

# Dispersive bounds for the three-dimensional Schrödinger equation with almost critical potentials

Michael Goldberg

September 15, 2004

## Abstract

We prove a dispersive estimate for the time-independent Schrödinger operator  $H = -\Delta + V$  in three dimensions. The potential  $V(x)$  is assumed to lie in the intersection  $L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ ,  $p < \frac{3}{2} < q$ , and also to satisfy a generic zero-energy spectral condition. This class, which includes potentials that have pointwise decay  $|V(x)| \leq C(1 + |x|)^{-2-\varepsilon}$ , is nearly critical with respect to the natural scaling of the Laplacian. No additional regularity, decay, or positivity of  $V$  is assumed.

## 1 Introduction

The propagator  $e^{-it\Delta}$  of the free Schrödinger equation in  $\mathbb{R}^3$  may be represented as a convolution operator with kernel  $(4\pi it)^{-3/2} e^{-i(|x|^2/4t)}$ . From this formula it is clear that the free evolution satisfies the dispersive bound  $\|e^{-it\Delta}\|_{1 \rightarrow \infty} \leq (4\pi|t|)^{-3/2}$  at all times  $t \neq 0$ . In this paper we consider the perturbed Hamiltonian  $H = -\Delta + V$  and seek to prove similar estimates on the time evolution operator  $e^{itH} P_{ac}(H)$ . The projection onto the absolutely continuous spectrum of  $H$ , denoted here by  $P_{ac}(H)$ , is needed to eliminate bound states which do not decay over any length of time. Our goal is to avoid placing excessive restrictions on the regularity, positivity, and decay of the potential  $V = V(x)$ . To that end we formulate the following theorem.

**Theorem 1.** *Let  $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ ,  $p < \frac{3}{2} < q$ . Assume also that zero is neither an eigenvalue nor a resonance of  $H = -\Delta + V$ . Then*

$$(1) \quad \|e^{itH} P_{ac}(H)\|_{1 \rightarrow \infty} \lesssim |t|^{-\frac{3}{2}}.$$

A precise definition of resonances is given in section 3. With this assumption the spectrum is known to be purely absolutely continuous on  $[0, \infty)$ , see [GS2] for details. We remark that if the zero-energy hypothesis is not satisfied, a dispersive estimate still holds for  $e^{itH} P_{[a, \infty)}(H)$  for any positive number  $a$ .

The original dispersive estimates expressed  $e^{itH}$  as a mapping between weighted  $L^2$  spaces, with the weights being exponential [Rau] or polynomial [JenKat]. A significant advance was made by Journé, Soffer, and Sogge [JSS], who proved the translation-invariant  $L^1 \rightarrow L^\infty$  bound in (1) for potentials satisfying  $|V(x)| \leq C(1 + |x|)^{-7-\varepsilon}$  and  $\hat{V} \in L^1(\mathbb{R}^3)$ . The pointwise decay and regularity hypotheses were subsequently weakened by Yajima [Yaj] and Goldberg and Schlag [GS1]. The ability to handle potentials with  $L^p$  singularities stems from recent results (e.g. [IonJer]) showing that  $-\Delta + V$  has no embedded eigenvalues at positive energies.

The hypotheses of Theorem 1 are nearly optimal in a number of respects. There exist compactly supported potentials  $V \in L_{\text{weak}}^{3/2}$  for which  $-\Delta + V$  admits bound states with positive energy [KocTat]. The inverse-square potential  $V(x) = A|x|^{-2}$  fails to be dispersive for the wave equation for any  $A < 0$  [PST]. It is possible that the decay criteria can be relaxed slightly to include all functions for which  $\sup_y \int_{\mathbb{R}^3} |x-y|^{-1} |V(x)| dx$  is finite. Such a condition is sufficient provided  $V$  is small [RodSch], or mostly positive [DanPie], and is critical with respect to scaling.

The proof of Theorem 1 begins by rewriting the operator  $e^{itH} P_{ac}(H)$  in terms of the resolvents  $R_0(z) = (-\Delta - z)^{-1}$ . In this manner the dispersive estimate can be reduced to a statement about the resolvents' mapping properties. As is frequently the case with dispersive phenomena, one needs to distinguish between high and low energies and make a separate calculation for each. The various pieces are then assembled back into the original theorem at the end.

