TIME INTEGRABLE WEIGHTED DISPERSIVE ESTIMATES FOR THE FOURTH ORDER SCHRÖDINGER EQUATION IN THREE DIMENSIONS

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ABSTRACT. We consider the fourth order Schrödinger operator $H = \Delta^2 + V$ and show that if there are no eigenvalues or resonances in the absolutely continuous spectrum of H that the solution operator e^{-itH} satisfies a large time integrable $|t|^{-\frac{5}{4}}$ decay rate between weighted spaces. This bound improves what is possible for the free case in two directions; both better time decay and smaller spatial weights. In the case of a mild resonance at zero energy, we derive the operator-valued expansion $e^{-itH}P_{ac}(H) = t^{-\frac{3}{4}}A_0 + t^{-\frac{5}{4}}A_1$ where $A_0 : L^1 \to L^\infty$ is an operator of rank at most four and A_1 maps between polynomially weighted spaces.

1. INTRODUCTION

In this paper we study the dynamics of the solution operator to the fourth order Schrödinger equation

$$(1) \qquad i\psi_t = (\Delta^2 + V)\psi, \quad \psi(0, x) = f(x), \qquad x \in \mathbb{R}^3.$$

Here V is a real-valued decaying potential. For convenience, we denote the fourth order Schrödinger operator $H := \Delta^2 + V$, and $P_{ac}(H)$ is projection onto the absolutely continuous subspace of $L^2(\mathbb{R}^3)$ associated to the self-adjoint operator H.

In the free case, when $V = 0$, it is well known (c.f. [2]) that the solution operator satsfies the dispersive estimate

$$\|e^{-it\Delta^2} f\|_{L^\infty(\mathbb{R}^3)} \lesssim |t|^{-\frac{3}{4}} \|f\|_{L^1(\mathbb{R}^3)}.$$

The notation $A \lesssim B$ means there exists an absolute constant C so that $|A| \leq C|B|$. In [9], it was shown that the solution operator $e^{-itH}P_{ac}(H)$ for (1) satisfies the same dispersive bound as above for a generic class of short range perturbations. In this paper we show that a generic perturbation can lead to faster time decay.

Theorem 1.1. *If zero is a regular point of the spectrum of H, and there are no embedded eigenvalues in the spectrum of H, then we have the following*

$$\|e^{-itH}P_{ac}(H)f\|_{L^{\infty,-1}(\mathbb{R}^3)} \lesssim |t|^{-\frac{5}{4}}\|f\|_{L^{1,1}(\mathbb{R}^3)}, \qquad |t| \geq 1$$

provided $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 7$.

Here $\langle x \rangle := (1+|x|^2)^{\frac{1}{2}}$. The polynomially weighted L^p spaces are defined by

$$L^{p,\sigma}(\mathbb{R}^3) = \{f \ : \ \langle \cdot \rangle^\sigma f \in L^p(\mathbb{R}^3)\}, \qquad 1 \leq p \leq \infty, \quad \sigma \in \mathbb{R}.$$

This result shows that that one can obtain faster, time-integrable decay if the perturbation V removes the natural zero energy resonance, at the cost of spatial weights. One can interpolate between the result in Theorem 1.1 and the result in [9] to show the bound

$$(2) \qquad \|e^{-itH}P_{ac}(H)f\|_{L^{\infty,-\sigma}(\mathbb{R}^3)} \lesssim |t|^{-\frac{3}{4}-\frac{\sigma}{2}}\|f\|_{L^{1,\sigma}(\mathbb{R}^3)}, \qquad |t| \geq 1, \quad 0 \leq \sigma \leq 1.$$

The condition that zero energy is regular prohibits the existence of solutions to $H\psi = 0$ with $\psi \in L^\infty(\mathbb{R}^3)$. If there is a non-trivial solution with $\psi \in L^\infty(\mathbb{R}^3)$ we say there is a zero energy resonance (or eigenvalue if $\psi \in L^2(\mathbb{R}^3)$), see [9] for a detailed study of these threshold phenomena. We say ψ is a resonance of the first kind if $\psi \in L^\infty(\mathbb{R}^3)$ but $\psi \notin L^p(\mathbb{R}^3)$ for any $p < \infty$.

The free operator $H_0 = \Delta^2$ is excluded from Theorem 1.1 due to the existence of the zero energy resonance of the first kind, namely the constant function $\psi(x) = 1$. Consequently, it does not obtain the same decay rate. There is an explicit solution formula in the free case

$$e^{-it\Delta^2}f(x) = t^{-\frac{3}{4}}\int_{\mathbb{R}^3}\mathcal{K}\big(t^{-\frac{1}{4}}(x-y)\big)f(y)\,dy,$$

with the kernel $\mathcal{K}(x)$ being the inverse Fourier transform of $e^{-i|\xi|^4}$. Since $\mathcal{K}(x)$ is bounded, smooth, and radially symmetric, it satisfies $|\mathcal{K}(x) - \mathcal{K}(0)| \lesssim |x|^2$, which in turn makes $|\mathcal{K}(t^{-1/4}(x-y)) - \mathcal{K}(0)| \lesssim t^{-\frac{1}{2}}|x-y|^2 \lesssim t^{-\frac{1}{2}}\langle x \rangle^2 \langle y \rangle^2$. If we define P_1 to be the rank-one projection onto the constant function ψ, it follows that

$$\|e^{-it\Delta^2}f - t^{-\frac{3}{4}}\mathcal{K}(0)P_1 f\|_{L^{\infty,-2}(\mathbb{R}^3)} \lesssim |t|^{-\frac{5}{4}}\|f\|_{L^{1,2}(\mathbb{R}^3)}.$$

Remark 2.4 below provides an alternate derivation of this bound for the free evolution. In Theorem 1.2 we show that this behavior is representative of what occurs when there is a resonance of the first kind. Comparing the two statements, we see that when a perturbation removes the resonance at zero, the result of Theorem 1.1 improves upon the free case in two

ways: both the overall time decay and the required spatial weights in the $|t|^{-\frac{5}{4}}$ part of the solution.

Theorem 1.2. *If there is a resonance of the first kind at zero, and there are no embedded eigenvalues in the spectrum of H, then we have the following*

$$\|(e^{-itH}P_{ac}(H) - F_t)f\|_{L^{\infty,-2}(\mathbb{R}^3)} \lesssim |t|^{-\frac{5}{4}}\|f\|_{L^{1,2}(\mathbb{R}^3)}, \qquad |t| \geq 1$$

with F_t a time-dependent operator of rank at most four that satisfies $\|F_t\|_{1\to\infty} \lesssim |t|^{-\frac{3}{4}}$. Provided $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 11$.

Effectively, the resonant perturbed evolution has two pieces. The time-decay of F_t matches the natural decay rate of the free equation, and it has rank at most four, the maximum dimension of the zero energy resonance space if there is only a resonance of the first kind. We explicitly construct this operator in (41) below. The remaining piece enjoys the same rate of time-decay as when zero energy is regular. In all cases the small time estimates are

$$\|e^{-itH}P_{ac}(H)f\|_{L^{\infty}(\mathbb{R}^3)} \lesssim |t|^{-\frac{3}{4}}\|f\|_{L^1(\mathbb{R}^3)}, \qquad |t| < 1,$$

due to the high energy portion of the evolution. Hence the dispersive bounds we prove are integrable in time over the entire real line. If integrability in time is the main concern, our proof of Theorem 1.2 is easily modified to show the bound

$$\|(e^{-itH}P_{ac}(H) - F_t)f\|_{L^{\infty,-2}(\mathbb{R}^3)} \lesssim |t|^{-1-}\|f\|_{L^{1,2}(\mathbb{R}^3)}, \qquad |t| \geq 1,$$

which necessitates only $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 9$, see Remark 4.4 below.

The fourth order operators $H = \Delta^2 - \epsilon\Delta + V$ with $\epsilon \in \{0, \pm 1\}$ were formulated to study the propagation of laser beams in a bulk medium with Kerr non-linearity, [26, 27]. Dispersive estimates were considered in [3, 2]. Much of the analysis of the dispersive estimates has been done for the case of $\epsilon = 0$, with recent work spurred on by the the weighted L^2 based analysis of Feng, Soffer and Yao in dimensions $n \geq 5$, [12]. The effect of zero energy resonances was then investigated by Toprak and the second author, [20] in four dimensions and Erdoğan, Toprak and the second author in three dimensions in the $L^1 \to L^\infty$ setting, [9]. Weighted L^2 estimates are considered in [14] in higher dimensions. The case of $\epsilon = 1$ is considered in [11], the case of $\epsilon = -1$ is open. Higher order Schrödinger operators $(-\Delta)^m + V$ are studied in [15]. In contrast to the second order Schrödinger operator, for higher order operators there can be embedded eigenvalues even for perturbations by compactly supported, smooth potentials. Examples can be constructed using the fact that the analogous Helmholtz

equation $(\Delta^2 - \lambda^4)\psi = 0$ admits exponentially decaying solutions in $\mathbb{R}^3$. Theorem 1.11 and Remark 1.12 in [15] establish a criteria to rule out embedded eigenvalues. Of particular relevance to the class of potentials we consider is the criterion that a repulsive potential, one satisfying $V(\gamma x) \le V(x)$ for all $\gamma > 1$, does not have any embedded eigenvalues.

