A DISPERSIVE BOUND FOR THREE-DIMENSIONAL SCHRÖDINGER OPERATORS WITH ZERO ENERGY EIGENVALUES

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Abstract. We prove a dispersive estimate for the evolution of Schrödinger operators $H = -\Delta + V(x)$ in $\mathbb{R}^3$. The potential is allowed to be a complex-valued function belonging to $L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, $p < \frac{3}{2} < q$, so that H need not be self-adjoint or even symmetric. Some additional spectral conditions are imposed, namely that no resonances of H exist anywhere within the interval $[0, \infty)$ and that eigenfunctions at zero (including generalized eigenfunctions) decay rapidly enough to be integrable.

1. Introduction

The Schrödinger equation is representative of a larger class of dispersive evolution equations, in which wave packets that are localized to distinct regions of Fourier space propagate with different group velocity. One consequence is that mass concentration tends to be transient in nature as the frequency components of a solution eventually separate from one another. Solutions of a dispersive equation are therefore expected to display an improvement in local regularity and/or a decrease in overall size over long time periods. We will concentrate here on evolution estimates that use the $L^1(\mathbb{R}^3)$ norm of initial data to control the supremum of the solution at later times. As with any map from L^1 to L^∞, this can also be understood as a pointwise bound on the propagator kernel.

The free Schrödinger equation propagates forward to time t through the action of $e^{it\Delta}$ on the initial data. By standard Fourier inversion identities, this is equivalent to convolution against the complex Gaussian kernel $(-4\pi it)^{-3/2} e^{i(|x|^2/4t)}$. It immediately follows that the free evolution satisfies a dispersive bound

$$\|e^{it\Delta} f\|_\infty \leq (4\pi|t|)^{-3/2} \|f\|_1 \tag{1}$$

at all times $t \neq 0$. In this paper we seek to prove similar estimates for the time evolution e^{-itH} induced by a perturbed Hamiltonian $H = -\Delta + V(x)$. We assume that $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ for a pair of exponents $p < \frac{3}{2} < q$, corresponding to homogeneous behavior on the order of $|x|^{-2+\varepsilon}$ for local singularities and $|x|^{-2-\varepsilon}$ as $|x| \to \infty$. There are no further restrictions on the size of V or on its negative or imaginary parts. In particular, there is no assurance that H is a symmetric or self-adjoint operator.

Eigenvalues of H present an immediate obstruction to the validity of dispersive estimates. If there exists a nonzero function $\Psi \in L^1(\mathbb{R}^3)$ that solves the eigenvalue equation $H\Psi = \lambda\Psi$, then the associated Schrödinger evolution

$$e^{-itH}\Psi = e^{-it\lambda}\Psi$$

certainly lacks the decay properties of (1). Indeed, such a solution either fails to decay at all (if $\lambda \in \mathbb{R}$) or experiences exponential growth according to the imaginary part of λ. Generalized eigenfunctions (i.e. solutions to $(H - \lambda)^k \Psi = 0$ for some $k > 1$) create a similar problem. The series expansion of e^{-itH} in powers of $t(H - \lambda)$ shows that the evolution of initial data Ψ must experience growth at a rate of $|e^{-it\lambda}|$ times a degree $(k-1)$ polynomial in t.

These bound states can often be avoided by introducing the appropriate spectral projection. A revised dispersive estimate for $H = -\Delta + V$ might take the form

$$\left\|e^{-itH}(I - \textstyle\sum_j P_{\lambda_j}(H))f\right\|_\infty \lesssim |t|^{-\frac{3}{2}} \|f\|_1 \tag{2}$$

where each $P_{\lambda_j}(H)$ is a projection onto the point spectrum of H at the eigenvalue $\lambda_j \in \mathbb{C}$. We use the term "projection" here to indicate a bounded linear operator satisfying $P^2 = P$, though not necessarily the

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canonical L^2-orthogonal projection. In general the correct choice for $P_\lambda(H)$ will depend on the eigenfunctions of H^* as well as those of H. Details of its construction are given in Section 5.

One additional concern here is the possible existence of resonances, which are solutions to the eigenvalue equation that do not decay rapidly enough to belong to $L^2(\mathbb{R}^3)$. Resonances exhibit enough persistence behavior (by virtue of their resemblance to L^2 bound states) to negate most dispersive estimates, but they cannot be so easily removed with a spectral projection.

Our main theorem proves that (2) remains valid in the presence of an eigenvalue at zero, so long as each of the eigenfunctions (and generalized eigenfunctions) belongs to $L^1(\mathbb{R}^3)$. Resonances must still be forbidden. To make a precise statement we introduce a classification system for the eigenspace of H lying over $\lambda = 0$.

Let X_1 represent the space of threshold eigenfunctions and resonances of H, which are distributional solutions of $(-\Delta+V)\Psi = 0$ belonging to $L^3_{\text{weak}}(\mathbb{R}^3)$, the same class as the Green's function of the Laplacian. Among weighted L^2 spaces, this places $X_1 \subset \langle x\rangle^\sigma L^2(\mathbb{R}^3)$ for each $\sigma > \frac{1}{2}$. Our assumptions on V indicate that VX_1 is a subset of $L^1(\mathbb{R}^3)$.

Because H is not a symmetric operator, its "Jordan block" structure at each eigenvalue may not be limited to a direct sum of eigenfunctions. We would like to classify the deeper structure by recursively setting X_{k+1} to be the space of functions Ψ such that $H\Psi \in X_k$, $k \geq 0$. This heuristic is imprecise with regard to the acceptable level of spatial growth/decay of $\Psi \in X_k$, so we introduce an inductive set of assumptions in order to define X_k more carefully. The base case assumption is

$$X_1 \subset L^1(\mathbb{R}^3).$$

Assuming at each step that X_k is a subspace of $L^1(\mathbb{R}^3)$, we are able to define

$$X_{k+1} := \{\Psi \in L^3_{\text{weak}}(\mathbb{R}^3) : (-\Delta + V)\Psi \in X_k\}.$$

We also denote by $\overline{X}_k$ the image of X_k under complex conjugation. These spaces describe the eigenvector structure of the adjoint operator $H*$ thanks to the fact that H and H^* are complex conjugates of one another. When the relationships between X_k and $\overline{X}_k$ are explored further in Section 5, only the connection to adjoint operators (rather than to conjugation) will play an essential role.

Now we are prepared to state the theorem.

Theorem 1. *Let $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, $p < \frac{3}{2} < q$ be a complex-valued potential. Assume that $H = -\Delta + V$ has no resonances, and that the generalized eigenspace at zero energy satisfies the assumptions*

(A1) *$X_k \subset L^1(\mathbb{R}^3)$ for each $k \geq 1$.*
(A2) *$X := \bigcup_k X_k$ is a finite-dimensional space.*

Then

$$\left\|e^{-itH}(I - P_{pp}(H))f\right\|_\infty \lesssim |t|^{-\frac{3}{2}}\|f\|_1. \tag{3}$$

Under these conditions, the spectral multiplier $P_{pp}(H) := \sum_j P_{\lambda_j}(H)$ is a finite-rank projection.

The propagator e^{-itH} is not a unitary operator unless V is real-valued, and indeed the evolution of complex eigenvalues and/or generalized eigenfunctions must be unbounded at large times. Once these exceptional vectors are projected away, then a global in time L^2 bound can be recovered. The following statement is adapted from Theorem 1 in [10], which is based on the endpoint inhomogeneous Strichartz estimate for the Laplacian [16].

Theorem 2. *Under the same assumptions on Theorem 1,*

$$\left\|e^{-itH}(I - P_{pp}(H))f\right\|_2 \lesssim \|f\|_2. \tag{4}$$

There is a slight difference in that the cited result assumes the absence of generalized eigenfunctions at zero and elsewhere. However the construction in Sections 5 and 6 here can be readily incorporated into the prevailing line of argument, making a separate proof unnecessary.

The first dispersive estimate of the form (3) was proved in [14] for real potentials satisfying a regularity hypothesis $\hat{V} \in L^1(\mathbb{R}^3)$ and pointwise decay bound $|V(x)| \lesssim \langle x\rangle^{-7-\varepsilon}$. Successive improvements ([25], [11], and [9] in chronological order) relaxed the requirements on V down to the $L^p \cap L^q$ condition found in the statement of Theorem 1. In all these results H is assumed not to possess an eigenvalue or resonance at zero.

The importance of the zero-energy spectral condition was already understood in earlier work on Schrödinger operators in weighted $L^2(\mathbb{R}^3)$. Notably, the asymptotic series expansion for e^{-itH} developed in [13] shows a leading term of order $|t|^{-1/2}$ (rather than the desired $|t|^{-3/2}$) whenever H has a resonance at zero. Furthermore, when $\lambda = 0$ is an eigenvalue this term typically does not vanish even after projecting away from the eigenfunction(s) with $I - P_0(H)$. The same phenomenon is observed in [7] and [26] in the $L^1 \to L^\infty$ setting. One general interpretation of these findings is that threshold eigenvalues of H cast a significant "shadow" onto adjacent parts of the continuous spectrum.

Both of the works cited above show how to compute the part of a solution that decays like $|t|^{-1/2}$ from information about the threshold eigenfunctions, with [26] giving an explicit formula modulo some scalar coefficients. If our conditions (A1) and (A2) are satisfied, then Theorem 1 greatly simplifies these calculations by showing that the leading term in fact vanishes.

It must be emphasized that the threshold spectral properties of a given Schrödinger operator $H = -\Delta + V$ are difficult to discern from V alone. For potentials with sufficiently large negative part there are no effective criteria to determine whether a resonance or eigenvalue is present at zero, only an understanding that these features are absent in the generic case. In Section 8 we construct a large family of prospective examples of potentials that generate a zero-energy eigenfunction with suitable decay. We believe that typically there are no other nontrivial elements of X, in which case (A1) and (A2) certainly hold. An argument in support of this assertion (though not a rigorous proof) is presented here as well.

Recent progress on the dynamical behavior of semilinear Schrödinger equations has created a need for dispersive estimates associated to non-selfadjoint Hamiltonians. With a model nonlinearity that depends on $|u(t,x)|^2$, the linearization around any nonzero standing wave solution will feature coupled equations for the discrepancy $w(t,x)$ and its complex conjugate $\overline{w(t,x)}$. The resulting 2×2 system of Schrödinger equations has a matrix "potential" whose diagonal elements are real and whose off-diagonal elements are skew-Hermitian. In any analogy to the scalar case these would correspond to the real and imaginary parts of our potential $V(x)$.