## 1.1 Resolvent Identities

Let  $H = -\Delta + V$  in  $\mathbb{R}^3$  and define the resolvents  $R_0(z) := (-\Delta - z)^{-1}$  and  $R_V(z) := (H - z)^{-1}$ . For each  $z \in \mathbb{C} \setminus \mathbb{R}^+$ , the operator  $R_0(z)$  is given by convolution with the kernel  $R_0(z)(x, y) = \frac{e^{i\sqrt{z}|x|}}{4\pi|x|}$ , where  $\sqrt{z}$  is taken to have positive imaginary part. While  $R_V(z)$  is not translation-invariant and does not possess an explicit representation of this form, it can be expressed in terms of  $R_0(z)$  via the identity

$$(2) \quad R_V(z) = (I + R_0(z)V)^{-1}R_0(z) = R_0(z)(I + VR_0(z))^{-1}$$

In the case where  $z = \lambda \in \mathbb{R}^+$ , the resolvent may be defined as a limit of the form  $R_0(\lambda \pm i0) := \lim_{\varepsilon \downarrow 0} R_0(\lambda \pm i\varepsilon)$ . The choice of sign determines which branch of the square-root function is selected in the formula above, therefore the two continuations do not agree with one another. In this paper we refer to resolvents along the positive real axis using the following notation.

$$R_0^\pm(\lambda) := R_0(\lambda \pm i0) \quad R_V^\pm(\lambda) := R_V(\lambda \pm i0)$$

Note that  $R_0^-(\lambda)$  is the formal adjoint of  $R_0^+(\lambda)$ , and a similar relationship holds for  $R_V^\pm(\lambda^2)$ . In fact these resolvents will be truly adjoint to one another, since our analysis takes place in settings where they are bounded operators. Establishing the self-adjointness of  $H$  is a more delicate matter, but it can be done because  $V \in L^{\frac{3}{2}}(\mathbb{R}^3)$ , see Theorem X.19 in [ReedSim].

The discrepancy between  $R_0^+(\lambda)$  and  $R_0^-(\lambda)$  provides a convenient characterization of the absolutely continuous part of the spectral measure of  $H$ . With  $E_{ac}(d\lambda)$  denoting the absolutely continuous spectral measure of  $H$ , the Stone formula states that

$$(3) \quad \langle E_{ac}(d\lambda)f, g \rangle = \frac{1}{2\pi i} \langle [R_V^+(\lambda) - R_V^-(\lambda)]f, g \rangle d\lambda.$$

Let  $\chi$  be a smooth, even, cut-off function on the line that is equal to one when  $|x| \leq 1$  and vanishes for all  $|x| \geq 2$ . Further assume that translations of  $\chi$  form a partition of unity, in other

words  $\chi(x) + \chi(x - 3) = 1$  for all  $x \in [-1, 4]$ . In order to prove Theorem 1 it suffices to show that

$$\begin{aligned}
& \sup_{L \geq 1} \left| \left\langle e^{itH} \chi^3(\sqrt{H}/L) P_{a.c.} f, g \right\rangle \right| \\
&= \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi^3(\lambda/L) \left\langle [R_V^+(\lambda^2) - R_V^-(\lambda^2)] f, g \right\rangle \frac{d\lambda}{\pi i} \right| \\
(4) \quad &= \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi^3(\lambda/L) \left\langle [R_0^+(\lambda^2)(I + VR_0^+(\lambda^2))^{-1} - R_0^-(\lambda^2)(I + VR_0^-(\lambda^2))^{-1}] f, g \right\rangle \frac{d\lambda}{\pi i} \right| \\
&\lesssim |t|^{-\frac{3}{2}} \|f\|_1 \|g\|_1.
\end{aligned}$$

The first equality is precisely (3), and we have also made the change of variable  $\lambda \mapsto \lambda^2$ . It is convenient to recall here that  $R_0(z)$  is a holomorphic family of operators on the domain  $\mathbb{C} \setminus \mathbb{R}^+$ , thus  $R_0(z^2)$  is holomorphic on the upper half-plane. Continuation onto the boundary  $\{z = \lambda \in \mathbb{R}\}$  is accomplished by taking limits from the interior.

$$R_0^+(\lambda^2) := \lim_{\varepsilon \rightarrow 0} R_0((\lambda + i\varepsilon)^2) = \lim_{\varepsilon \rightarrow 0} R_0(\lambda^2 + i \operatorname{sign}(\lambda)\varepsilon)$$

For all  $\lambda > 0$  this agrees with the previous definition of  $R_0^+(\lambda^2)$ , and for  $\lambda < 0$  we have the identity  $R_0^+(\lambda^2) = R_0^-((-\lambda)^2)$ .