The recent study of the dispersive bounds for fourth order equations builds heavily on results for the (second order) Schrödinger operator, see [23, 25, 33, 19, 36, 10, 16, 7, 6, 17, 18] for study of the effect of threshold obstructions on the dispersive estimates. Generically a threshold obstruction leads to a slower large time-decay in the dispersive estimate. The result of Theorem 1.1 mirrors the result of Erdoğan and the second author in [8] for the two dimensional Schrödinger operator, where one can obtain an integrable time-decay at the cost of spatial weights. The result of Theorem 1.2 mirrors the result of Toprak, [37], which connected results in [7, 8] to quantify the effect of a 'mild' resonance on the evolution of the two dimensional Schrödinger operator. Such estimates are of use in the study of asymptotic stablity of special solutions to nonlinear equations. Nonlinear fourth order equations have been studied, see for example [28, 29, 30, 21, 34, 35, 22, 31, 32, 4, 5, 13].

As is the standard set-up for proving dispersive estimates, we utilize the spectral theory of H to reduce to bounding oscillatory integrals in the spectral parameter. First, we relate the resolvent of the free fourth order operator to the free second order Schrödinger operator (c.f. [12] or apply the second resolvent identity) to see that

$$(3) \quad R(H_0; z) := (\Delta^2 - z)^{-1} = \frac{1}{2\sqrt{z}} \left((-\Delta - \sqrt{z})^{-1} - (-\Delta - \sqrt{-z})^{-1} \right), \quad z \in \mathbb{C} \setminus [0, \infty).$$

For convenience, we denote $R_0(z) = (-\Delta - z)^{-1}$ for the usual Schrödinger resolvent. For the potentials considered we have a Weyl criterion and hence $\sigma_{ac}(H) = \sigma_{ac}(\Delta^2) = [0, \infty)$. Let $\lambda \in \mathbb{R}^+$, we define the limiting resolvent operators by

$$(4) \qquad\qquad R^\pm(H_0; \lambda) := R(H_0; \lambda \pm i0) = \lim_{\epsilon \to 0^+} (\Delta^2 - (\lambda \pm i\epsilon))^{-1},$$

$$(5) \qquad\qquad R_V^\pm(\lambda) := R_V(\lambda \pm i0) = \lim_{\epsilon \to 0^+} (H - (\lambda \pm i\epsilon))^{-1}.$$

Note that using the representation (3) for $R(H_0; z)$ in definition (4) with $z = w^4$ for w in the first quandrant of the complex plane, and taking limits as w approaches the real $(w \to \lambda)$ and imaginary axes $(w \to i\lambda)$ from the first quadrant, we obtain

$$(6) \qquad\qquad R^\pm(H_0; \lambda^4) = \frac{1}{2\lambda^2} \left(R_0^\pm(\lambda^2) - R_0(-\lambda^2) \right), \quad \lambda > 0.$$

Note that $R_0(-\lambda^2)$ is a bounded operator on L^2 when $\lambda > 0$. Further, by Agmon's limiting absorption principle, [1], $R_0^\pm(\lambda^2)$ is well-defined between weighted L^2 spaces. Therefore,

$R^{\pm}(H_0; \lambda^4)$ is also well-defined between these weighted spaces. This property extends to $R_V^{\pm}(\lambda)$ provided there are no eigenvalues in the spectrum, [12]. While the absence of embedded eigenvalues is a standard assumption in the analysis of dispersive estimates, we note that [15] establishes classes of potentials for which embedded eigenvalues cannot exist but we choose to keep the absence of embedded eigenvalues as an overarching assumption. Using the functional calculus and Stone's formula we write

$$(7) \qquad e^{-itH} P_{ac}(H) f(x) = \frac{1}{2\pi i} \int_0^{\infty} e^{-it\lambda} [R_V^+(\lambda) - R_V^-(\lambda)] f(x) d\lambda.$$

Here the difference of the perturbed resolvents provides the spectral measure. We make the change of variables $\lambda \mapsto \lambda^4$ to bound

$$\frac{2}{\pi i} \int_0^{\infty} e^{-it\lambda^4} \lambda^3 [R_V^+(\lambda^4) - R_V^-(\lambda^4)] f(x) d\lambda.$$

For the convenience of the reader, we have gathered the notation and terminology we use throughout the paper. Similar notation is used in previous study of the fourth order Schrödinger operator in [20, 9]. For an operator $\mathcal{E}(\lambda)$, we write $\mathcal{E}(\lambda) = O_k(\lambda^{-\alpha})$ if its kernel $\mathcal{E}(\lambda)(x, y)$ has the property

$$(8) \qquad \sup_{x,y \in \mathbb{R}^3, \lambda > 0} \left[\lambda^{\alpha} |\mathcal{E}(\lambda)(x, y)| + \lambda^{\alpha+1} |\partial_\lambda \mathcal{E}(\lambda)(x, y)| + \cdots + \lambda^{\alpha+k} |\partial_\lambda^k \mathcal{E}(\lambda)(x, y)| \right] < \infty.$$

Similarly, we use the notation $\mathcal{E}(\lambda) = O_k(\lambda^{-\alpha} g(x, y))$ if $\mathcal{E}(\lambda)(x, y)$ satisfies

$$(9) \qquad |\mathcal{E}(\lambda)(x, y)| + \lambda |\partial_\lambda \mathcal{E}(\lambda)(x, y)| + \cdots + \lambda^k |\partial_\lambda^k \mathcal{E}(\lambda)(x, y)| \lesssim \lambda^{-\alpha} g(x, y).$$

For operators on $L^2(\mathbb{R}^3)$ we also recall the following terminology from [36, 7]: An operator $T : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ with kernel $T(\cdot, \cdot)$ is absolutely bounded if the operator with kernel $|T(\cdot, \cdot)|$ is bounded from $L^2(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$ as well. Finite rank and Hilbert-Schmidt operators are immediately absolutely bounded. The symbol Γ_θ denotes a λ dependent absolutely bounded operator satisfying

$$(10) \qquad \left\| |\Gamma_\theta| \right\|_{L^2 \to L^2} + \lambda \left\| |\partial_\lambda \Gamma_\theta| \right\|_{L^2 \to L^2} + \lambda^2 \left\| |\partial_\lambda^2 \Gamma_\theta| \right\|_{L^2 \to L^2} \lesssim \lambda^{\theta}, \quad \lambda > 0.$$

In other words it is similar to $O_2(\lambda^\theta)$ except in the family of bounded operators on L^2 instead of as maps from L^1 to L^∞. The operator represented by Γ_θ may be different in each occurrence.

We use the smooth, even low energy cut-off χ defined by $\chi(\lambda) = 1$ if $|\lambda| < \lambda_0 \ll 1$ and $\chi(\lambda) = 0$ when $|\lambda| > 2\lambda_0$ for some sufficiently small constant $0 < \lambda_0 \ll 1$. In analyzing the high energy we utilize the complementary cut-off $\widetilde{\chi}(\lambda) = 1 - \chi(\lambda)$. Throughout the paper, an

exponent $a-$ denotes $a - \epsilon$ for an arbitrarily small, but fixed $\epsilon > 0$. Similarly, $a+$ indicates $a + \epsilon$.

The paper is organized as follows. We begin in Section 2 by showing that the unperturbed solution operator satisfies the low energy bounds in Theorem 1.2. In Section 3 we prove the low energy portion of the evolution satisfies Theorem 1.1 and show that perturbing the free operator by a potential that removes the natural zero energy resonance leads to faster time decay at the cost of spatial weights. In Section 4 we establish the low energy portion of Theorem 1.2 and construct the operator F_t. Finally, in Section 5 we control the high energy portion of the evolution completing the proofs of Theorems 1.1 and 1.2.

2. The Free Evolution

In this section we obtain expansions for the free fourth order Schrödinger resolvent operators $R^\pm(H_0; \lambda^4)$, using the identity (3) and the Bessel function representation of the Schrödinger free resolvents $R_0^\pm(\lambda^2)$. We use these expansions to establish dispersive estimates for the free fourth order Schrödinger evolution, and throughout the remainder of the paper to study the spectral measure for the perturbed operator. Our expansions are similar to those obtained in [9], but our analysis requires control on further derivatives in all cases.

Recall the expression of the free Schrödinger resolvents in dimension three, (see [19] for example)

$$R_0^\pm(\lambda^2)(x, y) = \frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|}.$$

Therefore, by (6),

$$(11) \qquad R^\pm(H_0, \lambda^4)(x, y) = \frac{1}{2\lambda^2}\left(\frac{e^{\pm i\lambda|x-y|}}{4\pi|x-y|} - \frac{e^{-\lambda|x-y|}}{4\pi|x-y|} \right).$$

We have the following representation for the $R(H_0, \lambda^4)$.

$$(12) \quad R^\pm(H_0, \lambda^4)(x, y) = \frac{a^\pm}{\lambda} + \mathcal{E}_0(\lambda), \qquad \mathcal{E}_0(\lambda) = G_0 + \mathcal{E}_1(\lambda), \qquad \mathcal{E}_1(\lambda) = a_1^\pm \lambda G_1 + \mathcal{E}_2(\lambda).$$

Here

$$(13) \qquad a^\pm := \frac{1 \pm i}{8\pi}, \quad a_1^\pm = \frac{1 \mp i}{8\pi \cdot (3!)}, \quad G_0(x,y) = -\frac{|x-y|}{8\pi}, \quad G_1(x,y) = |x-y|^2.$$

Notice that $G_0 = (\Delta^2)^{-1}$. When $\lambda|x-y| \lesssim 1$, we have $\mathcal{E}_0(\lambda) = O_2(\lambda^{-1+\gamma}|x-y|^\gamma)$, $\mathcal{E}_1(\lambda) = O_2(\lambda^\gamma|x-y|^{1+\gamma})$, $0 \le \gamma \le 1$ and $\mathcal{E}_2(\lambda) = O_2(\lambda^{1+\gamma}|x-y|^{2+\gamma})$, for $0 \le \gamma \le 2$. The larger range of γ for $\mathcal{E}_2(\lambda)$ follows since the λ^2 terms in the series expansion of (11) cancel. When $\lambda|x-y| \gtrsim 1$, the expansion remains valid for the first derivative.