Stability of the nonlinear equation (or a characterization of its stable manifold) depends largely on the spectrum of this linearized operator, which is known for a limited number of examples. Dispersive estimates then play an important role in controlling the nonlinear interactions that appear in the course of a fixed-point argument. A representative sample of results can be found in [3], [5], [23], and [24].

Even in these settings it is currently necessary to assume that the Hamiltonian does not possess any eigenvalues at the threshold or embedded in its continuous spectrum. That is to say, the existing dynamical analysis depends on this spectral property but we lack a procedure to verify its validity. If embedded eigenvalues do exist, however, they must be algebraically simple and the associated eigenfunctions enjoy rapid polynomial decay [8, Lemma 4]. Thus they would satisfy a set of conditions analogous to (A1) and (A2), which suggests that a $|t|^{-3/2}$ linear dispersive bound is still achievable.

Returning to Theorem 1, much of the work is concentrated in choosing the correct set of projections in its statement. The structure of $P_0(H)$ will be examined in detail in Section 5, and is representative of all $P_\lambda(H)$. One characterization is that $P_\lambda(H)$ is the projection whose range is the generalized eigenspace of H over an eigenvalue λ, and whose adjoint has the generalized eigenspace of H^* over $\bar\lambda$ as its range. For isolated eigenvalues $\lambda \notin \mathbb{R}^+$ the analytic Fredholm theorem guarantees that $P_\lambda(H)$ has finite rank. These projections can also be defined as an element of the analytic functional calculus of H (as in [2], Chapter 1). Specifically, $P_\lambda(H)$ corresponds to a function that is identically 1 in a neighborhood of λ and vanishes near the remaining spectrum of H. The orthogonality relations $P_\lambda(H)P_\mu(H) = 0$ continue to hold for all $\lambda \neq \mu$ even when H is not self-adjoint.

The analytic functional calculus is less obviously applicable to eigenvalues embedded within the essential spectrum $[0,\infty)$ or at its endpoint. It is shown in [12] that Schrödinger operators with a complex potential $V \in L^{3/2}(\mathbb{R}^3)$ possess no eigenvalues along the positive real axis, however examples with an eigenvalue at $\lambda = 0$ are easily constructed as in Section 8

The proof of Theorem 1 is largely based on elementary observations, however several detours are needed in order to gather the relevant background. The road-map proceeds as follows: Section 2 describes a Fourier transform argument that reduces the problem to an operator estimate involving certain resolvents, whose basic properties are outlined in Section 3. Separate calculations are then required for high, intermediate,

and low energies, with the first two being borrowed nearly intact from [9]. Some details are sketched out in Section 4 for future reference. In Section 5 we examine the structure of X and its duality relationship to $\overline{X}$ in order to construct the projection $P_0(H)$. Sections 6 and 7 assemble the resolvent of H in the neighborhood of its threshold eigenvalue and recover the desired estimates once the poles associated to each (generalized) eigenfunction are projected away. The concluding section suggests a method for constructing potentials with an eigenvalue at zero satisfying the assumptions of Theorem 1.

2. Reduction to Resolvent Estimates

Let $H = -\Delta + V$ in $\mathbb{R}^3$ and define the resolvents $R_0(z) := (-\Delta - z)^{-1}$ and $R_V(z) := (H - z)^{-1}$. For each $z \in \mathbb{C} \setminus \mathbb{R}^+$, the operator $R_0(z)$ is given by convolution with the kernel

$$R_0(z)(x) = \frac{e^{i\sqrt{z}|x|}}{4\pi|x|},$$

where $\sqrt{z}$ is taken to have positive imaginary part. While $R_V(z)$ is not translation–invariant and does not possess an explicit representation of this form, it can be expressed in terms of $R_0(z)$ via the identity

$$R_V(z) = (I + R_0(z)V)^{-1}R_0(z) = R_0(z)(I + VR_0(z))^{-1} \tag{5}$$

In the case where $z = \lambda \in \mathbb{R}^+$, the resolvent may be defined as a limit of the form $R_0(\lambda \pm i0) := \lim_{\varepsilon \downarrow 0} R_0(\lambda \pm i\varepsilon)$. The choice of sign determines which branch of the square-root function is selected in the formula above, therefore the two continuations do not agree with one another. In this paper we refer to resolvents along the positive real axis using the following notation.

$$R_0^{\pm}(\lambda) := R_0(\lambda \pm i0) \qquad R_V^{\pm}(\lambda) := R_V(\lambda \pm i0)$$

Note that $R_0^-(\lambda)$ is the formal adjoint of $R_0^+(\lambda)$, and more generally $R_V^{\pm}(\lambda)$ is the formal adjoint of $R_{\overline{V}}^{\mp}(\lambda)$. In fact these resolvents will be truly adjoint to one another, since our analysis takes place in settings where they are bounded operators. Determining the adjoint of H is a more delicate matter because of technicalities related to its domain, but it can be done because $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$, see Theorem X.19 in [20].

If V were real-valued and satisfied the L^p condition in Theorem 1 (making H self-adjoint), then the Stone formula for the absolutely continuous spectral measure of H would dictate that

$$e^{-itH}f = \sum_j e^{-i\lambda_j t}\langle f, \psi_j\rangle\psi_j + \frac{1}{2\pi i}\int_0^\infty e^{-it\lambda}[R_V^+(\lambda) - R_V^-(\lambda)]f\, d\lambda.$$

The summation takes place over a countable number of eigenvalues λ_j with associated eigenfunctions ψ_j. A similar ansatz, modified to correctly portray the evolution of generalized eigenfunctions of a non-selfadjoint H, is valid for complex V as well. Here the spectral decomposition takes the form

$$\begin{aligned} e^{-itH}f = \sum_{\lambda_j \neq 0} e^{-itH}P_{\lambda_j}(H)f + e^{-itH}P_0(H)f \\ + \frac{1}{2\pi i}\int_0^\infty e^{-it\lambda}[R_V^+(\lambda) - R_V^-(\lambda)]f\, d\lambda. \end{aligned} \tag{6}$$

The range of each $P_{\lambda_j}(H)$ is a finite-dimensional subspace that is invariant under the action of H. The restriction of H to this space possesses a single eigenvalue λ_j, thus e^{-itH} behaves here like $e^{-i\lambda_j t}$ times a square matrix that grows polynomially in t.

In both cases the projection $I - P_{pp}(H)$ removes eigenfunctions in such a way that the initial sum vanishes. The success of dispersive estimates is then governed by the integral term, in particular by its regularity approaching the endpoint at zero. It is customary to view the right-hand integral, together with the discrete $\lambda = 0$ term, as a contour integral in the complex plane. Making a change of variables $\lambda \mapsto \lambda^2$ opens up the contour along the real axis, with the understanding that

$$R_V^+(\lambda^2) = \lim_{\varepsilon \downarrow 0} R_V((\lambda + i\varepsilon)^2) = \lim_{\varepsilon \downarrow 0} R_V(\lambda^2 + i\,\mathrm{sign}\,(\lambda)\varepsilon)$$

for all $\lambda \in \mathbb{R}$. Under this new notation, the integral term (representing the absolutely continuous part of the Schrödinger evolution) combines with the contribution of threshold eigenvalues to yield

$$\begin{aligned}\frac{1}{\pi i}\int_{-\infty}^{\infty} e^{-it\lambda^2}\lambda R_V^+(\lambda^2) f\, d\lambda \\ = \frac{1}{\pi i}\int_{-\infty}^{\infty} e^{-it\lambda^2}\lambda R_0^+(\lambda^2)(I+VR_0^+(\lambda^2))^{-1} f\, d\lambda.\end{aligned}$$

A formal integration by parts leads to the expression

$$\frac{1}{2\pi t}\int_{-\infty}^{\infty} e^{-it\lambda^2}\big(I+R_0^+(\lambda^2)V\big)^{-1}\frac{d}{d\lambda}\Big[R_0^+(\lambda^2)\Big]\big(I+VR_0^+(\lambda^2)\big)^{-1} f\, d\lambda.$$

If we adopt the shorthand notation $T^+(\lambda) := (I+VR_0^+(\lambda^2))^{-1}$ and also take $T^-(\lambda) := (I+\overline{V}R_0^-(\lambda^2))^{-1}$, Theorem 1 should be equivalent to the estimate

$$\Big|\int_{-\infty}^{\infty} e^{-it\lambda^2}\Big\langle \frac{d}{d\lambda}\Big[R_0^+(\lambda^2)\Big]T^+(\lambda)f,\, T^-(\lambda)g\Big\rangle\, d\lambda\Big| \lesssim |t|^{-\frac{1}{2}}\|f\|_1\|g\|_1 \tag{7}$$

holding for all $f \in \mathrm{ran}(I-P_{pp}(H)) \subset L^1$ and $g \in \mathrm{ran}\,(I-P_{pp}(H)^*) \subset L^1$. It is not necessary to test every $g \in L^1$ because of the operator identity

$$e^{-itH}(I-P_{pp}(H)) \;=\; (I-P_{pp}(H))e^{-itH}(I-P_{pp}(H))$$

which follows from $I-P_{pp}(H)$ being an idempotent function of H.

Remark 1. To conduct the formal steps properly, one should introduce a smooth cutoff $\chi(\lambda/L)$ into the integrand, then take limits as $L \to \infty$. The novel analysis in this paper takes place when λ is confined to a compact neighborhood of zero, where it is reasonable to assume that any such cutoff function is identically 1.

The integral in (7) can be evaluated via Plancherel's identity. First note that $e^{-it\lambda^2}\frac{d}{d\lambda}[R_0^+(\lambda^2)]$ is a family of convolution operators represented explicitly by the kernel $(-4\pi i)^{-1}e^{-it\lambda^2+i\lambda|x|}$. Taking ρ as the variable dual to λ, its Fourier transform should be a family of convolution operators with kernel

$$\big(e^{-it\lambda^2}\tfrac{d}{d\lambda}[R_0^+(\lambda^2)]\big)\hat{}\,(\rho)(x) = (16\pi i t)^{-\frac{1}{2}}e^{i\frac{(\rho-|x|)^2}{4t}}.$$

In particular this is bounded by $|t|^{-\frac{1}{2}}$ for every value of ρ and $|x|$. Theorem 1 then follows from as appropriate L^1 estimate on the Fourier transform of $T^\pm(\lambda)$. This is the approach taken in [9], whose principal technical result is stated below.

Theorem 3. *Let $V \in L^p(\mathbb{R}^3)\cap L^q(\mathbb{R}^3)$, $p < \frac{3}{2} < q$, be a real-valued potential and suppose that $H = -\Delta + V$ does not have a resonance or eigenvalue at zero. Then*

$$\big\|(T^\pm)\hat{}\,(\rho)f\big\|_{L^1_\rho L^1_x} \lesssim \|f\|_1$$

for all $f \in L^1(\mathbb{R}^3)$.