Using this extended definition, the integral in (4) can be rewritten as

$$(4') \quad \sup_{L \geq 1} \left| \int_{-\infty}^\infty e^{it\lambda^2} \lambda \chi^3(\lambda/L) \left\langle R_0^+(\lambda^2)(I + VR_0^+(\lambda^2))^{-1} f, g \right\rangle \frac{d\lambda}{\pi i} \right|$$

**Remark.** One can extend the domain of  $R_0^-(\lambda^2)$  to the entire real line by taking the domain of  $R_0(z^2)$  to be the lower half-plane in  $\mathbb{C}$ . The symmetry between  $R_0^+(\lambda^2)$  and  $R_0^-(\lambda^2)$  is reflected in the identity

$$R_0^\pm(\lambda^2) = R_0^\mp((-\lambda)^2) \quad \text{for all } \lambda \in \mathbb{R}$$

It will be shown that, provided zero energy is not an eigenvalue or resonance, the operators  $T(\lambda) = (I + VR_0^+(\lambda^2))^{-1}$  are bounded on  $L^1(\mathbb{R}^3)$ , uniformly in  $\lambda \in \mathbb{R}$ . We have included several copies of the cutoff function  $\chi(\lambda/L)$  in (4') so that one of them may be combined with  $T(\lambda)$  to form

$$(5) \quad T_L(\lambda) = \chi(\lambda/L)(I + VR_0^+(\lambda^2))^{-1}$$

For each  $L \geq 1$  we have  $\int_{\mathbb{R}} \|T_L(\lambda)\|_{1 \rightarrow 1} d\lambda < \infty$ , therefore it has a well-defined Fourier transform with respect to  $\lambda$ . Theorem 1 will eventually be derived from the following estimate on the Fourier transform of  $T_L$ .

**Theorem 2.** *If  $V$  satisfies the conditions of Theorem 1, then the family of operators  $T_L(\lambda)$  have the property*

$$(6) \quad \sup_{L \geq 1} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |\widehat{T}_L(\rho) f(x)| d\rho dx \lesssim \|f\|_1 \quad \text{for all } f \in L^1(\mathbb{R}^3).$$

**Proposition 3.** *The bound in (6) also holds for  $T_L^-(\lambda) = \chi(\lambda/L)(I + VR_0^-(\lambda^2))^{-1}$ .*

*Proof of Proposition 3.* The family of operators  $T_L^-(\lambda)$  is obtained from  $T_L(\lambda)$  by taking complex conjugates, so  $\widehat{T_L^-}(\rho)$  is just the complex conjugate of  $\widehat{T_L}(-\rho)$ . Neither conjugation nor reflection changes the value of the inner integral over  $\rho \in \mathbb{R}$ .  $\square$

In the next two sections we will prove Theorem 2 by splitting it into high-energy and low-energy cases. For high energies, the argument is a refinement of estimates found in [RodSch] for each individual term of the Born series. The key step is a differentiability estimate which enables us to control the geometric growth of the terms. For low energies the argument is an improvement of the one in [GS1], both in terms of the computations required and the result achieved. Finally, we show how the dispersive bound in Theorem 1 follows from Theorem 2.

## 2 The High-Energy Case

In this section we wish to show that Theorem 2 holds provided we introduce a cutoff at sufficiently high energy. A precise statement is formulated below.

**Theorem 4.** *Let  $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ ,  $p < \frac{3}{2} < q$ . There exist a number  $\lambda_1(V) < \infty$  and a constant  $A(V) < \infty$  so that the inequality*

$$(7) \quad \sup_{L \geq 1} \int_{\mathbb{R}^3} \int_{\mathbb{R}} |[(1 - \chi(\cdot/\lambda_1))T_L]^\wedge(\rho)f(x)| d\rho dx \leq A\|f\|_1$$

holds for all  $f \in L^1(\mathbb{R}^3)$ .

The general idea of the proof is to expand  $T_L(\lambda) = \chi(\lambda/L)(I + VR_0^+(\lambda^2))^{-1}$  as a power series and make estimates on each of the resulting terms. The high-energy cutoff will be needed only at the end to insure summability of the entire series. We begin with an elementary observation.