For control of the second derivative, we note that if $\lambda|x - y| \gtrsim 1$,

$$|\partial_\lambda^2 R^\pm(H_0, \lambda^4)(x,y)| \lesssim \frac{|x-y|}{\lambda^2}(\lambda|x-y|)^\ell, \qquad \text{for any } \ell \geq 0.$$

Hence, $\partial_\lambda^2 \mathcal{E}_j(\lambda)$, $j = 0, 1$ satisfies the same bounds when $\lambda|x - y| \gtrsim 1$. Thus, we may write

$$(14) \qquad \mathcal{E}_0(\lambda) = O_1(\lambda^{-1+\gamma}|x-y|^\gamma), \qquad \mathcal{E}_0(\lambda) = O_2(|x-y|), \quad 0 \leq \gamma \leq 1,$$

$$(15) \qquad \mathcal{E}_1(\lambda) = O_2(\lambda^\gamma|x-y|^{1+\gamma}), \quad 0 \leq \gamma \leq 1,$$

$$(16) \qquad \mathcal{E}_2(\lambda) = O_2(\lambda^{1+\gamma}|x-y|^{2+\gamma}), \quad 0 \leq \gamma \leq 2.$$

The following lemma is used repeatedly to obtain low energy dispersive estimates.

Lemma 2.1. *Fix $0 < \alpha < 4$. Assume that $\mathcal{E}(\lambda) = O_2(\lambda^\alpha)$ for $0 < \lambda \lesssim 1$, then we have the bound*

$$(17) \qquad \left| \int_0^\infty e^{-it\lambda^4} \chi(\lambda)\lambda^3 \mathcal{E}(\lambda)\, d\lambda \right| \lesssim \langle t \rangle^{-1-\frac{\alpha}{4}}.$$

Proof. By the support condition and since $0 < \alpha$, the integral is bounded. Now, for $|t| > 1$ we rewrite the integral in (17) as

$$\int_0^{t^{-\frac{1}{4}}} e^{-it\lambda^4}\lambda^3\chi(\lambda)\mathcal{E}(\lambda)\, d\lambda + \int_{t^{-\frac{1}{4}}}^\infty e^{-it\lambda^4}\lambda^3\chi(\lambda)\mathcal{E}(\lambda)\, d\lambda := I + II.$$

We see that

$$|I| \leq \int_0^{t^{-\frac{1}{4}}} \lambda^{3+\alpha}\, d\lambda \lesssim t^{-1-\frac{\alpha}{4}}.$$

For the second term, we use $\partial_\lambda e^{-it\lambda^4}/(-4it) = e^{-it\lambda^4}\lambda^3$ to integrate by parts twice,

$$II = \frac{e^{-it\lambda^4}\mathcal{E}(\lambda)}{-4it}\Big|_{t^{-\frac{1}{4}}} - \frac{e^{-it\lambda^4}\mathcal{E}'(\lambda)}{(4it)^2\lambda^3}\Big|_{t^{-\frac{1}{4}}} + \frac{1}{(4\pi it)^2}\int_{t^{-\frac{1}{4}}}^\infty e^{-it\lambda^4}\partial_\lambda\left(\frac{\mathcal{E}'(\lambda)}{\lambda^3}\right)\, d\lambda.$$

Then, we have

$$|II| \lesssim \frac{|\mathcal{E}(t^{-\frac{1}{4}})|}{t} + \frac{|\mathcal{E}'(t^{-\frac{1}{4}})|}{t^{\frac{5}{4}}} + \frac{1}{t^2}\int_{t^{-\frac{1}{4}}}^\infty \lambda^{\alpha-5}\, d\lambda \lesssim t^{-1-\frac{\alpha}{4}}.$$

$\square$

Another useful bound is

Lemma 2.2. [9, Lemma 3.1] *Fix $0 < \alpha < 4$. Assume that $\mathcal{E}(\lambda) = O_1(\lambda^{-\alpha})$ for $0 < \lambda \lesssim 1$, then we have the bound*

$$(18) \qquad \left| \int_0^\infty e^{-it\lambda^4}\chi(\lambda)\lambda^3\mathcal{E}(\lambda)\, d\lambda \right| \lesssim \langle t \rangle^{-1+\frac{\alpha}{4}}.$$

These can be combined to give a characterization of the free evolution.

Lemma 2.3. *For any* $-1 \leq \sigma \leq 1$ *We have the expansion*

$$\int_0^\infty e^{-it\lambda^4}\chi(\lambda)\lambda^3[R^+(H_0, \lambda^4) - R^-(H_0, \lambda^4)](x,y)\, d\lambda$$

$$= (a^+ - a^-)\int_0^\infty e^{-it\lambda^4}\chi(\lambda)\lambda^2\, d\lambda + O(\langle t\rangle^{-1-\frac{\sigma}{4}}\langle x\rangle^{1+\sigma}\langle y\rangle^{1+\sigma}).$$

Proof. Using (12) we have

$$[R^+(H_0, \lambda^4) - R^-(H_0, \lambda^4)](x,y) = \frac{a^+ - a^-}{\lambda} + [\mathcal{E}_0^+ - \mathcal{E}_0^-](\lambda).$$

Using (14), $\mathcal{E}_0^+ - \mathcal{E}_0^- = O_1(\lambda^{-1+\gamma}|x-y|^\gamma)$. Inserting this into the Stone's formula along with Lemma 2.2, noting $|x-y| \lesssim \langle x\rangle\langle y\rangle$, yields the claim. suffices to prove the desired bound for $-1+\gamma = \sigma < 0$. If $\sigma > 0$, we use (15) and (12) to write $\mathcal{E}_0^+ - \mathcal{E}_0^- = \mathcal{E}_1^+ - \mathcal{E}_1^-$ in Lemma 2.1. The remaining case of $\sigma = 0$ follows by interpolating between the two proven cases of $\sigma = 0+$ and $\sigma = 0-$ respectively.

$\square$

Remark 2.4. *It was shown in* [2, 9] *that the free operator satisfies the bound*

$$\|e^{it\Delta^2}\|_{L^1 \to L^\infty} \lesssim t^{-\frac{3}{4}}.$$

A consequence of the bound above is the operator-valued expansion

$$e^{it\Delta^2} = |t|^{-\frac{3}{4}}A_0 + |t|^{-1-\frac{\sigma}{4}}A_1$$

where $A_0 : L^1 \to L^\infty$ *is rank one and* $A_1 : L^{1,1+\sigma} \to L^{\infty,-1-\sigma}$ *for any* $-1 \leq \sigma \leq 1$.

Theorem 1.2 matches this expansion when $\sigma = 1$, except possibly for the rank of A_0. Comparing this expansion to the result in Theorem 1.1, we see that if zero energy is regular one obtains better time decay since $A_0 \equiv 0$ in this case. Further, the spatial weights required in Theorem 1.1 are one power smaller for the $|t|^{-\frac{5}{4}}$ term than for the free evolution.

3. WEIGHTED BOUNDS WHEN ZERO IS REGULAR

The main goal of this section is to establish the following

Proposition 3.1. *If zero is a regular point and* $|V(x)| \lesssim \langle x\rangle^{-\beta}$ *for some* $\beta > 7$, *then we have the following expansion for the perturbed resolvent in a sufficiently small neighborhood of* $\lambda = 0$:

(19) $$R_V^\pm(\lambda^4)(x,y) = C_0 + O_2(\lambda\langle x\rangle\langle y\rangle).$$

In particular,

$$(20) \qquad R_V^+ - R_V^- = O_2(\lambda \langle x \rangle \langle y \rangle).$$

The low energy portion of Theorem 1.1 follows from this bound and the oscillatory integral estimate in Lemma 2.1.

To understand (7) for small energies, i.e. $0 < \lambda \ll 1$, we use the symmetric resolvent identity. We define $U(x) = \text{sign}(V(x))$, $v(x) = |V(x)|^{\frac{1}{2}}$, and write

$$(21) \qquad R_V^\pm(\lambda^4) = R^\pm(H_0, \lambda^4) - R^\pm(H_0, \lambda^4) v (M^\pm(\lambda))^{-1} v R^\pm(H_0, \lambda^4),$$

where $M^\pm(\lambda) = U + v R^\pm(H_0, \lambda^4) v$. As a result, we need to obtain expansions for $(M^\pm(\lambda))^{-1}$. The behavior of these operators as $\lambda \to 0$ depends on the type of resonances at zero energy, see Section 4 of [9]. We first develop an approriate expansion when zero is regular to construct the low energy spectral measure in Stone's formula, (7).

