive and smoothing estimates. These inequalities place limits on the types of singularities that can arise as well as the length of time they are allowed to persist. On short time scales the dispersive bounds are a useful stepping stone toward a nonlinear local existence theory, and on long time scales they enable analysis of asymptotic properties and scattering behavior. We will examine the use of initial data in $L^1(\mathbb{R}^3)$ to control the supremum norm of the solution at later times. Mappings between those spaces are described fully by the pointwise size of the propagator kernel without regard to oscillations in sign.

To see that oscillatory integrals play a major role at every other step of the computation, consider the case of the free Schrödinger equation. Intial data at time zero is brought forward to time t through the action of a Fourier multiplier $e^{it\Delta}$. In spatial variables this is equivalent to convolution against the complex Gaussian kernel $(-4\pi it)^{-3/2}e^{i(|x|^2/4t)}$. The decay rate of $|t|^{-3/2}$ (more generally $|t|^{-n/2}$ when the equation is set in $\mathbb{R}^n$) arises as a consequence of stationary phase within the Fourier inversion integral. It immediately follows that the free evolution satisfies a dispersive bound

$$\|e^{it\Delta}f\|_\infty \leq (4\pi|t|)^{-3/2}\|f\|_1 \tag{1}$$

at all times $t \neq 0$. In this paper we seek to prove similar estimates for the time evolution e^{-itH} induced by a perturbed Hamiltonian $H = -\Delta + V(x)$. The class of admissible potentials will be defined with respect to the global Kato norm

$$\|V\|_{\mathcal{K}} = \sup_{y\in\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x)|}{|x-y|}\, dx. \tag{2}$$

Date: September 12, 2010.

The first author was supported by the ANR program PREFERED in the process of writing this paper.

The second author received support from NSF grants DMS-0600925 and DMS-0901063 during the preparation of this work.

In relation to other usual classes of functions, $L^{3/2-\epsilon} \cap L^{3/2+\epsilon} \subset L^{3/2,1} \subset \mathcal{K}$, where $L^{3/2,1}$ is a Lorentz space.

We assume that V belongs to $\mathcal{K}_0 \subset \mathcal{K}$, the norm-closure of the bounded, compactly supported functions within $\mathcal{K}$. This subspace carries the Kato norm, which is homogeneous with respect to the scaling $V_r(x) = r^2 V(rx)$, $r > 0$. There are no further restrictions on the size of V or its negative or imaginary parts.

Dispersive estimates cannot hold for all initial data if H possesses one or more bound states. Whenever there exists a nonzero function $\Psi \in L^1(\mathbb{R}^3)$ that solves the eigenvalue equation $H\Psi = \lambda\Psi$, the associated Schrödinger evolution

$$e^{-itH}\Psi = e^{-it\lambda}\Psi$$

maintains a constant amplitude for all times in violation of (1). Bound states are typically removed from consideration by applying a spectral projection to the initial data. A revised dispersive estimate for $H = -\Delta + V$ might take the form

$$\left\| e^{-itH}(I - \textstyle\sum_j P_{\lambda_j}(H))f \right\|_\infty \leq |t|^{-\frac{3}{2}} \|f\|_1 \tag{3}$$

where $P_{\lambda_j}(H)$ is a projection onto the point spectrum of H at the eigenvalue λ_j.

One additional concern here is the possible existence of resonances, which are solutions of the equation $\psi + (-\Delta - \lambda \pm i0)^{-1}V\psi = 0$ that do not decay rapidly enough to belong to $L^2(\mathbb{R}^3)$ but instead satisfy $(1 + |x|)^{-s}\psi \in L^2$ for every $s > \frac{1}{2}$. Applying $-\Delta - \lambda$ to both sides shows that $H\psi = \lambda\psi$ for these functions as well. Resonances exhibit enough persistence behavior (by virtue of their resemblance to L^2 bound states) to negate most dispersive estimates, but they cannot be so easily removed with a spectral projection.

Theorem 1. *Let $V \in \mathcal{K}_0$ be a real-valued potential for which the Schrödinger operator $H = -\Delta + V$ has no resonances or eigenvalues within the halfline $[0, +\infty)$. Then*

$$\left\| e^{-itH}(I - P_{pp}(H))f \right\|_\infty \lesssim |t|^{-\frac{3}{2}} \|f\|_1. \tag{4}$$

Under these conditions, the spectral multiplier $P_{pp}(H) := \sum_j P_{\lambda_j}(H)$ is a finite-rank projection.