By contrast, if there exists an eigenvalue or resonance at zero, then $T^\pm(\lambda)$ must have a corresponding pole. One cannot expect the Fourier transform of an unbounded family of operators to satisfy any kind of L^1 estimate. The proof of Theorem 1 therefore hinges on a modified L^1 bound which carefully avoids any poles.

Theorem 4. *Let $V \in L^p(\mathbb{R}^3)\cap L^q(\mathbb{R}^3)$, $p < \frac{3}{2} < q$, be a complex potential. Suppose that $H = -\Delta + V$ has no resonances along the interval $[0,\infty)$, and that the eigenspace at zero satisfies assumptions (A1) and (A2). Then*

$$\big\|(T^+)\hat{}\,(\rho)f\big\|_{L^1_\rho L^1_x} \lesssim \|f\|_1$$

for all $f \in L^1(\mathbb{R}^3)$ orthogonal to $\overline{X}$ in the sense that $\int f(x)\Psi(x)\,dx = 0$ for each $\Psi \in X$. By taking complex conjugates, the same estimate is true of $(T^-)\hat{}\,(\rho)$ applied to all integrable functions $g \perp X$.

The class of admissible f should be recognized as the range of $I - P_0(H)$. Working backward, Theorem 4 implies that the original integral term in (6) is always bounded in $L^\infty(\mathbb{R}^3)$ with norm controlled by $|t|^{-\frac{3}{2}}\|f\|_1$. The proof of Theorem 1 follows by adding in, then projecting away, the contributions of each discrete eigenvalue λ_j.

3. Resolvent Estimates and Resonances

We catalog several properties of the free Schrödinger resolvent for future use. These are all immediate consequences of the convolution representation

$$R_0^\pm(\lambda^2)(x) = \frac{e^{\pm i\lambda|x|}}{4\pi|x|}. \tag{8}$$

It will be assumed at all times that $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, $p < \frac{3}{2} < q$, and is measured according to the norm

$$\|V\| := \max\big(\|V\|_p, \|V\|_q\big).$$

Proposition 5. *The convolution kernel of $R_0^+(\lambda^2)$ belongs to $L^3_{\text{weak}}(\mathbb{R}^3)$ uniformly for all $\lambda \in \mathbb{R}$. Moreover, given any $p < \frac{3}{2}$, the convolution kernel of $R_0^+(\lambda^2) - R_0^+(\mu^2)$ belongs to $L^{p'}(\mathbb{R}^3)$ with a norm uniformly bounded by $|\lambda - \mu|^{1-\frac{3}{p'}}$.*

Given any function $V \in L^p \cap L^q$, $p < \frac{3}{2} < q$, the family of operators $I + VR_0^+(\lambda^2)$ are each bounded on $L^1(\mathbb{R}^3)$ and vary continuously with the parameter $\lambda \in \mathbb{R}$.

Proof. The first two assertions are verified by direct inspection of the free resolvent kernel in (8). They show that $R_0^+(\lambda^2)$ is a bounded map from $L^1(\mathbb{R}^3)$ to $L^{p'}(\mathbb{R}^3) + L^{q'}(\mathbb{R}^3)$, and provide a norm estimate for the difference $R_0^+(\lambda^2) - R_0^+(\mu^2)$. Multiplication by V then maps this space back to $L^1(\mathbb{R}^3)$ by Hölder's inequality. □

Proposition 6. *Suppose $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, $p < \frac{3}{2} < q$. For each $\lambda \in \mathbb{R}$, $VR_0^+(\lambda^2)$ is a compact operator from $L^1(\mathbb{R}^3)$ to itself.*

Proof. If V were a smooth, bounded function with compact support, then $(-\Delta + 1)VR_0^+(\lambda^2)$ would also map $L^1(\mathbb{R}^3)$ to itself thanks to the identity $(-\Delta + 1)R_0^+(\lambda^2) = I + (\lambda^2 + 1)R_0^+(\lambda^2)$. In that case the target space of $VR_0^+(\lambda^2)$ consists of functions in $W^{2,1}(\mathbb{R}^3)$ whose support is contained in $\operatorname{supp} V$. This embeds compactly into $L^1(\mathbb{R}^3)$ as desired.

The operator norm of $VR_0^+(\lambda^2)$ is always controlled by $\|V\|$. So for general potentials it suffices to express V as a norm limit of smooth compactly supported functions. □

It is less immediate, but still well known that $I + VR_0^+(\lambda^2)$ becomes a small perturbation of the identity once $|\lambda|$ is sufficiently large.

Proposition 7. *Suppose $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, $p < \frac{3}{2} < q$. Then*

$$\lim_{|\lambda|\to\infty} \|(VR_0^+(\lambda^2))^2\|_{1\to 1} = 0.$$

Proof. There are numerous mapping estimates for the free resolvent which incorporate some degree of decay as $\lambda \to \infty$. To name one example, [17] shows that $R_0^+(\lambda^2)$ is a bounded operator from $L^{\frac{4}{3}}$ to L^4 with norm comparable to $|\lambda|^{-\frac{1}{2}}$. By interpolation with the elementary kernel-derived bounds above, one concludes that $R_0^+(\lambda^2) : L^{1+\varepsilon} \to L^{3+\varepsilon'}$ with a polynomial decay rate determined by scaling.

Putting these pieces together, $VR_0^+(\lambda^2)$ maps L^1 to $L^{1+\varepsilon}$ with unit norm, and it takes $L^{1+\varepsilon}$ back to L^1 with an operator norm controlled by $|\lambda|^{-\varepsilon}$ for some $\varepsilon > 0$. □

In order for its Fourier transform to satisfy an estimate in L^1, the family of operators $(I + VR_0^+(\lambda^2))^{-1}$ should be uniformly bounded in norm. The above proposition suffices to show uniform boundedness for large λ, where a Neumann series for the inverse will be convergent. The continuity assertion of Proposition 5 then reduces the problem to one of pointwise existence.

Since each operator $I + VR_0^+(\lambda^2)$ is a compact perturbation of the identity, the Fredholm alternative stipulates that inverses must exist except in the case where $(I + VR_0^+(\lambda^2))g = 0$ has a solution $g \in L^1$.

It would follow that $\Psi = R_0^+(\lambda^2)g$, which belongs to various spaces such as $L^3_{\text{weak}}(\mathbb{R}^3)$ and $\langle x\rangle^\sigma L^2$ for all $\sigma > \frac{1}{2}$, is a distributional solution of $(-\Delta + V - \lambda^2)\Psi = 0$. In the event that $\Psi \notin L^2$, we say that H has a *resonance* at λ^2.

It is shown in [12] that H can only possess true eigenvalues at $\lambda^2 \in \mathbb{R}^+$ when $\lambda = 0$. Furthermore, for real valued potentials there is a self-adjointness argument [1] to rule out the possibility of resonances at any nonzero λ. We have adopted condition (A2) in Theorem 1 in order to avoid embedded resonances even for complex potentials. With the invertability of $I + VR_0^+(\lambda^2)$ now ensured for all $\lambda \in \mathbb{R} \setminus \{0\}$, Propositions 5 and 7 lead to the uniform bound

$$\sup_{|\lambda|>\lambda_1} \|(I + VR_0^+(\lambda^2))^{-1}\|_{1\to 1} < \infty \tag{9}$$

for every $\lambda_1 > 0$.

Our attention now turns to the eigenvalues that may occur at $\lambda = 0$. As before, the only obstruction to the invertability of $I + VR_0^+(0)$ is the existence of nonzero $g \in L^1(\mathbb{R}^3)$ such that $(I + VR_0^+(0))g = 0$. These solutions then generate our space X_1 via the relationship

$$X_1 = \{R_0^+(0)g : (I + VR_0^+(0))g = 0, g \in L^1\}. \tag{10}$$

One consequence of the Fredholm Alternative is that the space of admissible functions g must be finite-dimensional, so X_1 has finite dimension as well.

Proposition 8. *Let $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, $p < \frac{3}{2} < q$, and suppose that Assumption (A1) holds. Then $X_k \subset L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ for each $k \geq 1$.*

Proof. The kernel estimate in Proposition 5 immediately suggests that X_1 is contained in $L^3_{\text{weak}}(\mathbb{R}^3)$, based on the *a priori* information that $g \in L^1$. The assumptions on V then indicate that $V\Psi \in L^r(\mathbb{R}^3)$ for all $1 \leq r < 1 + \frac{2q-3}{q+3}$. Each function $\Psi \in X_1$ satisfies the recurrence relation

$$\Psi = -R_0^+(0)V\Psi = \Delta^{-1}V\Psi.$$

A bootstrapping argument, estimating the right-hand side with Sobolev inequalities and the assumption that $V \in L^q$ for some $q > \frac{3}{2}$, proves that $\Psi \in L^\infty$ (and even $W^{2,q}(\mathbb{R}^3)$). Meanwhile, $\Psi \in L^1$ by direct assertion of (A1).

The same approach applies equally well to X_k. By definition $\Psi \in X_k$ precisely if there is some element $\Psi_0 \in X_{k-1}$ for which

$$\Psi = -R_0^+(0)(V\Psi - \Psi_0) = \Delta^{-1}(V\Psi - \Psi_0).$$

Assuming inductively that Ψ_0 belongs to every $L^p(\mathbb{R}^3)$ makes it possible to place $\Psi \in L^\infty$ by bootstrapping. The fact that $X_k \subset L^1$ is again part of the statement (A1). □

This leads to a desirable state of affairs: Orthogonal projection away from the space of eigenfunctions X_1 (or generalized eigenfunctions X) acts as a bounded operator on both $L^1(\mathbb{R}^3)$ and $L^\infty(\mathbb{R}^3)$.

4. The High-Energy and Intermediate-Energy Cases

Theorem 4 is in fact a compilation of three statements, since distinct methods are used to analyze $T^\pm(\lambda)$ depending on the size of $|\lambda|$. Separating the different energy regimes is done by means of a smooth cutoff $\chi(\lambda)$ that is identically equal to 1 when $|\lambda| \leq 1$ and vanishes for $|\lambda| \geq 2$. We are able to borrow the high-energy and intermediate-energy results from [9] with minimal modification; the precise statements are formulated below.