Let $T := U + v G_0 v$, and recall (10), we have the following expansions.

Lemma 3.2. *For $0 < \lambda < 1$ define $M^\pm(\lambda) = U + v R^\pm(H_0, \lambda^4) v$. Let $P = v \langle \cdot, v \rangle \|V\|_1^{-1}$ denote the orthogonal projection onto the span of v. We have*

$$(22) \qquad M^\pm(\lambda) = A^\pm(\lambda) + M_0^\pm(\lambda),$$

$$(23) \qquad A^\pm(\lambda) = \frac{\|V\|_1 a^\pm}{\lambda} P + T,$$

where $T := U + v G_0 v$ and $M_0^\pm(\lambda) = v \mathcal{E}_1(\lambda) v = \Gamma_\gamma$, for any $0 \le \gamma \le 1$, provided that $v(x) \lesssim \langle x \rangle^{-\frac{5}{2} - \gamma-}$. Where $\mathcal{E}_0(\lambda)$ is the operator from (14). Moreover,

$$(24) \qquad M_0^\pm(\lambda) = a_1^\pm \lambda v G_1 v + \Gamma_{1+\gamma}, \qquad 0 \le \gamma \le 2$$

provided that $v(x) \lesssim \langle x \rangle^{-\frac{7}{2} - \gamma-}$. Here the operators and the error term are Hilbert-Schmidt, and hence absolutely bounded operators.

Proof. Using the expansion (12) and the bounds on $\mathcal{E}_1(\lambda)$ and $\mathcal{E}_2(\lambda)$ in (15) and (16) respectively in the definition of $M^\pm(\lambda)$ and $M_0^\pm(\lambda)$ suffices to establish the bounds. $\qquad \square$

The definition below classifies the type of resonances considered in this paper at the threshold energy. A more detailed study of the remaining types of resonances may be found in Definition 4.2 and Section 7 of [9].

Definition 3.3. *i) Let $Q := I - P$. Zero is a regular point of the spectrum of $\Delta^2 + V$ provided QTQ is invertible on QL^2. In that case we define $D_0 := (QTQ)^{-1}$ as an absolutely bounded operator on QL^2 provided $|V(x)| \lesssim \langle x \rangle^{-5-}$ by Lemma 4.3 in [9].*

ii) Assume that zero is not regular point of the spectrum. Let S_1 be the Riesz projection onto the kernel of QTQ. Then $QTQ + S_1$ is invertible on QL^2. Accordingly, we define $D_0 = (QTQ + S_1)^{-1}$, as an operator on QL^2. (This agrees with the previous definition since $S_1 = 0$ when zero is regular.) We say there is a resonance of the first kind at zero if the operator $T_1 := S_1 TPTS_1 - \frac{\|V\|_1}{3(8\pi)^2} S_1 v G_1 v S_1$ is invertible on $S_1 L^2$.

Note that T is a compact perturbation of U. Hence, the Fredholm alternative guarantees that S_1 is a finite-rank projection. Also, $S_1 \leq Q$ and $Qv = 0$, hence $S_1 v = 0$. Second, since T is a self-adjoint operator and S_1 is the Riesz projection onto its kernel, we have $S_1 D_0 = D_0 S_1 = S_1$.

Recall from (22) that $M^\pm(\lambda) = A^\pm(\lambda) + M_0^\pm(\lambda)$. If zero is regular then we have the following expansion for $(A^\pm(\lambda))^{-1}$.

Lemma 3.4. [9, Lemma 4.5] *Let $0 < \lambda \ll 1$. If zero is regular point of the spectrum of H. Then, we have*

$$(25) \qquad (A^\pm(\lambda))^{-1} = QD_0 Q + g^\pm(\lambda)S,$$

where $g^\pm(\lambda) = (\frac{a^\pm \|V\|_1}{\lambda} + c)^{-1}$ for some $c \in \mathbb{R}$, and

$$(26) \qquad S = \begin{bmatrix} P & -PTQD_0Q \\ -QD_0QTP & QD_0QTPTQD_0Q \end{bmatrix},$$

is a self-adjoint, finite rank operator. Moreover, the same formula holds for $(A^\pm(\lambda) + S_1)^{-1}$ with $D_0 = (Q(T + S_1)Q)^{-1}$ if zero is not regular.

This lemma and the definition preceeding it are of use in Section 4 when we consider a resonance of the first kind. For the rest of this section, we only consider the case when zero is regular and $S_1 = 0$. With the assumption that $v(x) \lesssim \langle x \rangle^{-\frac{7}{2}-}$, we are able to conclude from Lemma 3.2 that $M_0^\pm(\lambda) = \Gamma_1$.

By Lemma 3.4, $A^+(\lambda)^{-1} = \Gamma_0$, and then

$$M^+(\lambda)^{-1} = A^+(\lambda)^{-1} - A^+(\lambda)^{-1} M_0^+(\lambda)(I + A^+(\lambda)^{-1} M_0)^{-1} A^+(\lambda)^{-1}$$

$$(27) \qquad\qquad = A^+(\lambda)^{-1} - A^+(\lambda)^{-1} \Gamma_1 A^+(\lambda)^{-1}.$$

Another application of Lemma 3.4 leads

$$(28) \quad R_V^+(\lambda^4) = R^+(H_0, \lambda^4) - R^+(H_0, \lambda^4) v A^+(\lambda)^{-1} v R^+(H_0, \lambda^4)$$
$$+ R^+(H_0, \lambda^4) v A^+(\lambda)^{-1} \Gamma_1 A^+(\lambda)^{-1} v R^+(H_0, \lambda^4)$$

There is significant cancellation in that the leading part of $R^+(H_0, \lambda^4)v$ is orthogonal to the range of Q.

Lemma 3.5. *If $v(x) \lesssim \langle x \rangle^{-\frac{5}{2}-}$, then $R^+(H_0, \lambda^4)vQ$ is $O_2(1)$ as an operator from $L^2(\mathbb{R}^3)$ to $L^{\infty,-1}(\mathbb{R}^3)$.*

Proof. For all $\psi \in QL^2$, $\int v(y)\psi(y)\, dy = 0$, hence $R^+(H_0, \lambda^4)vQ = (R^+(H_0, \lambda^4) - \frac{a^+}{\lambda})vQ$. It follows from (12) and (14) that $(R^+(H_0, \lambda^4) - \frac{a^+}{\lambda}) = O_2(|x - y|)$, which is also $O_2(\langle x \rangle \langle y \rangle)$. The result follows as long as $\langle y \rangle v \in L^2(\mathbb{R}^3)$. $\qquad\square$

We first consider the final term in (28).

Lemma 3.6. *The kernel of $g^+(\lambda)R^+(H_0, \lambda^4)$ is $O_2(\langle x \rangle \langle y \rangle)$. As a consequence, if $v(x) \lesssim \langle x \rangle^{-\frac{5}{2}-}$ then $R^+(H_0, \lambda^4)vA^+(\lambda)^{-1} = O_2(1)$ as an operator from $L^2(\mathbb{R}^3)$ to $L^{\infty,-1}(\mathbb{R}^3)$, and*

$$R^+(H_0, \lambda^4)vA^+(\lambda)^{-1}\Gamma_1 A^+(\lambda)^{-1} v R^+(H_0, \lambda^4) = O_2(\lambda \langle x \rangle \langle y \rangle).$$

Proof. The scalar function $g^+(\lambda) = \lambda(a^+\|V\|_1 + c\lambda)^{-1}$ is both $O_2(\lambda)$ and $O_2(1)$. Recalling (14), we see that $R^+(H_0, \lambda^4) = \frac{a^+}{\lambda} + O_2(|x - y|)$. Then the products $\frac{a^+g^+(\lambda)}{\lambda} = O_2(1)$ and $O_2(|x - y|)g^+(\lambda) = O_2(|x - y|)$, and both of these are $O_2(\langle x \rangle \langle y \rangle)$.

Recall that $R^+(H_0, \lambda^4)vA^+(\lambda)^{-1} = R^+(H_0, \lambda^4)v(QD_0Q + g(\lambda)S)$. The first part of the product is $O_2(1)$ as a map from $L^2(\mathbb{R}^3)$ to $L^{\infty,-1}(\mathbb{R}^3)$ by Lemma 3.5. The second part of the product, using (14), satisfies the same bounds since $g^+(\lambda)R^+(H_0, \lambda^4) = O_2(\langle x \rangle \langle y \rangle)$ and $\langle y \rangle v \in L^2(\mathbb{R}^3)$.