Remark 1. No part of the proof of Theorem 1 relies on the potential $V(x)$ being real-valued or on H being self-adjoint (the use of spectral measures is just a convenience). As stated the theorem is equally true for complex $V \in \mathcal{K}_0$ so long as H satisfies the assumed spectral conditions. In the complex case the projections $P_{\lambda_j}(H)$ should be constructed to take into account the entire generalized eigenspace over λ_j. Even so they are still finite-rank operators that can be recovered from the (analytic) functional calculus of H in the neighborhood of each $\lambda_j \in \sigma(H)$.

Based on the commutation relation between dilations and the Laplacian, a potential of the form $V_r(x) = r^2 V(rx)$ is guaranteed to produce the same dispersive bounds as V itself. For this reason we regard an inverse-square law for potentials as being critical with respect to scaling. In the explicit case $V(x) = C|x|^{-2}$, dispersive bounds are true only when $C \geq 0$ [**?**]. If the pointwise decay rate is further relaxed to any lower power $|V(x)| \leq C|x|^{-2+\varepsilon}$, the dispersive bound may fail even for non-negative potentials [7]. We note that the global Kato norm, and membership in the class $\mathcal{K}_0$, are preserved among all members of a family V_r, $r > 0$.

The first dispersive estimate of the form (4) was proved in [8] for real potentials satisfying a regularity hypothesis $\hat{V} \in L^1(\mathbb{R}^3)$ and pointwise decay bound $|V(x)| \lesssim$

$\langle x\rangle^{-7-\varepsilon}$. Successive improvements ([11], [6], and [5] in chronological order) relaxed the requirements on V down to the condition $V \in L^{\frac{3}{2}-\varepsilon}(\mathbb{R}^3) \cap L^{\frac{3}{2}+\varepsilon}(\mathbb{R}^3)$. In terms of homogeneous functions, this permits local singularities on the order of $|x|^{-2+\varepsilon}$ and decay at the rate $|x|^{-2-\varepsilon}$ for large x. In all these results H is assumed not to possess an eigenvalue or resonance at zero.

Each of the above conditions leads to a situation where the measured "size" of a rescaled potential V_r increases without bound as $r \to 0$ and $r \to \infty$. Prior to the current work the only scale-invariant class of potentials known to produce a dispersive estimate is based on a smallness condition $\|V\|_{\mathcal{K}} < 4\pi$ that makes the perturbation series absolutely convergent [9]. Eigenvalues and resonances are not possible (at zero or elsewhere) for such small potentials.

The proof of Theorem 1 is based on a broad extension of Wiener's L^1-inversion theorem to operator-valued functions, first observed in [2]. In the one-dimensional setting, scaling-critical dispersive estimates were established with the help of the classical (scalar) Wiener inversion theorem instead [6]. There it is invoked at a crucial juncture to show that a particular quotient of functions has integrable Fourier transform. Keeping the denominator nonzero ends up being equivalent to the absence of resonances.

Our generalized inversion theorem plays a similar role here with operator inverses taking the place of a quotient. The precise statement and proof, which contain the result in [2] as a special case, are given in Section 3. Once again the spectral property required to apply the theorem to Schrödinger's equation is contingent on keeping the continuous spectrum of H free from eigenvalues and resonances.

The mathematical argument divides neatly into three parts. Section 2 outlines the decomposition of e^{-itH} according to the spectral measure of H and reduces the dispersive bound to a desired estimate in L^1. Section 3 introduces the machinery related to Wiener's inversion theorem, and in the concluding section we verify that the desired estimate fits into this abstract framework.

The same basic method also applies to the wave equation propagators $\cos(t\sqrt{H})$ and $\frac{\sin(t\sqrt{H})}{\sqrt{H}}$, yielding estimates in a variety of L^p and Sobolev spaces. The details of these cases will be presented in a separate paper [3] so that the present exposition can focus on operator-theoretic concerns with a relatively small number of side calculations.

2. Reduction to Resolvent Estimates

The derivation of dispersive estimates from properties of the free and perturbed resolvents is a standard practice for time-independent Schrödinger operators on $\mathbb{R}^n$. In fact it is formally equivalent to methods based on the Duhamel propagation formula, under the Fourier duality pairing of the time variable t with the spectral parameter λ. The most prominent feature in Duhamel's formula is a convolution integral (in t) against the free propagator kernel. In the dual setting this appears instead as a pointwise operator acting on functions of λ.