Theorem 9. *Let $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$, $p < \frac{3}{2} < q$ be a complex-valued potential. There exists a number $\lambda_1 < \infty$ depending only on $\|V\|$ so that the inequality*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}} \big|\big[(1 - \chi(\,\cdot\,/\,\lambda_1))T^+(\,\cdot\,)\big]^\wedge(\rho)f(x)\big|\,d\rho\,dx \lesssim \|f\|_1 \tag{11}$$

holds for all $f \in L^1(\mathbb{R}^3)$.

The proof is based on the Neumann series expansion $(I + VR_0^+(\lambda^2))^{-1} = \sum_m (-VR_0^+(\lambda^2))^m$. A finite number of terms at the beginning are computed directly from the resolvent kernel (8), and for small V it is even possible to control the entire series in this manner without any further restrictions or assumptions [22]. Otherwise the high-energy cutoff is needed so that the m^{th} term can be bounded by $(\lambda_1^{-\varepsilon}\|V\|)^m$ (estimates such as Proposition 7 play a large role here), leading to a convergent geometric series.

We refer the reader to [9] for the details. While only the case of real potentials is considered, the estimates are based solely on the size of V in $L^p \cap L^q$ and extend to the complex case without modification. Within the interval $\lambda \in (\lambda_1, \infty)$, convergence of the Neumann series for $(I+VR_0^+(\lambda^2))^{-1}$ expressly forbids the existence of embedded eigenvalues or resonances of H.

Theorem 10. *Suppose V satisfies the conditions of Theorem 1, and fix any $0 < \lambda_1 < \infty$. Then*

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}} \big| \big[(\chi(\cdot/\lambda_1) - \chi(\lambda_1 \cdot)) T^+(\cdot) \big]^\wedge(\rho) f(x) \big| \, d\rho \, dx \lesssim \|f\|_1 \tag{12}$$

holds for all $f \in L^1(\mathbb{R}^3)$.

This statement is almost identical to Theorem 13 in [9], only with a small interval around zero energy removed so as to avoid any possible eigenvalues there. We reproduce an summary of the proof because the more delicate low-energy analysis in Sections 6 and 7 will be adapted from these calculations.

Existence of $(I + VR_0^+(\lambda^2))^{-1}$ for each λ in this range is established via the Fredholm Alternative (with (A1) added by fiat as a supporting assumption) rather than by power series methods. This is a black-box procedure in that the outcome cannot be expressed more directly in terms of V and λ. As a result, the only known relationship between the various operators $T^+(\lambda)$ comes from the continuity statement in Proposition 5.

With this in mind, we introduce a secondary cutoff that localizes to much smaller intervals of λ where continuity arguments can be applied successfully. Theorem 10 is proved by adding up a finite number of local results.

Fix a "benchmark" energy $\lambda_0 \in \mathbb{R}$ and let $S_0 = (I + VR_0^+(\lambda_0^2))^{-1}$. For all values of λ sufficiently close to λ_0, we may regard $R_0^+(\lambda^2)$ as a perturbation of $R_0^+(\lambda_0^2)$ and treat the inverse of $I+VR_0^+(\lambda^2)$ as a perturbation of S_0 according to the formula

$$(I + VR_0^+(\lambda^2))^{-1} = (I + S_0 V B_{\lambda_0}^+(\lambda^2))^{-1} S_0$$

where $B_{\lambda_0}^+(\lambda^2)$ represents the difference $R_0^+(\lambda^2) - R_0^+(\lambda_0^2)$.

Let $\chi(\lambda)$ be a smooth cutoff that is identically equal to 1 when $|\lambda| \le 1$ and vanishes for all $|\lambda| \ge 2$. Suppose η is any smooth function supported in the interval $[\lambda_0 - r, \lambda_0 + r]$. The operator inverse of $I + VR_0^+(\lambda^2)$ has a local Neumann series expansion

$$\eta(\lambda)(I + VR_0^+(\lambda^2))^{-1} = \eta(\lambda) \sum_{m=0}^{\infty} \big(- S_0 V B_{\lambda_0}^+(\lambda^2) \chi\big(\tfrac{\lambda-\lambda_0}{r}\big)\big)^m S_0. \tag{13}$$

Convergence of the sum is guaranteed by Proposition 5 provided r is sufficiently small. Note that the role of S_0 is limited to its existence as a (fixed) bounded operator on $L^1(\mathbb{R}^3)$ whose norm is controlled by (9).

In order to take the Fourier transform of (13), the key step is to estimate the Fourier transform (with respect to λ) of $VB_{\lambda_0}^+(\lambda^2)\chi(r^{-1}(\lambda - \lambda_0))$. This is a family of integral operators with kernel

$$K(x, y, \lambda) = V(x) \frac{e^{i(\lambda-\lambda_0)|x-y|} - 1}{4\pi|x-y|} e^{i\lambda_0|x-y|} \chi\big(\tfrac{\lambda-\lambda_0}{r}\big)$$

Its Fourier transform is also a family of integral operators with kernel

$$\hat{K}(x, y, \rho) = V(x) e^{i\lambda_0(|x-y|-\rho)} \frac{r\hat{\chi}(r(\rho - |x-y|)) - r\hat{\chi}(r\rho)}{4\pi|x-y|}$$

where ρ is the Fourier variable dual to λ. Using the Mean Value inequality,

$$\int_{\mathbb{R}} r \big| \hat{\chi}(r(\rho - |x-y|)) - \hat{\chi}(r\rho) \big| \, d\rho \le \min(2\|\hat{\chi}\|_1, r|x-y|\|\hat{\chi}'\|_1)$$

therefore

$$\int_{\mathbb{R}} |\hat{K}(x,y,\rho)|\, d\rho \lesssim |V(x)| \min(r, |x-y|^{-1})$$

Then we can further integrate with respect to x by taking V in $L^p \cap L^q$, with the result

$$\sup_{y\in\mathbb{R}^3} \int_{R^4} |\hat{K}(x,y,\rho)|\, dx d\rho \leq C_{p,q} r^{\varepsilon} \|V\| \tag{14}$$

The exponent is $\varepsilon = \min(\frac{3}{p} - 2, 2 - \frac{3}{q}) > 0$. In other words, given any function $f \in L^1$,

$$\int_{\mathbb{R}} \|\hat{K}(\rho) f\|_1 \, d\rho \leq C_{p,q} \|V\| r^{\varepsilon} \|f\|_1. \tag{15}$$

Composing with the bounded operator S_0 has no effect except to change the value of the constant $C_{p,q}$. Multiplying the original family of operators by $\eta(\lambda)$ causes a convolution by $\hat{\eta}(\rho)$ on the Fourier side. This increases the constants again by a factor of $\|\hat{\eta}\|_1$, by the standard L^1 convolution arguments.

The Fourier transform of the m^{th} term of the series is subject to the estimate

$$\int_{\mathbb{R}} \|\hat{K}_m(\rho) f\|_1 \, d\rho \leq \left(C_{p,q} \|V\| r^{\varepsilon}\right)^m \|\hat{\eta}\|_1 \|f\|_1.$$

So long as r is small enough that $C_{p,q}\|V\| r^{\varepsilon} < \frac{1}{2}$, the Fourier transform of the Neumann series (13) will converge as desired.

We observe that convergence is not influenced by the specific profile of η, so any smooth partition of unity for $[\lambda_1^{-1}, \lambda_1]$ suffices to complete the proof of Theorem 10 so long as each component is supported in an interval of length less than $2r$.

5. Decomposition and Duality of X and $\overline{X}$

In the event that $I + VR_0^+(0)$ is an invertible operator on $L^1(\mathbb{R}^3)$, it is possible to choose $\lambda = 0$ as a benchmark energy as well. The zero-energy cutoff $\chi(\lambda_1 \cdot)$ in Theorem 10 could then be removed without consequence. This corresponds to the (frequently invoked) assumption that zero is neither an eigenvalue nor a resonance of H. In our notation, an equivalent condition is that $X = \{0\}$.

More generally, X captures the algebraic structure of H over its $\lambda = 0$ eigenvalue. We would like to choose a "Jordan basis" for X so that the action of H has a simple representation. In the process we will recover numerous duality properties regarding the pairing of functions from X and $\overline{X}$. The main (perhaps only) ingredient is the orthogonality relation between the range of a bounded operator and the kernel of its adjoint. To avoid problems stemming from the domain of H and H^k, we will only consider the restrictions $H : X \to X$ and $H^* : \overline{X} \to \overline{X}$, and formulate all other arguments in terms of resolvents. The original definition of X in Section 1 was carried out in a similar spirit.

Recall that X is a nested union of spaces $X_1 \subset X_2 \subset \ldots$. It will be assumed at all times that properties (A1) and (A2) are satisfied. By construction X_k consists of the distributional solutions to the equation $H^k\psi = 0$ that satisfy minimal regularity and decay properties. Assumption (A2) implies that in fact $X = X_k$ for some $k < \infty$. Let $K \geq 0$ be the smallest such k for which this holds.

Given the finite dimensional vector space X and nilpotent linear transformation $H : X \to X$ there are numerous methods for choosing a basis so that H appears in Jordan normal form. Loosely speaking this should subdivide X into a collection of subspaces $X_{j,k}$ whose elements belong to the image of H^{k-1} but not H^k and also belong to the kernel of H^j but not H^{j-1}. We describe one such construction here and then refine it so that the resulting basis vectors satisfy a set of orthogonality conditions.

First, Fix an embedding of X_1/HX_2 into X_1 and define this to be $X_{1,1}$. For each $2 \leq k \leq K$, let $X_{k,1}$ be an embedding of $X_k/(X_{k-1} + HX_{k+1})$ into X_k. To complete the triangular array, define $X_{j,k} = HX_{j+1,k-1}$ for each $1 < k \leq K + 1 - j$. Certain properties of the $X_{j,k}$ are independent of how each quotient space is embedded.

Lemma 11. *The following statements are valid for any set of spaces $X_{j,k}$ generated by the above construction.*

1. $HX \cap X_{k,1} = \{0\}$ *for each* k.
2. $H : X_{j,k} \to X_{j-1,k+1}$ *is a bijection for each* $j > 1$. *Thus* $\dim(X_{j,k}) = L_{j+k}$.

(3) $X_j/X_{j-1} \cong \bigoplus_{k=1}^{K+1-j} X_{j,k}$ *for each* $2 \leq j \leq K$, *and* $X_1 \cong \bigoplus_{k=1}^{K} X_{1,k}$.
(4) $X \cong \bigoplus_{j+k\leq K+1} X_{j,k}$.