**Proposition 5.** *If  $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ ,  $p < \frac{3}{2} < q$ , then*

$$\sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(x)|}{|x - y|} dx \leq C_{p,q}\|V\|$$

*Proof.* Here, and in the remainder of the discussion, we use  $\|V\|$  to indicate  $\max(\|V\|_p, \|V\|_q)$ . Inside the region  $\{|x - y| < 1\}$ , use Hölder's inequality with  $V \in L^q(\mathbb{R}^3)$  and  $|\cdot - y|^{-1} \in L^{q'}(\mathbb{R}^3)$ . In the region  $\{|x - y| \geq 1\}$ , consider  $V \in L^p(\mathbb{R}^3)$  and  $|\cdot - y|^{-1} \in L^{p'}(\mathbb{R}^3)$ .  $\square$

**Corollary 6.** *If  $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ ,  $p < \frac{3}{2} < q$ , then*

$$\|VR_0^+(\lambda^2)f\|_1 \leq C_{p,q}\|V\|\|f\|_1 \quad \text{for all } f \in L^1(\mathbb{R}^3), \lambda \in \mathbb{R}$$

*Proof.* Recall that the free resolvent in three dimensions can be represented explicitly by the integration kernel

$$(8) \quad R_0^+(\lambda^2)(x, y) = \frac{e^{i\lambda|x-y|}}{4\pi|x-y|}$$

The integration kernel for  $VR_0^+(\lambda^2)$  is therefore  $\frac{e^{i\lambda|x-y|}V(x)}{4\pi|x-y|}$ , which immediately satisfies the Schur criterion for boundedness as an operator on  $L^1$ .  $\square$

## 2.1 Integrability and Smoothness

The next lemma is a fundamental  $L^1$  estimate for the Fourier transform of  $VR_0^+(\lambda^2)$ . An unweighted version is implicit in the proof of Theorem 2.6 in [RodSch], however the extra decay assumption of  $V$  (we have  $V \in L^p(\mathbb{R}^3)$  instead of  $\sup_y \int_{\mathbb{R}^3} |x-y|^{-1} |V(x)| dx < \infty$ ) allows us to introduce a small polynomial weight.

**Lemma 7.** *If  $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ ,  $p < \frac{3}{2} < q$ , then there exist  $0 < \varepsilon < 1$  and  $C < \infty$  such that*

$$\sup_{L \geq 1} \int_{\mathbb{R}^3} \langle \rho \rangle^\varepsilon \left| [\chi(\cdot/L)(VR_0^+(\cdot^2))^k]^\wedge(\rho) f(x) \right| d\rho dx \lesssim (C\|V\|)^k \|f\|_1$$

for every  $f \in L^1(\mathbb{R}^3)$  and every  $k \geq 0$ .

*Proof.* The expression  $\langle \rho \rangle$  is given the usual meaning  $(1 + |\rho|^2)^{1/2}$ . After substituting the integration kernel (8) for each occurrence of  $R_0^+(\lambda^2)$ , we see that

$$\chi(\lambda/L)(VR_0^+(\lambda^2))^k f(x_0) = (4\pi)^{-k} \int_{\mathbb{R}^{3k}} \chi(\lambda/L) e^{i\lambda\Sigma} \left( \prod_{\ell=0}^{k-1} \frac{V(x_\ell)}{|x_\ell - x_{\ell+1}|} \right) f(x_k) dx_1 dx_2 \dots dx_k$$

where we have introduced the abbreviation  $\Sigma := \sum_{\ell=0}^{k-1} |x_\ell - x_{\ell+1}|$ . The integrand above is a function in  $L^1(\mathbb{R}^{3k+4})$ . This is most easily seen by integrating sequentially in the variables  $d\lambda, dx_0, dx_1, \dots, dx_{k-1}$  and applying Proposition 5 each time. We may therefore use Fubini's theorem to take the Fourier transform in  $\lambda$  before integrating in  $x_1, \dots, x_k$ . The resulting expression is

$$[\chi(\cdot/L)(VR_0^+(\cdot^2))^k]^\wedge(\rho) f(x_0) = (4\pi)^{-k} L \int_{\mathbb{R}^{3k}} \hat{\chi}(L(\rho - \Sigma)) \left( \prod_{\ell=0}^{k-1} \frac{V(x_\ell)}{|x_\ell - x_{\ell+1}|} \right) f(x_k) dx_1 dx_2 \dots dx_k$$