Let $H = -\Delta + V$ in $\mathbb{R}^3$ and for each $z \in \mathbb{C} \setminus \mathbb{R}^+$ define the resolvents $R_0(z) := (-\Delta - z)^{-1}$ and $R_V(z) := (H - z)^{-1}$. The operators $R_0(z)$ are all bounded on $L^2(\mathbb{R}^3)$ and act explicitly by convolution with the kernel

$$R_0(z)(x, y) = \frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|},$$

where $\sqrt{z}$ is taken to have positive imaginary part. While $R_V(z)$ is not translation–invariant and there is no simple formula for its integral representation, it can be expressed in terms of $R_0(z)$ via the identity

$$R_V(z) = (I + R_0(z)V)^{-1}R_0(z) = R_0(z)(I + VR_0(z))^{-1}. \tag{5}$$

In the case where $z = \lambda \in \mathbb{R}^+$, the resolvent may be defined as a limit of the form $R_0(\lambda \pm i0) := \lim_{\varepsilon\downarrow 0} R_0(\lambda \pm i\varepsilon)$. The choice of sign determines which branch of the square-root function is selected in the formula above, therefore the two continuations do not agree with one another. In this paper we refer to resolvents along the positive real axis using the following notation.

$$R_0^\pm(\lambda) := R_0(\lambda \pm i0) \qquad R_V^\pm(\lambda) := R_V(\lambda \pm i0)$$

Every perturbation $V \in \mathcal{K}_0$ satisfies the local Kato condition

$$\lim_{\delta\to 0} \sup_{y\in\mathbb{R}^3} \int_{|x-y|<\delta} \frac{|V(x)|}{|x-y|}\, dx = 0. \tag{6}$$

If V has compact support then (6) is even equivalent to membership in $\mathcal{K}_0$. This degree of local regularity is sufficient to to conlude that $H = -\Delta + V$ is essentially self-adjoint with spectrum bounded below by $-M$ for some $M < \infty$ [10]. The Stone formula for the absolutely continuous spectral measure of H then dictates that

$$e^{-itH} f = \sum_j e^{-i\lambda_j t} P_{\lambda_j}(H) f + \frac{1}{2\pi i} \int_0^\infty e^{-it\lambda} [R_V^+(\lambda) - R_V^-(\lambda)] f\, d\lambda. \tag{7}$$

Because zero is assumed to be a regular point of the spectrum, the summation contains only a finite set of eigenvalues λ_j which are removed by the projection $I - P_{pp}(H)$. Once the initial sum is forced to vanish, dispersive estimates will succeed or fail based on the behavior of the integral term.

It is customary to view the right-hand integral as a contour integral in the complex plane along a path that circles $\mathbb{R}^+$ clockwise. Making a change of variables $\lambda \mapsto \lambda^2$ opens up the contour to the entire real axis, with the understanding that

$$R_V^+(\lambda^2) = \lim_{\varepsilon\downarrow 0} R_V((\lambda + i\varepsilon)^2) = \lim_{\varepsilon\downarrow 0} R_V(\lambda^2 + i \operatorname{sign}(\lambda)\varepsilon)$$

for all $\lambda \in \mathbb{R}$. Written out this way the integral term in (7) simplifies to

$$\begin{aligned} &\frac{1}{\pi i} \int_{-\infty}^\infty e^{-it\lambda^2} \lambda R_V^+(\lambda^2) f\, d\lambda \\ &\quad = \frac{1}{\pi i} \int_{-\infty}^\infty e^{-it\lambda^2} \lambda R_0^+(\lambda^2)(I + VR_0^+(\lambda^2))^{-1} f\, d\lambda. \end{aligned}$$

A formal integration by parts leads to the expression

$$\frac{1}{2\pi t} \int_{-\infty}^\infty e^{-it\lambda^2} \big(I + R_0^+(\lambda^2)V\big)^{-1} \frac{d}{d\lambda}\Big[R_0^+(\lambda^2)\Big] \big(I + VR_0^+(\lambda^2)\big)^{-1} f\, d\lambda.$$

If we adopt the shorthand notation $\hat{T}^\pm(\lambda) := VR_0^\pm(\lambda^2)$, Theorem 1 should follow from the estimate

$$\Big| \int_{-\infty}^\infty e^{-it\lambda^2} \Big\langle \frac{d}{d\lambda}\Big[R_0^+(\lambda^2)\Big](I + \hat{T}^+(\lambda))^{-1} f,\, (I + \hat{T}^-(\lambda))^{-1} g\Big\rangle\, d\lambda \Big| \lesssim |t|^{-\frac{1}{2}} \|f\|_1 \|g\|_1. \tag{8}$$

In fact Theorem 1 only requires (8) to hold for all $f, g \in \mathrm{ran}(I - P_{pp}(H)) \subset L^1$. The extra restriction on the class of test functions is crucial when H possesses an eigenvalue at zero but it is not needed here.