The function spaces $\overline{X}_{j,k}$ form an identical decomposition of $\overline{X}$, with H^* taking the place of H. We now turn our attention to the relationships between X and $\overline{X}$ in the setting of the L^2 inner product. It will soon become apparent that $\overline{X}$ is a dual space to X (and vice versa) and the subdivision structure of $X_{j,k}$ is preserved by this identification. The primary goal of this section is to construct an "orthonormal" basis, as follows.

Lemma 12. *There exists a basis for* X *of the form*

$$\{\psi_{j,k}^{(\ell)} : j,k \geq 1,\ j+k \leq K+1,\ 1 \leq \ell \leq L_{j+k}\}$$

with the following properties.

(1) $\{\psi_{j,k}^{(\ell)} : 1 \leq \ell \leq L_{j+k}\}$ *is a basis for* $X_{j,k}$.
(2) $H\psi_{j,k}^{(\ell)} = \psi_{j-1,k+1}^{(\ell)}$ *for each triple* (j,k,ℓ) *with* $j > 1$, *and* $H\psi_{1,k}^{(\ell)} = 0$.
(3) $\langle \psi_{j_1,k_1}^{(\ell_1)}, \bar{\psi}_{j_2,k_2}^{(\ell_2)} \rangle = \begin{cases} 1, & \text{if } (j_1,k_1) = (k_2,j_2) \text{ and } \ell_1 = \ell_2. \\ 0, & \text{otherwise.} \end{cases}$

The third statement asserts a duality relationship between X and $\overline{X}$, and at the level of subspaces it identifies $\overline{X}_{k,j}$ with the dual space of $X_{j,k}$. Using the basis produced by Lemma 12, we define the spectral projection of H onto the eigenspace at zero to be

$$P_0(H)f = \sum_{\substack{j+k\leq K+1 \\ 1\leq \ell \leq L_{j+k}}} \langle f, \bar{\psi}_{k+1-j,k}^{(\ell)} \rangle \psi_{j,k}^{(\ell)}. \tag{16}$$

This operator is bounded as a map from $L^p(\mathbb{R}^3)$ to itself for every $1 \leq p \leq \infty$, thanks to Proposition 8. We list two additional properties of $P_0(H)$ that follow readily from its definition.

Proposition 13. $P_0(H)$ *is the unique finite-rank projection whose range is given by* X *and whose adjoint has range* $\overline{X}$, *therefore it does not depend on the particular choice of basis* $\psi_{j,k}^{(\ell)}$ *employed during its construction. The operators* H *and* $P_0(H)$ *commute with one another when applied to any function in the domain of* H.

The same constructions can be carried out at any eigenvalue λ_j of H, with $H - \lambda_j$ replacing H in the statement of Lemma 12. Some arguments and definitions are actually made simpler by the fact that $(-\Delta - \lambda)^{-1}$ is always an L^2-bounded operator for $\lambda \notin [0,\infty)$. To give one example, Condition (A2) is automatically satisfied for each nonzero eigenvalue as a result of the Analytic Fredholm Theorem (e.g. Theorem VI.14 in [21]). Condition (A1) is also satisfied because the extended collection of free resolvent estimates allows for much stronger bootstrapping arguments than were used in Proposition 8.

The resulting operators $P_{\lambda_j}(H)$ are precisely the pure-point spectral restrictions indicated in Theorem 1.

Returning to the eigenspace at zero, we define a pair of limited projections that deal only with $\overline{X}_1$ (the kernel of H^*) and its dual, $X^1 := \bigoplus_k X_{k,1}$.

$$\tilde{P}_0 f = \sum_{\substack{1\leq k\leq K \\ 1\leq \ell \leq L_{1+k}}} \langle f, \bar{\psi}_{1,k}^{(\ell)} \rangle \psi_{k,1}^{(\ell)} \tag{17}$$

$$\tilde{Q}_0 = I - \tilde{P}_0 \tag{18}$$

The range of $\tilde{P}_0$ is X^1, which is a (non-canonical) embedding of X/HX into X. In contrast, the range of $\tilde{P}_0^*$ is necessarily $\overline{X}_1$ regardless of the choice of basis $\psi_{j,k}^{(\ell)}$. The range of $\tilde{Q}_0$ consists of all functions orthogonal to $\overline{X}_1$.

The remainder of this section is dedicated to proving Lemma 12. The proof is subdivided into four smaller steps that amount to a series of exercises in linear algebra and functional analysis.

Remark 2. For our purposes it would suffice to construct a basis $\psi_{j,k}^{(\ell)}$ for X and a dual basis $\varphi_{j,k}^{(\ell)}$ for $\overline{X}$, each satisfying the first two statements of Lemma 12 (or their formulation with $\overline{X}$ and H^*, as appropriate).

There is no compelling reason other than convenience to demand that $\varphi_{j,k}^{(\ell)} = \bar{\psi}_{j,k}^{(\ell)}$, as we do. It is therefore unwise to draw any correspondence between $\psi \in X$ and $\bar{\psi} \in \overline{X}$ in the remaining discussion except where such a connection is explicitly mentioned.

Proposition 14. *For $i = 1, 2$, choose a pair of numbers $1 \le j_i + k_i \le K + 1$ such that $j_2 < k_1$ or $j_1 < k_2$. Then the spaces X_{j_1,k_1} and $\overline{X}_{j_2,k_2}$ are mutually orthogonal.*

Proof. Consider the case where $j_2 < k_1$. Any function $\psi \in X_{j_1,k_1}$ can be expressed as $H^{j_2}\psi'$ for some $\psi' \in X_{j_1+j_2,k_1-j_2}$. It follows that for every $\varphi \in \overline{X}_{j_2,k_2}$,

$$\langle \psi, \varphi \rangle = \langle \psi', (H^*)^{j_2}\varphi \rangle = \langle \psi', 0 \rangle.$$

The proof is identical for the case $j_1 < k_2$, with the roles of ψ and φ reversed. □

Lemma 15. *For each $j + k \le K + 1$, the space $\overline{X}_{k,j}$ is naturally identified with the dual of $X_{j,k}$ via the ordinary L^2 pairing*

$$\langle \psi, \varphi \rangle = \int_{\mathbb{R}^3} \psi(x)\bar{\varphi}(x)\, dx \qquad \psi \in X_{j,k}, \varphi \in \overline{X}_{k,j}$$

This identification is independent of the particular realization of $X_{j,k}$.

Proof. It suffices to consider only the cases where $k = 1$. Whenever $k > 1$ we have the identity

$$\langle \psi, \varphi \rangle = \langle \psi', (H^*)^{k-1}\varphi \rangle \tag{19}$$

where $\psi' \in X_{j+k-1,1}$ is the unique solution of $H^{k-1}\psi' = \psi$. Recall that $(H^*)^{k-1}$ is an invertible map between $\overline{X}_{k,j}$ and $\overline{X}_{1,j+k-1}$.

To show that pairings between $X_{k,1}$ and $\overline{X}_{1,k}$ are independent of their respective realizations, recall that $X_{k,1}$ is an embedding of the quotient space $X_k/(X_{k-1} + HX_{k+1})$. Meanwhile, $\overline{X}_{1,k}$ is an embedding of the quotient $(H^*)^{k-1}\overline{X}_k/(H^*)^k\overline{X}_{k+1}$. Consider the resulting pairing of cosets

$$\begin{aligned}
&\langle \psi + X_{k-1} + HX_{k+1},\, \varphi + (H^*)^k\overline{X}_{k+1} \rangle \\
&\quad = \langle \psi + \bigoplus_{j<k} X_{j,k} + \bigoplus_{k'>1} X_{k,k'},\, \varphi + \bigoplus_{k'>k} \overline{X}_{1,k'} \rangle \\
&\quad = \langle \psi, \varphi \rangle
\end{aligned}$$

Every one of the possible cross-terms, including those involving $\psi \in X_{k,1}$ and $\varphi \in \overline{X}_{1,k}$, vanishes by Proposition 14.

Duality of $X_{k,1}$ and $\overline{X}_{1,k}$ is proved by induction starting with the base case $k = 1$. We may assume that $X_{1,1}$ is nontrivial. Since it is finite dimensional, it suffices to check that no nonzero element of $X_{1,1}$ is orthogonal to the entirety of $\overline{X}_{1,1}$.

Suppose that $\psi \in X_{1,1}$ were orthogonal to $\overline{X}_{1,1}$. Then ψ is orthogonal as well to all of $\overline{X}_1 = \bigoplus_k \overline{X}_{1,k}$ by Proposition 14. This suggests that, since $\psi \perp \ker H^*$, it should belong to the range of H. In particular it belongs to HX_2 and by definition it must be the zero element in $X_{1,1} \cong X_1/HX_2$.

Now suppose that duality between $X_{j,1}$ and $\overline{X}_{1,j}$ has been established for each $1 \le j < k$, and there exists a function $\psi \in X_{k,1}$ that is orthogonal to $\overline{X}_{1,k}$. Then ψ is also orthogonal to $\bigoplus_{k' \ge k} \overline{X}_{1,k'}$ and pairing with ψ induces a linear functional on each of $\overline{X}_{1,j}$, $1 \le j < k$. Using the inductive hypothesis, there exists $\psi_{k-1} \in X_{k-1,1}$ so that $\psi - \psi_{k-1} \perp \bigoplus_{k' \ge k-1} \overline{X}_{1,k'}$.

This construction can be continued until there is a collection of functions $\psi_j \in X_{j,1}$, $1 \le j < k$, so that $\psi' = \psi - \sum_j \psi_j \perp \overline{X}_1$. Initially we have $\psi' \in X_k$, but by following the previous argument (orthogonality to $\ker H^*$) we arrive at the stronger conclusion $\psi' \in HX_{k+1}$. This implies ψ itself belongs to $HX_{k+1} + X_{k-1}$, in which case $\psi = 0$. □

It is already a consequence of Lemma 15 and Proposition 14 that given a nonzero $\psi \in X$, the pairings $\langle \psi, \varphi \rangle$ cannot vanish for all $\varphi \in \overline{X}$. In that sense X and $\overline{X}$ are mutually identified with each other's dual space. The last bit of work is to generate a decomposition of X into $\bigoplus_{j,k} X_{j,k}$ that emphasizes the duality between $X_{j,k}$ and $\overline{X}_{k+1-j,k}$. Once this is done the desired basis $\psi_{j,k}^{(\ell)}$ can be formed using a variant of the Gram-Schmidt process.