Finally, the operator $A^+(\lambda)^{-1}vR^+(H_0, \lambda^4)$ is $O_2(1)$ as a map from $L^{1,1}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. So the composition of these pieces together with Γ_1 is an $O_2(\lambda)$ map from $L^{1,1}(\mathbb{R}^3)$ to $L^{\infty,-1}(\mathbb{R}^3)$, or in other words it is $O_2(\lambda \langle x \rangle \langle y \rangle)$. $\qquad\square$

Before proceeding further, we recall from (12) that the expansion of the free resolvent $R^+(H_0, \lambda^4)$ is $\frac{a^+}{\lambda} + G_0 + O_2(\lambda|x - y|^2)$. Its remainder term is worse than the $O_2(\lambda \langle x \rangle \langle y \rangle)$ required for Proposition 3.1. Controlling the first two terms of (28) requires finding additional cancellation within $\mathcal{E}_1 = O_2(\lambda|x - y|^2)$. Using (6) and (12), the kernel of $\mathcal{E}_1(\lambda)$ has the form

$$(29) \qquad \mathcal{E}_1(\lambda, x, y) = \frac{K(\lambda|x - y|)}{\lambda}, \qquad K(z) = \frac{e^{iz} - e^{-z}}{8\pi z} - a^+ + \frac{z}{8\pi}.$$

The functions $K_1(z) = zK'(z)$ and $K_2(z) = zK_1'(z)$ are useful in a moment. By the Leibniz rule,

$$\frac{d}{d\lambda}\mathcal{E}_1(\lambda, x, y) = \frac{-K(\lambda|x - y|) + K_1(\lambda|x - y|)}{\lambda^2},$$

$$\frac{d^2}{d\lambda^2}\mathcal{E}_1(\lambda, x, y) = \frac{2K(\lambda|x - y|) - 3K_1(\lambda|x - y|) + K_2(\lambda|x - y|)}{\lambda^3}.$$

All three functions have the property $|K_j(z)| \lesssim z^2$ and $|K_j'(z)| \lesssim z$ for all positive z. We note that the bound $|K_j(z)| \lesssim z$ is also true. By the Mean Value Theorem,

$$(30) \qquad |K_j(\lambda|x - y|) - K_j(\lambda|x|)| \lesssim \lambda^2\langle y\rangle \max(|x - y|, |x|) \lesssim \lambda^2\langle y\rangle \max(\langle x\rangle, \langle y\rangle).$$

We may insert $K_j(\lambda|y|)$ into this estimate as below without changing the upper bound,

$$\left|K_j(\lambda|x - y|) - K_j(\lambda|x|) - K_j(\lambda|y|)\right| \lesssim \lambda^2\langle y\rangle \max(\langle x\rangle, \langle y\rangle).$$

However the left side is symmetric in x and y. thus it is also bounded by $\lambda^2\langle x\rangle \max(\langle x\rangle, \langle y\rangle)$. We may conclude that

$$(31) \qquad \left|K_j(\lambda|x - y|) - K_j(\lambda|x|) - K_j(\lambda|y|)\right| \lesssim \lambda^2 \min(\langle x\rangle, \langle y\rangle) \max(\langle x\rangle, \langle y\rangle) = \lambda^2\langle x\rangle\langle y\rangle.$$

Lemma 3.7. *If $v(x) \lesssim \langle x\rangle^{-\frac{7}{2}-}$, then $\mathcal{E}_1 vQ$ is $O_2(\lambda)$ as an operator from $L^2(\mathbb{R}^3)$ to $L^{\infty, -1}(\mathbb{R}^3)$.*

Proof. For all $\psi \in QL^2$, $\int v(y)\psi(y)\, dy = 0$, hence using (29),

$$|\mathcal{E}_1 v\psi(x)| = \left|\int_{\mathbb{R}^3} \lambda^{-1}\big(K(\lambda|x - z|) - K(\lambda|x|)\big)v(z)\psi(z)\, dz\right|$$

$$\lesssim \lambda \int_{\mathbb{R}^3} \langle z\rangle \max(\langle x\rangle, \langle z\rangle)v(z)|\psi(z)|\, dz \lesssim \langle x\rangle,$$

provided $\langle z\rangle^2 v \in L^2(\mathbb{R}^3)$. Similar estimates can made for the derivatives of $\mathcal{E}_1(\lambda)$ using the bounds on K_1 and K_2. $\qquad\square$

For clarity in the calculations to follow, we denote $\mathbf{1}$ to be the operator with integral kernel $\mathbf{1}(x, y) = 1$. That way $R^+(H_0, \lambda^4)$ is $\frac{a^+}{\lambda}\mathbf{1} + \mathcal{E}_0 = \frac{a^+}{\lambda}\mathbf{1} + G_0 + \mathcal{E}_1$. The crucial step in the proof of Proposition 3.1 is in the estimate below for the first two terms of (28).

Lemma 3.8. *If $v(x) \lesssim \langle x\rangle^{-\frac{7}{2}-}$, then*

$$R^+(H_0, \lambda^4) - R^+(H_0, \lambda^4)vA^+(\lambda)^{-1}vR^+(H_0, \lambda^4)$$

$$= G_0 - \left[G_0 vQD_0 QvG_0 + \frac{c}{\|V\|_1} - \frac{G_0 vSv\mathbf{1} + \mathbf{1}vSvG_0}{\|V\|_1}\right] + O_2(\lambda\langle x\rangle\langle y\rangle)$$

This ultimately explains why the statement of Theorem 1.1 has a lower order of polynomial weights than Theorem 1.2 or the evolution of the unperturbed solution operator.

Proof. By (25), we may write

$$(32) \quad R^+(H_0, \lambda^4) - R^+(H_0, \lambda^4)vA^+(\lambda)^{-1}vR^+(H_0, \lambda^4)$$

$$= R^+(H_0, \lambda^4) - g^+(\lambda)R^+(H_0, \lambda^4)vSvR^+(H_0, \lambda^4) - R^+(H_0, \lambda^4)vQD_0QvR^+(H_0, \lambda^4)$$

The last term can be further expanded as

$$(G_0 + \mathcal{E}_1(\lambda))vQD_0Qv(G_0 + \mathcal{E}_1(\lambda))$$

$$= G_0vQD_0QvG_0 + \mathcal{E}_1(\lambda)vQD_0QvG_0 + G_0vQD_0Qv\mathcal{E}_1(\lambda) + \mathcal{E}_1(\lambda)vQD_0Qv\mathcal{E}_1(\lambda).$$

Here we used the orthogonality of Q to the span of v. Using Lemma 3.7 as needed, we conclude that

$$R^+(H_0, \lambda^4)vQD_0QvR^+(H_0, \lambda^4) = G_0vQD_0QvG_0 + O_2(\lambda\langle x\rangle\langle y\rangle).$$

The function $g^+(\lambda)$ has the power series expansion

$$(33) \quad g^+(\lambda) = \lambda(a^+\|V\|_1 + c\lambda)^{-1} = \frac{\lambda}{a^+\|V\|_1} - \frac{c\lambda^2}{(a^+\|V\|_1)^2} + O_2(\lambda^3),$$

which allows us to write

$$(34) \quad R^+(H_0, \lambda^4) - g^+(\lambda)R^+(H_0, \lambda^4)vSvR^+(H_0, \lambda^4)$$

$$= R^+(H_0, \lambda^4) - \frac{\lambda}{a^+\|V\|_1}R^+(H_0, \lambda^4)vSvR^+(H_0, \lambda^4)$$

$$+ \frac{c\lambda^2}{(a^+\|V\|_1)^2}R^+(H_0, \lambda^4)vSvR^+(H_0, \lambda^4) + O_2(\lambda\langle x\rangle\langle y\rangle).$$

The error term follows since Lemma 3.6 implies that $\lambda R^+(H_0, \lambda^4) = O_2(\langle x\rangle\langle y\rangle)$ and $\lambda S = \Gamma_1$.

Now we take inventory of the order λ^{-1}, λ^0, and remainder terms in the rest of (34). Recalling (12) and Lemma 3.4 we see that the only term of order λ^0 arising from $\lambda^2 R^+(H_0, \lambda^4)vSvR^+(H_0, \lambda^4)$ comes from the projection P in the upper-left corner of (26). All other parts of S introduce expressions $R^+(H_0, \lambda^4)vQ$ or $QvR^+(H_0, \lambda^4)$ which neutralizes the leading order part of $R^+(H_0, \lambda^4)$ by Lemma 3.5. More specifically, writing $\lambda R^+(H_0, \lambda^4) = a^+ + \lambda\mathcal{E}_0(\lambda)$, we have

$$(35) \quad \lambda^2 R^+(H_0, \lambda^4)vSvR^+(H_0, \lambda^4)$$

$$= (a^+)^2\mathbf{1}vSv\mathbf{1} + a^+\lambda(\mathcal{E}_0(\lambda)vSv\mathbf{1} + \mathbf{1}vSv\mathcal{E}_0(\lambda)) + \lambda^2\mathcal{E}_0(\lambda)vSv\mathcal{E}_0(\lambda)$$

$$= (a^+)^2 \|V\|_1 \mathbf{1} + O_2(\lambda \langle x \rangle \langle y \rangle).$$

We have used (26) and the fact that $Qv\mathbf{1} = \mathbf{1}vQ = 0$ to see $\mathbf{1}vSv\mathbf{1} = \mathbf{1}vPv\mathbf{1}$, the identity $\mathbf{1}vPv\mathbf{1} = \|V\|_1 \mathbf{1}$, as well as Lemma 3.5.