The choice of notation is designed to indicate that $\hat{T}^{\pm}(\lambda)$ exists as the Fourier transform of an important family of operators $T^{\pm}(\rho)$, with ρ serving as the variable dual to λ. In particular, note that the integral in (8) can be estimated with the help of Plancherel's identity. Within the integrand there is a factor of $e^{-it\lambda^2}\frac{d}{d\lambda}[R_0^+(\lambda^2)]$. This family of integral operators has an explicit kernel representation

$$K(\lambda, x, y) = (-4\pi i)^{-1} e^{-it\lambda^2 + i\lambda|x-y|}.$$

Its inverse Fourier transform is also a family of convolution operators, with kernel

$$\check{K}(\rho, x, y) = (64\pi^3 it)^{-\frac{1}{2}} e^{-i\frac{(\rho+|x-y|)^2}{4t}}.$$

The expression for $\check{K}$ is bounded by $|t|^{-\frac{1}{2}}$ for every value of ρ, x, and y. Therefore it suffices to prove that the inverse Fourier transforms of $(I + \hat{T}^+(\lambda))^{-1}f$ and $(I + \hat{T}^-(\lambda))^{-1}g$ both satisfy a corresponding L^1 estimate. Our goal is then to prove the following mapping properties for $T^{\pm}(\rho)$.

Theorem 2. *Let $V \in \mathcal{K}_0$ be a scalar potential in $\mathbb{R}^3$ and suppose that $H = -\Delta + V$ has no resonances or eigenvalues along the interval $[0, \infty)$. Then*

$$\|T^{\pm}(\rho)f\|_{L^1_\rho L^1_x} \leq \left(\frac{\|V\|_{\mathcal{K}}}{4\pi}\right)\|f\|_1$$

$$\text{and} \quad \left\|\left((I + \hat{T}^{\pm})^{-1} - I\right)^{\vee}(\rho)f\right\|_{L^1_\rho L^1_x} \leq C\|f\|_1$$

for all $f \in L^1(\mathbb{R}^3)$.

The same approach is taken in [5] where the potential is instead assumed to belong to $L^{\frac{3}{2}-\varepsilon}(\mathbb{R}^3) \cap L^{\frac{3}{2}+\varepsilon}(\mathbb{R}^3)$. The scale-invariant L^p space for Schrödinger potentials is $L^{\frac{3}{2}}(\mathbb{R}^3)$ exactly, so the intersection condition demands more decay at infinity and better local regularity than scaling arguments alone would suggest. We present Theorem 2 as an application of the operator-valued Wiener Theorem in Section 3. This method has two principal advantages over the previous work. First, the result is sharper: The space $\mathcal{K}_0$ includes all other known admissible classes of potentials and the global Kato norm remains invariant under scaling transformations $V(x) \mapsto r^2 V(rx)$. Second, the proof is considerably cleaner as many of the delicate L^p resolvent bounds are replaced with a crude but effective limiting argument.

For small potentials with $\|V\|_{\mathcal{K}} < 4\pi$, the constant in the second ineqality can be bounded by $(1 - \|V\|_{\mathcal{K}}/4\pi)^{-1}$. Dispersive estimates for time-dependent potentials below this size threshold are proved in [9]. When the potential is large the constant depends more heavily on spectral properties (e.g. the avoidance of resonances) and is not directly tied to the size of V.

3. An Operator-Valued Wiener Theorem

Given a Banach space X, let $\mathcal{W}_X$ be the space of bounded linear maps $T : X \to L^1(\mathbb{R}; X)$. In the event that X is not separable, the target space $L^1(\mathbb{R}; X)$ is defined to be the norm-closure of simple functions.

$\mathcal{W}_X$ contains, for example, all measurable operator-valued functions $T : \mathbb{R} \to \mathcal{B}(X)$ for which $\|T(\rho)\|_{\mathcal{B}(X)}$ is a.e. finite and the integral

$$\int_{\mathbb{R}} \|T(\rho)\|_{\mathcal{B}(X)}\, d\rho \tag{9}$$

is also finite. However, not all elements of $\mathcal{W}_X$ have this form, as indicated in [2].