Lemma 16. *There exists a particular realization of each $X_{k,1}$ so that the dual space to $X = \bigoplus_{j,k} X_{j,k}$ is naturally identified with $\bigoplus_{j,k} \overline{X}_{k,j}$ in the sense that $\overline{X}_{k,j}$ is dual to $X_{j,k}$ and*

$$\langle \sum_{j,k} \psi_{j,k}, \sum_{j,k} \varphi_{j,k} \rangle = \sum_{j,k} \langle \psi_{j,k}, \varphi_{k,j} \rangle \tag{20}$$

for any pairing with $\psi_{j,k} \in X_{j,k}$ and $\varphi_{j,k} \in \overline{X}_{j,k}$.

Proof. Lemma 15 showed the duality of $\overline{X}_{k,j}$ and $X_{j,k}$. The remaining task is to construct spaces $X_{k,1}$ so that $X_{j,k} \perp \overline{X}_{j',k'}$ in every case where $k \neq j'$ or $j \neq k'$. Thanks to (19) it will suffice to make $X_{k,1}$ orthogonal to each $\overline{X}_{j',k'}$, $(j',k') \neq (1,k)$.

For convenience the construction here is presented inductively, though in fact the steps need not be taken in strict order. Start with an initial embedding of X_1/HX_2 into X_1 whose image is $\tilde{X}_{1,1}$. By Proposition 14 $\tilde{X}_{1,1}$ is already orthogonal to all spaces $\overline{X}_{j,k}$ with $k > 1$. Now suppose that it is also orthogonal to $\overline{X}_{j,1}$ for each $2 \leq j < J$ (which is vacuously true when $J = 2$).

There is an induced linear map

$$T_{1,J} : \psi \mapsto \langle \psi, \cdot \rangle$$

from $\tilde{X}_{1,1}$ to the dual space of $\overline{X}_{J,1}$, which can also be presented in the form $\psi \in X_{1,1} \mapsto \psi_{1,J} \in X_{1,J}$ by duality of $X_{J,1}$ and $\overline{X}_{1,J}$. The the collection of vectors

$$\{\psi - \psi_{1,J} : \psi \in \tilde{X}_{1,1}\}$$

is another realization of X_1/HX_2 that is additionally orthogonal to $\overline{X}_{J,1}$. Repeating this process through the step $J = K$ constructs the desired space $X_{1,1}$.

To construct $X_{k,1}$ start with a embedding of $X_k/X_{k-1} + HX_{k+1}$ whose image is $\tilde{X}_{k,1}$. Proposition 14 esures that $\tilde{X}_{k,1}$ is aready orthogonal to each $\overline{X}_{j',k'}$ with $k' > k$. The first induced linear map to consider is

$$T_{k,2} : \psi \mapsto \psi_{k,2}$$

where $\psi_{k,2}$ is the unique element of $X_{k,2}$ that generates the same inner products as ψ when paired with any element in $\overline{X}_{2,k}$. It follows that the refined collection of vectors

$$\{\psi - \psi_{k,2} : \psi \in \tilde{X}_{k,1}\}$$

is a realization of $X_k/X_{k-1} + HX_{k+1}$ that is additionally orthogonal to $\overline{X}_{2,k}$. By making the adjustments directed by the analogous maps $T_{k,3}, T_{k,4}, \ldots, T_{k,K+1-k}$ (in order), followed by $T_{(k-1),1} \ldots T_{(k-1),K+2-k}$ and continuing all the way to $T_{1,1} \ldots T_{1,K}$, one recovers a suitable construction of $X_{k,1}$.

$T_{k,j}$, $j = 3, 4, \ldots, K+1-k$ (in order), followed by $T_{k-1,j}$, $j = 1, 2, \ldots K+2-k$ and continuing all the way to $T_{1,j}$, $j = 1, 2, \ldots K$, one recovers a suitable construction of $X_{k,1}$.

The order of operations is chosen with some deference to Proposition 14. The vectors $\psi_{j,k}$ added at any one step are guaranteed not to disturb the orthogonality relations established in any of the previous steps. As with any row-reduction algorithm, there are other possible orderings that produce an identical result. □

Any basis of $X_{k,1}$ naturally induces bases for each space $X_{j,1+k-j}$ by taking its image under H^{k-j}. Bases for $\overline{X}_{j,1+k-j}$, which represent the dual of $X_{1+k-j,j}$, are obtained through complex conjugation. We verify that a Gram-Schmidt process exists for creating a basis of $X_{k,1}$ that induces its own dual basis in $\overline{X}_{1,k}$.

Proposition 17. *For each $k = 1, 2, \ldots, K$ there exists a basis of functions $\psi^{(\ell)} \in X_{k,1}$ such that*

$$\langle \psi^{(\ell)}, \overline{H^{k-1}\psi^{(m)}} \rangle = \langle H^{k-1}\psi^{(\ell)}, \bar{\psi}^{(m)} \rangle = \delta_{\ell m}$$

Proof. Start by choosing any nonzero $\psi \in X_{k,1}$. Lemma 15 guarantees the existence of $\varphi \in \overline{X}_{k,1}$ so that $\langle H^{k-1}\psi, \varphi \rangle = A \neq 0$. Taking linear combinations of ψ and $\bar{\varphi}$, we see that the value of $\langle H^{k-1}(z\psi + \bar{\varphi}), \bar{z}\bar{\psi} + \varphi \rangle$ is a nonconstant linear or quadratic function of the complex variable z. Choose $\psi^{(1)}$ to be an element of the form $z_0\psi + \bar{\varphi}\mathrm{S}$ such that $\langle H^{k-1}\psi^{(1)}, \bar{\psi}^{(1)} \rangle = 1$.

Now suppose $\psi^{(\ell)}$, $1 \le \ell < L$, have been chosen so that $\langle H^{k-1}\psi^{(\ell)}, \bar\psi^{(m)}\rangle = \delta_{\ell m}$. Given a linearly independent element $\tilde\psi$, let

$$\psi = \tilde\psi - \sum_{\ell=1}^{L-1} \langle H^{k-1}\tilde\psi, \bar\psi^{(\ell)}\rangle \psi^{(\ell)}.$$

All pairings between $H^{k-1}\psi$ and $\bar\psi^{(\ell)}$ vanish by construction. Once again there exists $\tilde\varphi \in \overline{X}_{k,1}$ so that $\langle H^{k-1}\psi, \tilde\varphi\rangle = A \neq 0$. Moreover, it does not change the value if $\tilde\varphi$ is replaced by

$$\varphi = \tilde\varphi - \sum_{\ell=1}^{L-1} \overline{\langle H^{k-1}\psi^{(\ell)}, \tilde\varphi\rangle} \bar\psi^{(\ell)},$$

the difference being that $\langle H^{k-1}\bar\varphi, \bar\psi^{(\ell)}\rangle = 0$ as well. Then it is possible to choose $\psi^{(L)} = z_0\psi + \bar\varphi$ in the same manner as before to continue building out the "self-dual" basis. □

6. Construction of $(I + VR_0^+(\lambda^2))^{-1}$.

Even though $I + VR_0^+(0)$ may not be an invertible operator on $L^1(\mathbb{R}^3)$, it is still possible to determine the inverse of $I + VR_0^+(\lambda^2)$ for nearby λ by perturbation. The first step is to use the techniques from Section 4 to construct an approximate inverse over a large subspace of $L^1(\mathbb{R}^3)$.

As a compact perturbation of the identity, $I + VR_0^+(0)$ has a finite dimensional kernel (namely VX_1) and its range is a closed subspace of $L^1(\mathbb{R}^3)$ of equal codimension. By the Open Mapping theorem, the exists a continuous linear map

$$S_0 : \operatorname{range}(I + VR_0^+(0)) \to L^1(\mathbb{R}^3)/\ker(I + VR_0^+(0))$$

with the property that

$$(I + VR_0^+(0))S_0 = I_{\operatorname{range}(I+VR_0^+(0))}.$$

Both the domain and range of S_0 can be expressed in terms of the spaces X and $\overline{X}$. The range of $I+VR_0^+(0)$ in L^1 consists of functions that vanish when paired with any element of $\overline{X}_1 = \ker(I + R_0^-(0)\overline{V})$. Meanwhile, since $VX_1 = \ker(I+VR_0^+(0))$ is finite dimensional there exist subspaces of $L^1(\mathbb{R}^3)$ isomorphic to the quotient L^1/VX_1. We will concentrate on one such embedding that is canonical in the context of this discussion.

Observe that $R_0^-(0)\overline{X}^1 = \bigoplus_k R_0^-(0)\overline{X}_{k,1}$ is dual to VX_1 in that

$$\langle V\psi, R_0^-(0)\varphi\rangle = \langle R_0^+(0)V\psi, \varphi\rangle = -\langle \psi, \varphi\rangle$$

for any $\psi \in X_{1,k}$, $\varphi \in \overline{X}_{k',1}$. Given a function $g \in L^1(\mathbb{R}^3)$ there is a unique choice of $\psi_k \in X_{1,k}$ so that $\langle g, R_0^-(0)\varphi_k\rangle = \langle V\psi_k, R_0^-(0)\varphi_k\rangle$ for all $\varphi_k \in \overline{X}_{k,1}$. Then the mapping $g \mapsto g - \sum_k V\psi_k$ has VX_1 as its kernel. Its range is an embedding of L^1/VX_1 inside of $L^1(\mathbb{R}^3)$ consisting of functions orthogonal to $R_0^-(0)\overline{X}^1$.

In summary, there exists a linear map $S_0 : \overline{X}_1^\perp \to (R_0^-(0)\overline{X}^1)^\perp$ that is a one-sided inverse to $I + VR_0^+(0)$ over the domain $\overline{X}_1^\perp$. Both the domain and range of S_0 are understood to be subspaces of $L^1(\mathbb{R}^3)$. We express this property in the slightly redundant form

$$\tilde Q_0(I + VR_0^+(0))S_0 = \text{Identity of } \overline{X}_1^\perp \subset L^1(\mathbb{R}^3).$$

The next step is to find a perturbation of S_0 that will serve as the inverse of $\tilde Q_0(I + VR_0^+(\lambda^2))$, taking the latter as a map from $(R_0^-(0)\overline{X}^1)^\perp$ to $\overline{X}_1^\perp$. If λ is sufficiently small this can be done via a convergent Neumann series. Start with the decomposition

$$\tilde Q_0(I + VR_0^+(\lambda^2)) = S_0^{-1} + \tilde Q_0 VB_0^+(\lambda^2).$$

From this it follows that

$$S(\lambda) = \sum_{m=0}^{\infty} (-1)^m \big(S_0\tilde Q_0 VB_0^+(\lambda^2)\big)^m S_0 \tag{21}$$

will be the desired inverse of $\tilde Q_0(I + VR_0^+(\lambda^2))$. Proposition 5 asserts that $VB_0^+(\lambda^2)$ vanishes as an operator on $L^1(\mathbb{R}^3)$ when λ approaches 0, and the same is true after composing with $\tilde Q_0$ and S_0. Thus the series in (21) converges absolutely for small values of λ.