The next term requires carrying out the expansion a little farther than in (35) by writing $\mathcal{E}_0(\lambda) = G_0 + \mathcal{E}_1(\lambda)$ in the expansion of the resolvent.

$$\lambda R^+(H_0, \lambda^4) vSv R^+(H_0, \lambda^4)$$

$$= \lambda^{-1}(a^+)^2 \mathbf{1}vSv\mathbf{1} + a^+(\mathcal{E}_0(\lambda)vSv\mathbf{1} + \mathbf{1}vSv\mathcal{E}_0(\lambda)) + \lambda \mathcal{E}_0(\lambda)vSv\mathcal{E}_0(\lambda)$$

$$= \lambda^{-1}(a^+)^2 \|V\|_1 \mathbf{1} + a^+(G_0 vSv\mathbf{1} + \mathbf{1}vSvG_0)$$

$$+ a^+ \mathcal{E}_1(\lambda)vSv\mathbf{1} + a^+ \mathbf{1}vSv\mathcal{E}_1(\lambda) + \lambda \mathcal{E}_0(\lambda)vSv\mathcal{E}_0(\lambda).$$

Using (26), $Qv\mathbf{1} = 0$, (14), (15) and Lemmas 3.5 and 3.7 as needed, we conclude that

$$(36) \quad \frac{\lambda}{a^+ \|V\|_1} R^+(H_0, \lambda^4) vSv R^+(H_0, \lambda^4) = \frac{a^+ \mathbf{1}}{\lambda} + \frac{G_0 vSv\mathbf{1} + \mathbf{1}vSvG_0}{\|V\|_1}$$

$$+ \frac{\mathcal{E}_1(\lambda)vPv\mathbf{1} + \mathbf{1}vPv\mathcal{E}_1(\lambda)}{\|V\|_1} + O_2(\lambda \langle x \rangle \langle y \rangle).$$

We note that by (15) the $\mathcal{E}_1(\lambda)$ terms are order $O_2(\lambda \langle x \rangle^2 \langle y \rangle^2)$, which is too large to be included in the remainder.

The free resolvent has the expansion $R^+(H_0, \lambda^4) = \frac{a^+ \mathbf{1}}{\lambda} + G_0 + \mathcal{E}_1(\lambda)$. The leading-order term cancels immediately with its counterpart in (36) in their contribution to (32). The lemma, and Proposition 3.1, concludes with one last estimate,

$$(37) \quad \mathcal{E}_1(\lambda) - \frac{\mathcal{E}_1(\lambda)vPv\mathbf{1} + \mathbf{1}vPv\mathcal{E}_1(\lambda)}{\|V\|_1} = O_2(\lambda \langle x \rangle \langle y \rangle).$$

Using the facts that $Pv\mathbf{1} = v\mathbf{1}$, and $\int v^2(z)\, dz = \|V\|_1$, along with (29), the operator on the left side of (37) has kernel

$$(38) \quad \frac{1}{\lambda}\left(K(\lambda|x-y|) - \int K(\lambda|x-z|)\frac{v^2(z)}{\|V\|_1}\, dz - \int K(\lambda|y-z|)\frac{v^2(z)}{\|V\|_1}\, dz \right)$$

$$= \frac{K(\lambda|x-y|) - K(\lambda|x|) - K(\lambda|y|)}{\lambda} - \int \frac{K(\lambda|x-z|) - K(\lambda|x|)}{\lambda}\frac{v^2(z)}{\|V\|_1}\, dz$$

$$- \int \frac{K(\lambda|y-z|) - K(\lambda|y|)}{\lambda}\frac{v^2(z)}{\|V\|_1}\, dz.$$

From the inequalities (30), (31) for K we can see that

$$\left| \mathcal{E}_1(\lambda) - \mathcal{E}_1(\lambda)\frac{v^2}{\|V\|_1}\mathbf{1} - \mathbf{1}\frac{v^2}{\|V\|_1}\mathcal{E}_1(\lambda) \right| \lesssim \lambda \langle x \rangle \langle y \rangle + \lambda \int (\langle x \rangle + \langle y \rangle)\frac{\langle z \rangle^2 v^2(z)}{\|V\|_1}\, dz \lesssim \lambda \langle x \rangle \langle y \rangle$$

provided $|v(x)| \lesssim \langle x \rangle^{-\frac{5}{2}-}$. Bounds for the first two derivatives can be computed from (38) by using the Leibniz rule formulas for $\frac{d}{d\lambda}\mathcal{E}_1(\lambda)$ and $\frac{d^2}{d\lambda^2}\mathcal{E}_1(\lambda)$ and the analogous properties of K_1 and K_2. $\qquad \square$

Finally, we prove the main Proposition.

of Proposition 3.1. Inserting the results of Lemma 3.6, 3.8 into (21), specifically the expansion (28) suffices to prove the desired result.

$\qquad \square$

We collect a couple of notes for future reference when handling the resonant case in the next section. First, the cancellation of order λ^{-1} terms between $R^+(H_0, \lambda^4)$ and $R^+(H_0, \lambda^4)vg^+(\lambda)SvR^+(H_0, \lambda^4)$ takes place in the same manner because the projection in the upper left corner of S in (26) is the same in either case.

The fact that the constant, order λ^0 terms, are the same for $R_V^+(\lambda^4)$ and $R_V^-(\lambda^4)$ in the resonant case is much more difficult to check by hand. However, note that the original definition of $R_V^\pm(\lambda^4) = R_V(\lambda^4 \pm i0)$ implies that $R_V^-(\lambda^4) = R_V^+((i\lambda)^4)$. As a consequence, the power series coefficients for $R_V^\pm(\lambda^4)$ should coincide for each term of order λ^{4k} and for the constant term in particular.

4. Weighted bounds when there is a resonance of the first kind

The main goal of this section is to establish the following

Proposition 4.1. *If there is a resonance of the first kind at zero and $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for some $\beta > 11$, then we have the following expansion for the perturbed resolvent in a sufficiently small neighborhood of $\lambda = 0$:*

$$(39) \qquad R_V^\pm(\lambda^4)(x,y) = \frac{a^\pm \|V\|_1}{\lambda} C_{-1}^\pm + C_0 + O_2(\lambda \langle x \rangle^2 \langle y \rangle^2).$$

Where $C_{-1}^\pm$ is an operator of rank at most four given by

$$C_{-1}^\pm = (G_0 v + a^\pm \mathbf{1} v F_L^\pm) D_1 (v G_0 + a^\pm F_R^\pm v \mathbf{1})$$

where $F_R^\pm := \frac{1}{a^\pm \|V\|_1} S + a_1^\pm v G_1 v Q D_0 Q$, and $F_L^\pm := \frac{1}{a^\pm \|V\|_1} S + a_1^\pm Q D_0 Q v G_1 v$. In particular,

$$(40) \qquad R_V^+ - R_V^- = \frac{C_{-1}^+ - C_{-1}^-}{\lambda} + O_2(\lambda \langle x \rangle^2 \langle y \rangle^2).$$

From this theorem we can write an explicit formulation of the time dependent operator F_t in the statement of Theorem 1.2,

$$(41) \qquad F_t = \frac{2}{\pi i} \int_0^\infty e^{-it\lambda^4} \lambda^3 \chi(\lambda) \frac{C_{-1}^+ - C_{-1}^-}{\lambda} \, d\lambda$$

Since $D_1 = S_1 D_1 S_1$, we can see that $C_{-1}^\pm$ is an operator of rank at most the rank of S_1. By Remark 7.9 in [9], in the case of a resonance of the first kind the rank of S_1 is at most four. As in the case when zero is regular, we need to understand the resolvent as $\lambda \to 0$, hence we study the operators $M^\pm(\lambda)^{-1}$.

Lemma 4.2. *If there is a resonance of the first kind at zero and if $v(x) \lesssim \langle x \rangle^{-\frac{11}{2}-}$, then*

$$[M^\pm(\lambda)]^{-1} = (M^\pm(\lambda) + S_1)^{-1} - \frac{a^\pm \|V\|_1 D_1}{\lambda} + \widetilde{M_0^\pm} + \lambda \widetilde{M_1^\pm} + \lambda^2 \widetilde{M_2^\pm} + \Gamma_3.$$

Furthermore, D_1, $\widetilde{M_0^\pm}$, $\widetilde{M_1^\pm}$ and $\widetilde{M_2^\pm}$ are finite-rank operators.

Our expansions vary from those presented in [9] as we need further information on the second derivative for our goals. We begin with the following lemma from [24].