$\mathcal{W}_X$ is an algebra under the formal convolution-composition product

$$S * T(\rho)f = \int_{\mathbb{R}} S(\rho-\sigma)T(\sigma)f\, d\sigma \tag{10}$$

whose boundedness follows from standard L^1 arguments including approximating T by an operator for which Tf is a simple function.

There is a well-defined Fourier transform for elements of $\mathcal{W}_X$, namely

$$\hat{T}(\lambda)f = \int_{\mathbb{R}} e^{-i\lambda\rho}T(\rho)f\, d\rho.$$

With no further assumptions on T one can determine that $\hat{T}$ is a strongly continuous family of operators with $\sup_\lambda \|\hat{T}(\lambda)\|_{\mathcal{B}(X)} \leq \|T\|_{\mathcal{W}_X}$. In addition $\hat{T}(\lambda)$ converges to zero in the strong operator topology as $|\lambda| \to \infty$, by the Riemann-Lebesgue lemma.

There is no naturally occuring multiplicative identity in $\mathcal{W}_X$. Let $\overline{\mathcal{W}}_X$ denote the extension of $\mathcal{W}_X$ to include complex multiples of an identity element, with the norm $\|z\mathbf{1}+T\|_{\overline{\mathcal{W}}_X} = |z|+\|T\|_{\mathcal{W}_X}$. If necessary one can write $\mathbf{1}(\rho) = \delta_0(\rho)I$, where I is the identity in $\mathcal{B}(X)$, and its Fourier transform is the constant function $\hat{\mathbf{1}}(\lambda) = I$. We note (as an aside) that $\mathcal{W}_X$ and $\overline{\mathcal{W}}_X$ both embed isometrically as subalgebras of $\mathcal{B}(L^1(\mathbb{R};X))$.

As usual the Fourier transform intertwines convolution products in $\overline{\mathcal{W}}_X$ with pointwise multiplication in $C_0^{\text{strong}}(\mathbb{R};\mathcal{B}(X))$. For any pair of elements $z_1\mathbf{1}+S$, $z_2\mathbf{1}+T \in \overline{\mathcal{W}}_X$ we have

$$\big[(z_1\mathbf{1}+S)*(z_2\mathbf{1}+T)\big]\hat{}\,(\lambda) = (z_1I+\hat{S}(\lambda))(z_2I+\hat{T}(\lambda)). \tag{11}$$

Based on the product formula, any invertible element of $\overline{\mathcal{W}}_X$ must possess a Fourier transform that is invertible at every $\lambda \in \mathbb{R}$, with uniformly bounded inverses. An ideal Wiener theorem might show that these conditions are sufficient for invertibility in $\overline{\mathcal{W}}_X$. The version that we prove here includes modest assumptions about the continuity and locality of T.

Theorem 3. *Suppose T is an element of $\mathcal{W}_X$ satisfying the properties*

(C1) $\lim_{\delta\to 0} \|T(\rho)-T(\rho-\delta)\|_{\mathcal{W}_X} = 0.$

(C2) $\lim_{R\to\infty} \|T\chi_{|\rho|\geq R}\|_{\mathcal{W}_X} = 0.$

If $I+\hat{T}(\lambda)$ is an invertible element of $\mathcal{B}(X)$ for every $\lambda \in \mathbb{R}$, then $\mathbf{1}+T$ possesses an inverse in $\overline{\mathcal{W}}_X$ of the form $\mathbf{1}+S$.

In fact it is only necessary for some finite power $T^N \in \mathcal{W}_X$ (using the definition of products in $\mathcal{W}_X$ given by (10)*) to satisfy the translation-continuity condition* (C1) *rather than T itself.*

Proof. It suffices to show that $(I+\hat{T}(\lambda))^{-1}$ is the Fourier transform of an element $\mathbf{1}+S \in \overline{\mathcal{W}}_X$. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a standard cutoff function. For any real number L

one can express $(1-\eta(\lambda/L))\hat{T}$ as the Fourier transform of

$$S_L(\rho) = \big(T - L\check{\eta}(L\,\cdot\,) * T\big)(\rho) = \int_{\mathbb{R}} L\check{\eta}(L\sigma)[T(\rho) - T(\rho-\sigma)]\,d\sigma$$

Thanks to condition (C1), the $\mathcal{W}_X$ norm of the right-hand integral vanishes as $L\to\infty$. This makes it possible to construct an inverse Fourier transform for

$$(1-\eta(\lambda/2L))\big(I+\hat{T}(\lambda)\big)^{-1} = (1-\eta(\lambda/2L))\sum_{k=0}^{\infty}(-1)^k\Big(\big(1-\eta(\lambda/L)\big)\hat{T}(\lambda)\Big)^k$$

via a convergent power series expansion in S_L provided $L \geq L_1$. The $k=0$ term is exactly $\mathbf{1} - 2L\check{\eta}(2L\rho)I$, and every subsequent term belongs to $\mathcal{W}_X$.