Of course $S(\lambda)$ is not generally a true one-sided inverse of $I + VR_0^+(\lambda^2)$ over the domain $\overline{X}_1^\perp$. The difference between $(I+VR_0^+(\lambda^2))S(\lambda)f$ and f itself will be a linear combination of the $\psi_{k,1}^{(\ell)}$, with coefficients determined by its pairings with the corresponding $\bar\psi_{1,k}^{(\ell)} \in \overline{X}_1$. If we can identify solutions to

$$(I + VR_0^+(\lambda^2))\Psi_k^{(\ell)} = \psi_{k,1}^{(\ell)} \tag{22}$$

then the construction of $(I + VR_0^+(\lambda^2))^{-1}$ will be complete.

In fact there is an explicit formula for each $\Psi_k^{(\ell)}$ based on the defined relationships between the $\psi_{j,k}^{(\ell)}$.

Proposition 18. *For each $2 \le j \le k \le K$ and each $1 \le \ell \le L_k$, there is the identity*

$$(I + R_0^+(\lambda^2)V)\psi_{j,k}^{(\ell)} = R_0^+(\lambda^2)\big(\psi_{j-1,k+1}^{(\ell)} - \lambda^2\psi_{j,k}^{(\ell)}\big) \tag{23}$$

and $(I + R_0^+(\lambda^2))\psi_{1,k}^{(\ell)} = -\lambda^2 R_0^+(\lambda^2)\psi_{1,k}^{(\ell)}$ in the case $j = 1$.

The analogous identities for $(I + R_0^-(\lambda^2)\overline{V})\bar\psi_{j,k}^{(\ell)}$ are also valid.

Proof. This is a direct consequence of the operator identity

$$R_0^+(\lambda^2) = (I + \lambda^2 R_0^+(\lambda^2))R_0^+(0). \tag{24}$$

As a consequence, $I + R_0^+(\lambda^2)V$ can be rewritten as

$$(I + \lambda^2 R_0^+(\lambda^2))(I + R_0^+(0)V) - \lambda^2 R_0^+(\lambda^2).$$

By definition $(I + R_0^+(0)V)\psi_{j,k}^{(\ell)} = R_0^+(0)\psi_{j-1,k+1}^{(\ell)}$. The proposition is then proved with one more application of (24). □

Corollary 19. *For each $1 \le k \le K$ and $1 \le \ell \le L_k$ there is the identity*

$$(I + R_0^+(\lambda^2)V)\sum_{j=1}^{k} \lambda^{2(k-j)}\psi_{1+k-j,j}^{(\ell)} = -\lambda^{2k} R_0^+(\lambda^2)\psi_{k,1}^{(\ell)}.$$

Once again the "adjoint" statement relating $R_0^-(\lambda^2)$, $\overline{V}$, and $\bar\psi_{j,k}^{(\ell)}$ holds as well. We state the solution formula for (22) as a final corollary.

Corollary 20. *For each $1 \le k \le K$ and $1 \le \ell \le L_k$,*

$$(I + VR_0^+(\lambda^2))\Big(\psi_{k,1}^{(\ell)} + \sum_{j=1}^{k} \lambda^{-2j} V\psi_{1+k-j,j}^{(\ell)}\Big) = \psi_{k,1}^{(\ell)}. \tag{25}$$

Putting the pieces together, we conclude that for any $f \in L^1(\mathbb{R}^3)$,

$$\begin{aligned}(I + VR_0^+(\lambda^2))^{-1}f &\\ = \; & S(\lambda)\tilde{Q}_0 f \\ & + \sum_{k,\ell} \big(\langle f, \bar\psi_{1,k}^{(\ell)}\rangle - F_k^{(\ell)}(\lambda)\big)\Big[\Big(\sum_{1\le j\le k} \lambda^{-2j} V\psi_{1+k-j,j}^{(\ell)}\Big) + \psi_{k,1}^{(\ell)}\Big]\end{aligned} \tag{26}$$

where $F_k^{(\ell)}(\lambda)$ stands for the inner product $\big\langle (I + VR_0^+(\lambda^2))S(\lambda)\tilde{Q}_0 f, \bar\psi_{1,k}^{(\ell)}\big\rangle$. The difference between $\langle f, \bar\psi_{1,k}^{(\ell)}\rangle$ and $F_k^{(\ell)}(\lambda)$ is always a bounded function of λ, hence the operator inverse of $I + VR_0^+(\lambda^2)$ has an isolated singularity at $\lambda = 0$. The highest degree of blowup appears in each of the $j = k$ terms and is potentially of order λ^{-2k}.

In fact we can be much more precise about the type of singular behavior of $(I + VR_0^+(\lambda^2))^{-1}f$ in the vicinity of the origin. The key observation here is that for each $j \ge 2$,

$$\begin{aligned}\big\langle f, \bar\psi_{j,k}^{(\ell)}\big\rangle &= \big\langle \tilde{Q}_0 f, \bar\psi_{j,k}^{(\ell)}\big\rangle \\ &= \big\langle \tilde{Q}_0(I + VR_0^+(\lambda^2))S(\lambda)\tilde{Q}_0 f, \bar\psi_{j,k}^{(\ell)}\big\rangle \\ &= \big\langle (I + VR_0^+(\lambda^2))S(\lambda)\tilde{Q}_0 f, \bar\psi_{j,k}^{(\ell)}\big\rangle.\end{aligned}$$

The middle equation is due to $S(\lambda)$ being the one-sided inverse of $\tilde{Q}_0(I + VR_0^+(\lambda^2))$ and the others hold because the adjoint of $\tilde{Q}_0$ acts as the identity on all functions orthogonal to X^1. It then follows that

$$\begin{aligned} F_k^{(\ell)}(\lambda) + & \sum_{1\le j<k} \lambda^{2(k-j)} \big\langle f,\, \bar\psi^{(\ell)}_{1+k-j,j} \big\rangle \\ &= \Big\langle (I + VR_0^+(\lambda^2))S(\lambda)\tilde{Q}_0 f,\, \sum_{1\le j\le k} \lambda^{2(k-j)} \bar\psi^{(\ell)}_{1+k-j,j} \Big\rangle \\ &= \Big\langle S(\lambda)\tilde{Q}_0 f,\, (I + R_0^-(\lambda^2)\overline{V}) \sum_{1\le j\le k} \lambda^{2(k-j)} \bar\psi^{(\ell)}_{1+k-j,j} \Big\rangle \\ &= -\lambda^{2k} \big\langle S(\lambda)\tilde{Q}_0 f,\, R_0^-(\lambda^2)\bar\psi^{(\ell)}_{k,1} \big\rangle \end{aligned}$$

with the last equation being a restatement of Corollary 19. Substituting this expression in place of $F_k^{(\ell)}(\lambda)$ in (26) yields a formula for $(I + VR_0^+(\lambda^2))^{-1}$ that isolates the coefficients of the pole at the origin.

Lemma 21. *Suppose V satisfies the conditions of Theorem 1. Then for all $\lambda \in \mathbb{R}\setminus\{0\}$ sufficiently close to zero,*

$$\begin{aligned} (I + VR_0^+(\lambda^2))^{-1} f \;&= S(\lambda)\tilde{Q}_0 f \\ &\quad + \sum_{k,\ell} \big\langle S(\lambda)\tilde{Q}_0 f,\, R_0^-(\lambda^2)\bar\psi^{(\ell)}_{k,1} \big\rangle \Big[\Big(\sum_{1\le j\le k} \lambda^{2(k-j)} V\psi^{(\ell)}_{1+k-j,j} \Big) + \lambda^{2k}\psi^{(\ell)}_{k,1} \Big] \\ &\quad + \sum_{k,\ell} \Big(\sum_{1\le i\le k} \lambda^{2(k-i)} \big\langle f,\, \bar\psi^{(\ell)}_{1+k-i,i} \big\rangle \Big) \Big[\Big(\sum_{1\le j\le k} \lambda^{-2j} V\psi^{(\ell)}_{1+k-j,j} \Big) + \psi^{(\ell)}_{k,1} \Big] \end{aligned} \tag{27}$$

Note that the second line is a bounded function, and poles occur for each term in the third line that has $i + j > k$.

7. Proof of the Low-Energy Estimate

We are now in a position to prove Theorem 4 in the low-energy regime by using (27) to characterize $T^+(\lambda) = (I + VR_0^+(\lambda^2))^{-1}$ in terms of known functions.

Theorem 22. *Suppose V satisfies the conditions of Theorem 1. There exists $r > 0$ such that*

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}} \big| \big[\chi(\,\cdot\,/r)T^+(\,\cdot\,)\big]^\wedge(\rho) f(x) \big|\, d\rho\, dx \lesssim \|f\|_1 \tag{28}$$

holds for all $f \in L^1(\mathbb{R}^3)$ that are orthogonal to $\overline{X} \subset L^\infty(\mathbb{R}^3)$.

Proof. This is a statement about the Fourier transform of $(I+VR_0^+(\lambda^2))^{-1}$, so it makes sense to consider the Fourier transform of each term in (27) individually. The initial term $\chi(\lambda/r)S(\lambda)\tilde{Q}_0 f$ requires us to retrace the construction of $S(\lambda)$ in (21). Start with the power series

$$\chi(\lambda/r)S(\lambda)\tilde{Q}_0 f = \sum_{m=0}^{\infty} (-1)^m \big(\chi(2\lambda/r)S_0\tilde{Q}_0 V B_0^+(\lambda^2)\big)^m \chi(\lambda/r)S_0\tilde{Q}_0 f.$$

Aside from the restricted domain of S_0, and its associated projection $\tilde{Q}_0$, this is the same series as (13) with the value $\lambda_0 = 0$. The L^1 bounds on its Fourier transform can be computed in an identical manner as well.

For functions $f \in L^1(\mathbb{R}^3)$ orthogonal to $\overline{X}$, each of the pairings $\langle f, \bar\psi^{(\ell)}_{1+k-i,i}\rangle$ must be zero. Consequently the third line of (27) vanishes, taking all of the most dangerous terms with it.