Lemma 4.3. [24, Lemma 2.1] *Let M be a closed operator on a Hilbert space $\mathcal{H}$ and S a projection. Suppose $M + S$ has a bounded inverse. Then M has a bounded inverse if and only if*

$$B := S - S(M + S)^{-1} S$$

has a bounded inverse in $S\mathcal{H}$, and in this case

$$(42) \qquad M^{-1} = (M + S)^{-1} + (M + S)^{-1} S B^{-1} S (M + S)^{-1}.$$

We use this lemma with $M = M^\pm(\lambda)$ and $S = S_1$ to prove Lemma 4.2. Then, we have $B^\pm(\lambda) = S_1 - S_1(M^\pm(\lambda) + S_1)^{-1} S_1$ on $S_1 L^2$. We note that since S_1 is a finite rank projection, both $B^\pm(\lambda)$ and $B^\pm(\lambda)^{-1}$ are finite rank operators. In the case of a resonance of the first kind, S_1 has rank at most four.

of Lemma 4.2. Using (24) and (25), via a longer Neumann series expansion than in (27) when $S_1 = 0$, we obtain

$$(43) \quad (M^\pm(\lambda) + S_1)^{-1} = QD_0Q + g^\pm(\lambda)S - a_1^\pm \lambda QD_0QvG_1vQD_0Q$$

$$- a_1^\pm \lambda g^\pm(\lambda) \big[QD_0QvG_1vS + SvG_1vQD_0Q \big] + (a_1^\pm)^2 \lambda^2 QD_0Q(vG_1vQD_0Q)^2 + \Gamma_3,$$

provided that $v(x) \lesssim \langle x \rangle^{-\frac{11}{2}-}$. Here we note that all the operators are λ independent, and we carefully note any $\pm$ dependence.

$$(44) \quad S_1(M^\pm(\lambda) + S_1)^{-1}S_1 = S_1 + g^\pm(\lambda)T_1 - a_1^\pm \lambda g^\pm(\lambda)S_1 v G_1 v S_1$$
$$- a_1^\pm \lambda g^\pm(\lambda)S_1 \left[v G_1 v S + S v G_1 v \right] S_1 + (a_1^+)^2 \lambda^2 S_1 v G_1 v Q D_0 Q v G_1 v S_1 + \Gamma_3.$$

Therefore

$$B^\pm(\lambda) = S_1 - S_1(M^\pm(\lambda) + S_1)^{-1}S_1 = -g^\pm(\lambda)T_1$$
$$+ a_1^\pm \lambda g^\pm(\lambda)S_1(v G_1 v + S v G_1 v + v G_1 v S)S_1 - (a_1^\pm)^2 \lambda^2 S_1 v G_1 v Q D_0 Q v G_1 v S_1 + S_1 \Gamma_3 S_1.$$

Recalling (33), a careful Neumann series computation yield the expansion

$$(45) \quad B^\pm(\lambda)^{-1} = -\frac{a^\pm \|V\|_1}{\lambda} D_1 - c D_1 - a_1^\pm a^\pm D_1 \tilde{B}_0 D_1 + (a_1^\pm a^\pm)^2 D_1 \tilde{B}_{0,1} D_1 + D_1 \Gamma_1 D_1.$$

Here the operators $\tilde{B}_0$ and $\tilde{B}_{0,1}$ are independent of both λ and the choice of $\pm$. Substituting (45) and (43) into (42) finishes the proof.

$\square$

We are now ready to prove Proposition 4.1.

of Proposition 4.1. By the symmetric resolvent identity, we need to understand the behavior of

$$R_V^\pm(\lambda^4) = R^\pm(H_0, \lambda^4) - R^\pm(H_0, \lambda^4)(M^\pm(\lambda) + S_1)^{-1}R^\pm(H_0, \lambda^4)$$
$$- R^\pm(H_0, \lambda^4)(M^\pm(\lambda) + S_1)^{-1}S_1 B^\pm(\lambda)^{-1} S_1(M^\pm(\lambda) + S_1)^{-1}R^\pm(H_0, \lambda^4).$$

The contribution of the first two terms, and the cancellation of their most singular λ terms follow exactly as in the case when there is no resonance at zero, that is when $S_1 = 0$. The expansion in Proposition 3.1 implies that

$$R^\pm(H_0, \lambda^4) - R^\pm(H_0, \lambda^4)(M^+(\lambda) + S_1)^{-1}R^\pm(H_0, \lambda^4) = C_{0,1} + O_2(\lambda \langle x \rangle \langle y \rangle).$$

This follows since the leading terms of the expansion in $(M^\pm(\lambda) + S_1)^{-1}$ and $M^\pm(\lambda)^{-1}$, on which the delicate cancellation relies, are identical. Namely, by (44)

$$(M^\pm(\lambda) + S_1)^{-1} = Q D_0 Q + g^\pm(\lambda)S + Q \Gamma_1 Q.$$

The remaining terms in (43) is controlled using that Lemma 3.5 implies that

$$R^\pm(H_0, \lambda^4)vQ\Gamma_1 Qv R^\pm(H_0, \lambda^4) = O_2(\lambda \langle x \rangle \langle y \rangle).$$

For the remainder of the proof, we seek to establish that the order zero terms in (39) are independent of the choice of '+' or '-'. For simplicity, we drop the superscripts and consider the '+' case unless otherwise noted. We write

$$(M + S_1)^{-1} = QD_0Q + \lambda M_1 + \lambda^2 M_2 + \Gamma_3.$$

Here $M_1 = \frac{1}{a\|V\|_1}S + a_1 QD_0QvG_1vQD_0Q$ and a careful Neumann series computation yields

$$(46) \quad M_2 = \frac{-c}{(a\|V\|_1)^2}S - \frac{a_1}{a\|V\|_1}QD_0QvG_1vS - \frac{a_1}{a\|V\|_1}SvG_1vQD_0Q$$
$$+ a_1^2 QD_0QvG_1vQD_0QvG_1vQD_0Q.$$

Since $B = [g(\lambda)T_1 + O_2(\lambda^2)]$, and using (33), as in (45) we may write

$$B^{-1} = \frac{1}{\lambda}B_{-1} + B_0 + \lambda B_1,$$

here $B_{-1} = a\|V\|_1 D_1$ where $D_1 = S_1 T_1^{-1} S_1$ the operators are explicit in (45).

Then, we have the following as expansion for sufficiently small λ:

$$(M + S_1)^{-1}S_1 B^{-1} S_1 (M + S_1)^{-1}$$

$$= \left[QD_0Q + \lambda M_1 + \lambda^2 M_2\right]\left[\lambda^{-1}B_{-1} + B_0 + \lambda B_1\right]\left[QD_0Q + \lambda M_1 + \lambda^2 M_2\right]$$

$$= \lambda^{-1}QD_0QB_{-1}QD_0Q + \left[QD_0QB_{-1}M_1 + M_1B_{-1}QD_0Q + QD_0QB_0QD_0Q\right]$$

$$+ \lambda\left[M_2B_{-1}QD_0Q + QD_0QB_{-1}M_2 + M_1B_0QD_0Q + QD_0QB_0M_1\right.$$

$$\left. + M_1B_{-1}M_1 + QD_0QB_1QD_0Q\right]$$

$$+ \lambda^2\left[M_1B_0M_1 + M_1B_{-1}M_2 + M_2B_{-1}M_1 + QD_0Q\Gamma_0 + \Gamma_0QD_0Q\right] + \Gamma_3.$$

Using

$$R(H_0, \lambda^4) = \frac{a\mathbf{1}}{\lambda} + G_0 + a_1\lambda G_1 + \mathcal{E}_2(\lambda),$$

and that $\mathbf{1}vQ = Qv\mathbf{1} = 0$, we arrive at the following expansion

$$(47) \quad R(H_0, \lambda^4)v(M + S_1)^{-1}S_1 B^{-1} S_1 (M + S_1)^{-1}vR(H_0, \lambda^4) =$$

$$\frac{1}{\lambda}\left[G_0vQD_0QB_{-1}QD_0QvG_0 + G_0vQD_0QB_{-1}M_1va\mathbf{1}\right.$$

$$\left. + a\mathbf{1}vM_1B_{-1}QD_0QvG_0 + a\mathbf{1}vM_1B_{-1}M_1va\mathbf{1}\right]$$

$$+ \Big[a_1 G_1 v Q D_0 Q B_{-1} Q D_0 Q v G_0 + a_1 G_0 v Q D_0 Q B_{-1} Q D_0 Q v G_1 + G_0 v Q D_0 Q B_{-1} M_1 v G_0$$

$$+ G_0 v M_1 B_{-1} Q D_0 Q v G_0 + a_1 G_1 v Q D_0 Q B_{-1} M_1 v a \mathbf{1} + a \mathbf{1} v M_1 B_{-1} Q D_0 Q v G_1 a_1$$

$$+ G_0 v Q D_0 Q B_0 Q D_0 Q v G_0 + a \mathbf{1} v M_2 B_{-1} Q D_0 Q v G_0 + G_0 v Q D_0 Q B_{-1} M_2 v a \mathbf{1}$$

$$+ a \mathbf{1} v M_1 B_0 Q D_0 Q v G_0 + G_0 v Q D_0 Q B_0 M_1 v a \mathbf{1} + G_0 v M_1 B_{-1} M_1 v a \mathbf{1} + a \mathbf{1} v M_1 B_{-1} M_1 v G_0$$

$$+ a \mathbf{1} v M_1 B_0 M_1 v a \mathbf{1} + a \mathbf{1} v M_1 B_{-1} M_2 v a \mathbf{1} + a \mathbf{1} v M_2 B_{-1} M_1 v a \mathbf{1} \Big] + O_2(\lambda \langle x \rangle^2 \langle y \rangle^2).$$

Our goal is to show that the order λ^0 terms are all independent of the '+' or '-' in the resolvent. Recalling (13), we see $a^+ a_1^+ = a^- a_1^-$, which allows us to conclude the following pairs from the above expansion are $\pm$ independent: $a_1 B_{-1}$, $M_1 B_{-1}$. This also implies that $(aa_1)^2$, and hence the combinations $a_1 B_{-1} M_1 a$, and $a M_1$ are $\pm$ independent. Further, it follows from (45) that B_0 is $\pm$ independent. We may also conclude that the combination $a M_1 B_0$ is $\pm$ independent. Finally, using (46) we may conclude that the combinations $a M_2 B_{-1}$ and $a M_2 B_{-1} M_1 a$ are $\pm$ independent.