If only T^N satisfies (C1) then one constructs an inverse Fourier transform for $(1-\eta(\lambda/2L))(I \pm \hat{T}^N(\lambda))^{-1}$ via the above process and observes that

$$(1-\eta(\lambda/2L))\big(I+\hat{T}(\lambda)\big)^{-1} = (1-\eta(\lambda/2L))\big(I+(-\hat{T}(\lambda))^N\big)^{-1}\sum_{k=0}^{N-1}(-1)^k\hat{T}^k(\lambda).$$

A similar approach works for finding a local inverse in the neighborhood of any $\lambda_0\in\mathbb{R}$. For simplicity, consider the representative case $\lambda_0 = 0$, and let $A_0 = I + \hat{T}(0) \in \mathcal{B}(X)$. One can write $\eta(\lambda/L)(I+\hat{T}(\lambda) - A_0)$ as the Fourier transform of

$$\begin{aligned} S_L(\rho) &= L\check{\eta}(L\,\cdot\,) * T(\rho) - L\check{\eta}(L\rho)(I - A_0) \\ &= \int_{\mathbb{R}} L\big(\check{\eta}(L(\rho-\sigma)) - \check{\eta}(L\rho)\big)T(\sigma)\,d\sigma. \end{aligned}$$

Here we have used the fact that $A_0 = I + \int_{\mathbb{R}} T(\rho)\,d\rho$.

By the mean value theorem, $\int_{\mathbb{R}} L|\check{\eta}(L(\rho-\sigma)) - \check{\eta}(L\rho)|\,d\rho \lesssim \min(L|\sigma|, 1)$. Recall that for any fixed unit vector $f\in X$, the function $T(\sigma)f \in L^1(\mathbb{R};X)$ has norm bounded by $\|T\|_{\mathcal{W}_X}$. If assumption (C2) holds then there is R_ε so that $\|\chi_{|\sigma|>R_\varepsilon}T(\sigma)f\| < \varepsilon\|f\|$ uniformly in the choice of f. It follows that

$$\|S_L(\rho)f\|_{L^1(\mathbb{R};X)} \lesssim (LR_\varepsilon\|T\|_{\mathcal{W}_X} + \varepsilon)\|f\|_X$$

and consequently that $\lim\limits_{L\to 0}\|S_L\|_{\mathcal{W}_X} = 0$.

For any smooth function ϕ supported in $[-\frac{L}{2}, \frac{L}{2}]$, there exists a local Neumann series

$$\begin{aligned} \phi(\lambda)(I+\hat{T}(\lambda))^{-1} &= \phi(\lambda)\big(A_0 + \eta(\lambda/L)(1+\hat{T}(\lambda) - A_0)\big)^{-1} \\ &= \phi(\lambda)A_0^{-1}\big(I + \hat{S_L}(\lambda)A_0^{-1}\big)^{-1} = \phi(\lambda)A_0^{-1}\sum_{k=0}^{\infty}(-1)^k\big(\hat{S_L}(\lambda)A_0^{-1}\big)^k. \end{aligned}$$

So long as L is chosen small enough that $\|S_L\|_{\mathcal{W}_X}\|A_0^{-1}\|_{\mathcal{B}(X)} < 1$, the inverse Fourier transform of this series is convergent in the $\mathcal{W}_X$ norm.

Over the compact interval $\lambda_0\in[-2L_1, 2L_1]$, there is a nonzero lower bound on the length L required for convergence of the above power series. Therefore it is possible to choose a partition of unity on $[-2L_1, 2L_1]$ so that the support of each cutoff ϕ_j has diameter small enough that $\phi_j(\lambda)(I+\hat{T}(\lambda))^{-1}$ is the Fourier transform of an element of $\mathcal{W}_X$. Finally, the inverse Fourier transform of $1-\eta(\lambda/2L)(I+\hat{T}(\lambda))^{-1}$ belongs to the affine space $\mathbf{1}+\mathcal{W}_X \subset \overline{\mathcal{W}}_X$, completing the construction of $(\mathbf{1}+T)^{-1}$. □

4. Proof of Theorem 2

We give the proof for $T^-(\rho)$. Since $R_0^-(\lambda^2) = R_0^+((-\lambda)^2)$ there is no difference between $T^+(\rho)$ and $T^-(\rho)$ except for a reflection along the ρ-axis.