Finally, the second line can be controlled in its entirety provided the Fourier transform (in λ) of each function

$$\chi(\lambda/r)\big\langle S(\lambda)\tilde{Q}_0 f,\, R_0^-(\lambda^2)\bar\psi^{(\ell)}_{k,1} \big\rangle$$

is integrable, with its $L^1(\mathbb{R})$ norm bounded by $\|f\|_1$. Each side of the pairing is itself a function of λ, so its Fourier transform should appear as the convolution

$$\int_{\mathbb{R}} \big\langle K_1(\,\cdot\,,\sigma), K_2(\,\cdot\,,\sigma-\rho) \big\rangle\, d\sigma = \int_{\mathbb{R}} \int_{\mathbb{R}^3} K_1(x,\sigma)\overline{K_2(x,\sigma-\rho)}\, dx d\sigma$$

where $K_1(x,\rho)$ represents the Fourier transform of $\chi(\lambda/r)S(\lambda)\tilde{Q}_0 f$ and $K_2(x,\rho)$ is the Fourier transform of $\chi(2\lambda/r)R_0^-(\lambda^2)\bar{\psi}_{k,1}^{(\ell)}$. It suffices to show that

$$\|K_1\|_{L^1_{x,\rho}} \lesssim \|f\|_1 \qquad \text{and} \qquad \|K_2\|_{L^\infty_x L^1_\rho} \lesssim 1 \tag{29}$$

and therefore

$$\int_{\mathbb{R}} \Big| \int_{\mathbb{R}} \int_{\mathbb{R}^3} K_1(x,\sigma) K_2(x,\sigma-\rho)\, dx\, d\sigma \Big|\, d\rho \lesssim \|f\|_1.$$

The estimate for K_1 has already been established by expanding out the Neumann series for $S(\lambda)$. The estimate for K_2 is straightforward, using only the explicit resolvent kernel $R_0^-(\lambda^2)_{(x,y)} = e^{-i\lambda|x-y|}/(4\pi|x-y|)$. This leads to the formula

$$K_2(x,\rho) = \frac{r}{4} \int_{\mathbb{R}^3} \frac{\hat{\chi}(r(\rho+|x-y|)/2)}{|x-y|}\, \bar{\psi}_{k,1}^{(\ell)}(y)\, dy$$

Taking the $L^1(\mathbb{R})$ norm in the ρ variable and bring in absolute values yields

$$\|K_2(x,\,\cdot\,)\|_1 \le \|\hat{\chi}\|_1 \int_{\mathbb{R}^3} \frac{|\bar{\psi}_{k,1}^{(\ell)}(y)|}{2|x-y|}\, dy \lesssim 1$$

because $\psi_{k,1}^{(\ell)}$ is assumed to belong to $L^1(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. □

Remark 3. The bounds on $\langle S(\lambda)\tilde{Q}_0 f,\ R_0^-(\lambda^2)\bar{\psi}_{k,1}^{(\ell)}\rangle$ can be strengthened further. Recall that the range of S_0 could be any embedding of $L^1(\mathbb{R}^3)/VX_1$ into $L^1(\mathbb{R}^3)$, and we chose it to be the subspace of functions orthogonal to $R_0^-(0)\overline{X}^1$. By construction this subspace serves as the range of $S(\lambda)$ as well.

Therefore the value of the pairing should not change if the operator $R_0^-(\lambda^2)$ is replaced by $(B_0^+(\lambda^2))^*$, which is just $R_0^-(\lambda^2) - R_0^-(0)$. Following the same factorization method as above, but using the explicit kernel of $B_0^+(\lambda^2)$ rather than $R_0^+(\lambda^2)$, it is possible to achieve the bound $\|K_2\|_{L^\infty_x L^1_\rho} \lesssim r$.

This gives some evidence that our definition of S_0 leads to the "best" possible approximate inverse operator $S(\lambda)$.

8. Prospective examples of $V(x)$

Any solution of the equation $(-\Delta + V)\Psi = 0$ can be written out as $\Psi = -(4\pi|x|)^{-1} * (V\Psi)$. If V has compact support then Ψ is a harmonic function for large x and its decay rate is controlled by the amount of cancellation inside the integral. Choose any collection $x_i \in \mathbb{R}^3$, $\mu_i \in \mathbb{C}$ so that the point masses $\mu_i\delta_{x_i}$ have vanishing moments up to fourth order. Then

$$\big|\Delta^{-1}\big(\textstyle\sum_i \mu_i\delta_{x_i}\big)\big| = -\frac{1}{4\pi}\sum_i \mu_i|x-x_i|^{-1} \lesssim |x|^{-6}.$$

Call this function $F(x)$. Surrounding each of the points x_i is a neighborhood Ω_i where $F(x)$ is dominated by $\mu_i|x-x_i|^{-1}$ and is therefore nonzero. Now let $\Psi(x)$ be any function that agrees with $F(x)$ outside of the union of Ω_i and is a nonvanishing C^2 continuation inside. Then $\Delta\Psi$ is continuous with compact support inside $\cup_i\overline{\Omega}_i$. As a result,

$$V(x) = \frac{\Delta\Psi(x)}{\Psi(x)}$$

belongs to $C_c(\mathbb{R}^3)$. The Schrödinger operator $H = -\Delta + V$ has Ψ as a rapidly decaying eigenfunction at $\lambda = 0$.

Starting from a fixed choice of Ψ it is possible to construct a larger family of prospective examples of the form $\Psi_z(x) = \Psi(x)e^{iz\eta(x)}$, where $\eta \in C_c^\infty(\Omega_i)$ and z varies over $\mathbb{C}$. The resulting potentials $V_z(x)$ have complex-analytic dependence on z. The analytic Fredholm theorem applied to $(I + R_0^+(0)V_z)$ indicates that the dimension of the nullspace for $-\Delta + V_z$ should be constant for generic z with only a discrete set of exceptions. We believe the prevailing dimension is 1 (i.e. Ψ_z is the unique eigenfunction), in which case the conditions of Theorem 1 are satisfied. A complete proof of this point is lacking, however we can offer a formal argument that reaches the same conclusion.

At issue is whether the nonlinear mapping $\Psi \mapsto V(\Psi) = \frac{\Delta\Psi}{\Psi}$ is one-to-one at most points in the target space. Looking at small perturbations of the form $\Psi + s\phi$, the variational derivative is seen to be

$$\begin{aligned}\tfrac{d}{ds}V(\Psi + s\phi)\big|_{s=0} &= \frac{\Delta\phi}{\Psi} - \frac{\Delta\Psi}{\Psi^2}\phi \\ &= -\frac{\big(-\Delta + V(\Psi)\big)\phi}{\Psi}\end{aligned}$$

Suppose that Ψ_1 and Ψ_2 (not necessarily constructed according to the recipe above) both generate the same potential $V = \frac{\Delta\Psi_i}{\Psi_i}$. Let $H = (-\Delta + V)$ for this particular choice of potential. We would like to find a small localized perturbation $\Psi_1 + s\phi_1$ whose first-order effect on V cannot be recreated by any corresponding perturbation $\Psi_2 + \phi_2$. The basic requirement is that

$$\frac{H\phi_1}{\Psi_1} \neq \frac{H\phi_2}{\Psi_2}$$

for any choice of ϕ_2, which is to say that $\frac{\Psi_2}{\Psi_1}H\phi_1$ should lie outside the range of H. It suffices to show that $\frac{\Psi_2}{\Psi_1}H\phi_1$ is not entirely orthogonal to $\ker H^*$.

Here $H^* = -\Delta + \overline{V} = \overline{H}$ so the vector spaces $\ker H$ and $\ker H^*$ are related by complex conjugation. Pairing with $\overline{\Psi}_2 \in \ker H^*$ yields the relation

$$\big\langle \tfrac{\Psi_2}{\Psi_1}H\phi_1,\, \overline{\Psi}_2\big\rangle = \big\langle \phi_1,\, H^*\Big(\tfrac{\overline{\Psi}_2^2}{\overline{\Psi}_1}\Big)\big\rangle = \big\langle H\Big(\tfrac{\Psi_2^2}{\Psi_1}\Big),\, \overline{\phi}_1\big\rangle.$$

So long as the ratio $\frac{\Psi_2^2}{\Psi_1}$ is not another eigenfunction of H one can choose any number of compactly supported ϕ_1 to make the inner product nonzero.

On the other hand, if $\Psi_3 = \frac{\Psi_2^2}{\Psi_1}$ is an eigenfunction of H, then $\overline{\Psi}_3$ also belongs to $\ker H^*$ and it would suffice to make $\langle H(\frac{\Psi_2}{\Psi_1}\Psi_3), \overline{\phi}_1\rangle$ nonzero. As before, it can be done easily so long as $(\frac{\Psi_2}{\Psi_1})^2\Psi_1$ is not an eigenfunction of H as well. The eigenspace of H at $\lambda = 0$ has only finite multiplicity, so the process generally terminates after some number of iterations, at which point the choice of ϕ_1 can be made. The only scenario where this method fails to identify a suitable ϕ_1 is if $(\frac{\Psi_2}{\Psi_1})^N$ is constant for some N, but since the Ψ_i are continuous functions that can only happen if Ψ_2 is a scalar multiple of Ψ_1.

The suggested conclusion is that whenever $(-\Delta + V)\Psi = 0$ has several linearly independent solutions, there should be a generic family of perturbations that reduce the multiplicity to one. These include potentials for which the unique perturbed eigenfunction differs from a selected element of the Ψ_i only on a compact set in $\mathbb{R}^3$. Thus it retains the same spatial decay properties as Ψ_i.

The quantity $\langle\Psi, \overline{\Psi}\rangle$ holds particular interest. By choosing the local C^2 continuations in Ψ appropriately it can be made to vanish on a codimension-1 set of examples. When this occurs, the generalized eigenspace for H must contain additional elements in order to satisfy the third assertion in Lemma 12. In other words, there exists a function Φ that solves $(-\Delta + V)\Phi = K\Psi$ with $K \in \{0, 1\}$. Applying Δ^{-1} to both sides yields

$$\Phi = \Delta^{-1}(V\Phi - K\Psi).$$

By construction $\Delta^{-1}\Psi \lesssim |x|^{-4}$ still belongs to $L^1(\mathbb{R}^3)$. The leading order decay of Φ will generally be determined by the term $\Delta^{-1}V\Phi$, which is subject to the same moment considerations as $V\Psi$ was earlier. While the implicit description of Φ obscures its dependence on the original construction paremeters, it is certainly plausible that a further subset (perhaps of codimension 10) would lead to vanishing moments of $V\Phi$ up to second order, allowing $\Phi \in L^1(\mathbb{R}^3)$ as desired. The creation of more elaborate generalized eigenspaces appears to follow inductively by adding a finite number of extra orthogonality conditions (e.g.$\langle\Psi, \overline{\Phi}\rangle = 0$) to increase the dimension of X and moment conditions to insure that each known element of X remains integrable.

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