These computations suffice to establish the claim that the order λ^0 terms cancel in the $+/-$ difference. Using that $S_1 Q D_0 Q = Q D_0 Q S_1 = S_1$ on the order λ^{-1} operator in (47) yields the formula for $C_{-1}^{\pm}$ given in Proposition 4.1.

$\square$

Proposition 4.1, along with oscillatory integral bounds in Lemmas 2.1 and 2.2 yields the dispersive bound

$$e^{-itH} P_{ac}(H) \chi(H) = A_{-1} \langle t \rangle^{-\frac{3}{4}} + A_1 \langle t \rangle^{-\frac{5}{4}},$$

where A_{-1} is an operator from $L^1 \to L^\infty$ and $A_1 : L^{1,2} \to L^{\infty, -2}$.

Remark 4.4. *We note that the error term in Lemma 4.2, specifically (43) may be replaced by Γ_{2+}, which requires only that $v(x) \lesssim \langle x \rangle^{-\frac{9}{2}-}$. This filters through the argument, allowing one to replace the error terms Γ_k with Γ_{k-1+}, and the error term in (47) is now $O_2(\lambda^{0+} \langle x \rangle^2 \langle y \rangle^2)$. This yields the expansion*

$$e^{-itH} P_{ac}(H) \chi(H) = A_{-1} \langle t \rangle^{-\frac{3}{4}} + A_1 \langle t \rangle^{-1-},$$

where A_{-1} is an operator from $L^1 \to L^\infty$ and $A_1 : L^{1,2} \to L^{\infty,-2}$ requiring only $|V(x)| \lesssim \langle x \rangle^{-\beta}$ for $\beta > 9$.

5. High Energy

To complete the proofs of Theorem 1.1 and 1.2, we need to control the contribution of the high energy portion, $\lambda \gtrsim 1$, in (7). The effect of a zero energy resonance is only felt in a neighborhood of zero. Hence, we need only prove one bound for the high energy.

Proposition 5.1. *Let* $|V(x)| \lesssim \langle x \rangle^{-5-}$, *and assume there are no embedded eigenvalues in the continuous spectrum of* H, *then for* $|t| > 1$

$$(48) \qquad \left| \int_0^\infty e^{-it\lambda^4} \lambda^3 \widetilde{\chi}(\lambda) R_V^\pm(\lambda^4)(x,y) \, d\lambda \right| \lesssim |t|^{-2} \langle x \rangle \langle y \rangle.$$

Recall that $\widetilde{\chi}(\lambda) = 1 - \chi(\lambda)$ is a cut-off away from a small neighborhood of $\lambda = 0$. In Proposition 6.1 of [9], the uniform bound:

$$(49) \qquad \sup_{x,y \in \mathbb{R}^3} \left| \int_0^\infty e^{-it\lambda^4} \lambda^3 \widetilde{\chi}(\lambda) R_V^\pm(\lambda^4)(x,y) \, d\lambda \right| \lesssim |t|^{-\frac{3}{4}}$$

was established. This bound holds true under the hypotheses of Proposition 5.1. The bound in (48) decays faster for large t at the cost of spatial weights. This bound can be established directly by integration by parts. Using the support condition of $\widetilde{\chi}(\lambda)$ and the decay of $R_V^\pm(\lambda^4)$ as $\lambda \to \infty$ established below, there will be no boundary terms and it suffices to control

$$(50) \qquad \left| \int_0^\infty e^{-it\lambda^4} \lambda^3 \widetilde{\chi}(\lambda) R_V^\pm(\lambda^4)(x,y) \, d\lambda \right| \lesssim \frac{1}{t^2} \int_0^\infty \left| \partial_\lambda \left(\frac{\partial_\lambda [\widetilde{\chi}(\lambda) R_V^\pm(\lambda^4)(x,y)]}{\lambda^3} \right) \right| d\lambda.$$

To prove the Proposition 5.1 we use the resolvent identities and write,

$$(51) \quad R_V(\lambda^4) = R^\pm(H_0, \lambda^4) - R^\pm(H_0, \lambda^4) V R^\pm(H_0, \lambda^4) + R^\pm(H_0, \lambda^4) V R_V(\lambda^4) V R^\pm(H_0, \lambda^4).$$

Recall by (12), we have

$$(52) \qquad R^\pm(H_0, \lambda^4)(x,y) = O_1(\lambda^{-1}), \qquad \partial_\lambda^2 R^\pm(H_0, \lambda^4)(x,y) = O(\lambda^{-2} \langle x \rangle \langle y \rangle).$$

This, along with the fact that $\lambda \gtrsim 1$, shows that

$$(53) \qquad \partial_\lambda^k [R^\pm(H_0, \lambda^4) V R^\pm(H_0, \lambda^4)] = O(\lambda^{-2} \langle x \rangle \langle y \rangle), \qquad k = 0, 1, 2.$$

This suffices to establish the desired bound for the contribution of the first summand of (51) to (50). In fact, the spatial weights are only necessary when $k = 2$. Furthermore, these bounds suffice to control the second summand of (51) to (50) as they imply (for $k = 0, 1, 2$)

$$|\partial_\lambda^k [R^\pm(H_0, \lambda^4) V R^\pm(H_0, \lambda^4)(x,y)]| \lesssim \lambda^{-2} \langle x \rangle \langle y \rangle \int_{\mathbb{R}^3} \langle x_1 \rangle |V(x_1)| dx_1 \lesssim \lambda^{-2} \langle x \rangle \langle y \rangle.$$

The assumed decay on V suffices to ensure the integral converges.

For the final summand in (51), we utilize the limiting absorption principle.

Theorem 5.2. [12, Theorem 2.23] *Let* $|V(x)| \lesssim \langle x \rangle^{-k-1}$. *Then for any* $\sigma > k + 1/2$, $\partial_z^k R_V(z) \in \mathcal{B}(L^{2,\sigma}(\mathbb{R}^d), L^{2,-\sigma}(\mathbb{R}^d))$ *is continuous for* $z \notin 0 \cup \Sigma$. *Further,*

$$\|\partial_z^k R_V(z)\|_{L^{2,\sigma}(\mathbb{R}^d) \to L^{2,-\sigma}(\mathbb{R}^d)} \lesssim z^{-(3+3k)/4}.$$

Note that first by (52), and using that $L^\infty \subset L^{2,-\frac{3}{2}-}$, we have

(54) $$\|[R^\pm(H_0, \lambda^4)]\|_{L^1 \to L^{2,-\sigma}} = O_1(\lambda^{-1}), \quad \sigma > 3/2,$$

which implies the dual estimate as an operator from $L^{2,\sigma} \to L^\infty$. Hence, by Theorem 5.2 we have the following estimate

$$|[R^\pm(H_0, \lambda^4) V R_V(\lambda^4) V R^\pm(H_0, \lambda^4)]|$$

$$\lesssim \|R^\pm(H_0, \lambda^4)\|_{L^{2,\sigma} \to L^\infty} \|V R_V(\lambda^4) V\|_{L^{2,-\sigma} \to L^{2,\sigma}} \|R^\pm(H_0, \lambda^4)\|_{L^1 \to L^{2,-\sigma}} \lesssim \lambda^{-2},$$

provided $|V(x)| \lesssim \langle x \rangle^{-2-}$. Similarly, by (52) and Theorem 5.2 (with $z = \lambda^4$) one obtains the same bound for $\partial_\lambda[R^\pm(H_0, \lambda^4) V R_V(\lambda^4) V R^\pm(H_0, \lambda^4)]$. Finally,

(55) $$|\partial_\lambda^2 \{R^\pm(H_0, \lambda^4) V R_V(\lambda^4) V R^\pm(H_0, \lambda^4)\}| \lesssim \lambda^{-2} \langle x \rangle \langle y \rangle$$

provided $|V(x)| \lesssim \langle x \rangle^{-5-}$. Here, we note that the extra decay on V is needed when the derivatives fall on the perturbed resolvent R_V so that V maps $L^{2,-\frac{5}{2}-} \to L^{2,\frac{5}{2}+}$. The weights arise from when both derivatives fall on a single free resolvent, (52).

of Proposition 5.1. Substituting (52), (53) and (55) into (51) we have shown that

$$|\partial_\lambda^k R_V(\lambda^4)| \lesssim \lambda^{-2} \langle x \rangle \langle y \rangle, \qquad k = 0, 1, 2.$$

Hence, by (50), we have

$$\left| \int_0^\infty e^{-it\lambda^4} \lambda^3 \widetilde{\chi}(\lambda) R_V^\pm(\lambda^4)(x, y) \, d\lambda \right| \lesssim \frac{\langle x \rangle \langle y \rangle}{t^2} \int_1^\infty \lambda^{-5} \, d\lambda \lesssim \frac{\langle x \rangle \langle y \rangle}{t^2}.$$

$\square$

We are now ready to prove the main theorems.

of Theorems 1.1 and 1.2. Interpolating between the high energy bounds (48) and (49) suffice to show that the large energy portion of the solution satsifies the required bounds in either case.

For the low energy portion when zero is regular, we note that Proposition 3.1, specifically (20) may be inserted into the Stone's formula (7). Lemma 2.1 provides the desired time

decay. When there is a resonance of the first kind at zero, we use (40) in Proposition 4.1 in the Stone's formula. Now, the operator F_t is controlled by Lemma 2.2 and the remaining piece is controlled by Lemma 2.1.

$\square$

Acknowledgments

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