The first statement in Theorem 2 is a direct calculation. Using the fact that in three dimensions $R_0^-(\lambda^2)$ is a convolution against the kernel $(4\pi|x|)^{-1}e^{-i\lambda|x|}$ we can compute that

$$T^-(\rho)f(x) = (4\pi\rho)^{-1}V(x)\int_{|x-y|=\rho} f(y)\,dy. \tag{12}$$

Suppose f is a bounded compactly-supported function in $\mathbb{R}^3$. Then its convolution with the surface measure of a sphere is still bounded and compactly supported. Meanwhile $V \in \mathcal{K}_0$ is locally integrable so $T^-(\rho)f$ belongs to $L^1(\mathbb{R}^3)$ for all $\rho \neq 0$. Finally,

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{\mathbb{R}} |T^-(\rho)f(x)|\,dx\,d\rho &\leq \frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|V(x)|}{|x-y|}|f(y)|\,dy\,dx \\ &\leq \frac{1}{4\pi}\|V\|_{\mathcal{K}}\|f\|_1.\end{aligned}$$

The inequality extends by continuity to all $f \in L^1(\mathbb{R}^3)$, which shows that $\|T^-\|_{\mathcal{W}_{L^1}} \leq \|V\|_{\mathcal{K}}/4\pi$. We remark once again that the individual operators $T^-(\rho)$ need not belong to $\mathcal{B}(L^1(\mathbb{R}^3))$ for the integral inequality to hold.

The second inequality in Theorem 2 is the stronger conclusion. It will be an immediate consequence of Theorem 3, for $X = L^1(\mathbb{R}^3)$, once we verify that T^- satisfies condition (C2) and $(T^-)^N$ satisfies (C1) for some finite N. Pointwise invertibility of $I+\hat{T}^-(\lambda)$ comes from the Fredholm alternative and the assumption that H has no eigenvalues or resonances along its continuous spectrum.

To summarize the invertibility argument: $\hat{T}^-(\lambda) = VR_0^-(\lambda^2)$ is a compact operator on $L^1(\mathbb{R}^3)$ for each λ, improving regularity by two derivatives, if V belongs to C_c^∞. Compactness is preserved by taking norm limits to $V \in \mathcal{K}_0$. By the Fredholm alternative, $I+\hat{T}^-(\lambda_0)$ fails to be invertible only if there exists a function $\varphi \in L^1$ such that $(I+\hat{T}^-(\lambda_0))\varphi = 0$. Then $\psi = R_0^-(\lambda_0^2)\varphi$ would be a solution to $\psi + R_0^-(\lambda_0^2)V\psi = 0$ satisfying $(1+|x|)^{-s}\psi \in L^2(\mathbb{R}^3)$ for all $s > \frac{1}{2}$ because the free resolvent is a bounded map from L^1 into these weighted function spaces.

Either way, H would have an eigenvalue or resonance at λ_0^2 according to whether or not $\psi \in L^2(\mathbb{R}^3)$.

The locality condition (C2) is rather straightforward as well. Suppose V is a bounded function with compact support in a set of diameter D. From (12) it follows that for $R > 2D$

$$\begin{aligned}\int_{\mathbb{R}^3}\int_{|\rho|>R} |T^-(\rho)f(x)|\,dx\,d\rho &\leq \frac{1}{4\pi}\iint_{|x-y|>R}\frac{|V(x)|}{|x-y|}|f(y)|\,dy\,dx \\ &\lesssim R^{-1}\|V\|_1\|f\|_1\end{aligned}$$

Then $\|T^-\chi_{|\rho|>R}\|_{\mathcal{W}_{L^1}} \to 0$ as $R \to \infty$, and this property is preserved under a limiting sequence in $V \in \mathcal{K}$.

Remark 2. Condition (C2) is actually satisfied for the larger class of potentials that possess the "distal Kato property"

$$\lim_{R\to\infty} \sup_{y\in\mathbb{R}^3} \int_{|x-y|>R} \frac{|V(x)|}{|x-y|}\, dx = 0$$

Unlike the assumption $V \in \mathcal{K}_0$, the distal Kato property does not guarantee that operators $VR_0^-(\lambda)$ act compactly on $L^1(\mathbb{R}^3)$. Most technical elements of the dispersive estimate are not affected except for the question of whether $I + VR_0^-(\lambda)$ is invertible pointwise in λ. Normally one uses the Fredholm Alternative to derive it from the absence of embedded eigenvalues and resonances. With this argument unavailable one needs to strengthen the spectral assumptions on H accordingly.

The remaining task is to verify that some power of T^- in $\mathcal{W}_{L^1}$, $(T^-)^N$, satisfies the translation-continuity hypothesis (C1). We proceed using the value $N = 4$. Choose a bounded compactly supported approximation V_ε with $\|V - V_\varepsilon\|_{\mathcal{K}} < \varepsilon$ and $\|V_\varepsilon\|_{\mathcal{K}} \le \|V\|_{\mathcal{K}}$. Let T_ε^- denote the corresponding element of $\mathcal{W}_{L^1}$.

As before the compact support makes it possible to choose R so that $\|T_\varepsilon^- \chi_{|\rho|>R}\|_{\mathcal{W}_{L^1}}$ is smaller than $\varepsilon\|V\|^{-3}$. Then by convolution of support (in ρ) one also has $\|(T_\varepsilon^-)^4 \chi_{|\rho|>4R}\|_{\mathcal{W}_{L^1}} < 4\varepsilon$, with the result that

$$\big\|(T_\varepsilon^-)^4(\rho) - (T_\varepsilon^-)^4(\rho-\delta)\big\|_{\mathcal{W}_{L^1}} \le \big\|((T_\varepsilon^-)^4(\rho) - (T_\varepsilon^-)^4(\rho-\delta))\chi_{|\rho|<4R+1}\big\|_{\mathcal{W}_{L^1}} + 8\varepsilon$$

for any $\delta < 1$. Over the finite interval $|\rho| < 4R + 1$ it will suffice to show that $(T_\varepsilon^-)^4(\rho)$ is in fact a continuously differentiable function taking values in $\mathcal{B}(L^1(\mathbb{R}^3))$.

The kernel of the free resolvent has the pointwise absolute value $|R_0^-(\lambda^2)_{(x,y)}| = \frac{1}{4\pi|x-y|}$ which shows it to be a bounded operator from $L^1(\mathbb{R}^3)$ to $L^2(\mathrm{supp}(V_\varepsilon))$. Meanwhile there is also a known family of weighted L^2 estimates

$$\|\langle x\rangle^{-\alpha} R_0^-(\lambda^2)\langle x\rangle^{-\alpha} f\|_2 \le C_\alpha(1+|\lambda|)^{-1}\|f\|_2$$

for any exponent $\alpha > \frac{1}{2}$ (cf. [1, Theorem 5.1]). Taken in combination these imply that

$$\big\|\big((T_\varepsilon^-)^4\big)\hat{}\,(\lambda)f\big\|_1 = \big\|\big(V_\varepsilon R_0^-(\lambda^2)\big)^4 f\big\|_1 \lesssim (1+|\lambda|)^{-3}\|f\|_1$$

with the constant determined by the maximum size of V_ε and the diameter of its support. With this much decay present as $\lambda \to \infty$ it follows from Fourier inversion that $(T_\varepsilon^-)^4(\rho)$ has a derivative bounded in size by the same constant. Therefore

$$\big\|((T_\varepsilon^-)^4(\rho) - (T_\varepsilon^-)^4(\rho-\delta))\chi_{|\rho|<4R+1}\big\|_{\mathcal{W}_{L^1}} \le C(V_\varepsilon)(4R+1)\delta$$

and δ can be chosen sufficiently small to keep this quantity less than ε. The norm difference between $(T_\varepsilon^-)^4$ and its translation is no greater than 9ε.

The triangle inequality permits a step from T_ε^- back to T^- by giving the bound

$$\begin{aligned}\big\|(T^-)^4(\rho) - (T^-)^4(\rho-\delta)\big\|_{\mathcal{W}_{L^1}} &\le 2\|(T^-)^4 - (T_\varepsilon^-)^4\|_{\mathcal{W}_{L^1}} + 9\varepsilon \\ &\le (C\|V\|_{\mathcal{K}}^3 + 9)\varepsilon\end{aligned}$$

for all δ sufficiently small. Taking ε to zero verifies that condition (C1) is satisfied.

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