Multiply by the weight  $\langle \rho \rangle^\varepsilon$  and integrate with respect to  $d\rho$ . It is an elementary fact, proven below, that for any  $\Sigma \in \mathbb{R}$ ,  $\sup_{L \geq 1} L \int_{\mathbb{R}} \langle \rho \rangle^\varepsilon |\hat{\chi}(L(\rho - \Sigma))| d\rho \lesssim \langle \Sigma \rangle^\varepsilon$ . Recall that  $\Sigma = \sum_{\ell=0}^{k-1} |x_\ell - x_{\ell+1}|$  by definition. Repeated application of the inequalities  $\langle A+B \rangle < \langle A \rangle + \langle B \rangle$  and  $(A+B)^\varepsilon \leq A^\varepsilon + B^\varepsilon$  for nonnegative  $A, B$  and  $0 < \varepsilon < 1$  shows that  $\langle \Sigma \rangle^\varepsilon \leq \sum_{\ell=0}^{k-1} \langle |x_\ell - x_{\ell+1}| \rangle^\varepsilon$ . It follows that

$$\begin{aligned} & \sup_{L \geq 1} \int_{\mathbb{R}} \langle \rho \rangle^\varepsilon \left| [\chi(\cdot/L)(VR_0^+(\cdot^2))^k]^\wedge(\rho) f(x_0) \right| d\rho \\ & \lesssim (4\pi)^{-k} \sum_{\ell=0}^{k-1} \int_{\mathbb{R}^{3k}} \frac{|V(x_0)|}{|x_0 - x_1|} \dots \frac{|V(x_\ell)| \langle |x_\ell - x_{\ell+1}| \rangle^\varepsilon}{|x_\ell - x_{\ell+1}|} \dots \frac{|V(x_{k-1})|}{|x_{k-1} - x_k|} |f(x_k)| dx_1 \dots dx_k \end{aligned}$$

When the  $L^1(\mathbb{R}^3)$  norm is taken in the  $x_0$  variable, it is possible to integrate sequentially with respect to  $dx_0, dx_1, \dots, dx_{k-1}$ , applying Proposition 5 each time. The integral in  $dx_\ell$  is slightly different, however it too is uniformly bounded provided  $1 - \varepsilon > \frac{3}{p}$ . Summing over  $\ell$  introduces an extra factor of  $k$ , which may be absorbed into the constant  $C$  because  $k \leq 2^k$  for all  $k \geq 0$ .

By Fubini's theorem, the same result would be achieved had we first integrated  $dx_1 \dots dx_k$ , then  $d\rho$  and  $dx_0$  as suggested in the statement of the lemma.  $\square$

**Proposition 8.** *Let  $\eta : \mathbb{R} \rightarrow \mathbb{R}$  satisfy the size bound  $|\eta(y)| \lesssim \langle y \rangle^{-2}$ . Then for any  $0 < \varepsilon < 1$ ,*

$$\sup_{L \geq 1} L \int_{\mathbb{R}} \langle y \rangle^\varepsilon |\eta(L(y - \Sigma))| \lesssim \langle \Sigma \rangle^\varepsilon$$

*Proof.* If  $|\Sigma| \leq 1$ , then the integral over the domain  $y \in [-2, 2]$  is comparable to 1, as desired. For  $|y| > 2$ ,  $|\eta(L(y - \Sigma))| \lesssim |Ly|^{-2}$ . Thus the tail of the integral is controlled by  $\frac{1}{L} \leq 1$ .

If  $|\Sigma| > 1$ , integrate first on the domain  $y \in [-2\Sigma, 2\Sigma]$ , taking  $\langle y \rangle^\varepsilon$  in  $L^\infty$  and  $L\eta(L(\cdot - \Sigma))$  in  $L^1$ . Using the same estimates as above, the tail integral contributes no more than  $L^{-1}\Sigma^{\varepsilon-1} \leq \langle \Sigma \rangle^\varepsilon$ .  $\square$

The next order of business is to show that the Fourier transform of  $(VR_0^+(\lambda^2))^k$  becomes differentiable for sufficiently large  $k$ . This corresponds to polynomial decay in  $\lambda$  of  $(VR_0^+(\lambda^2))^k$  as an operator on  $L^1(\mathbb{R}^3)$ . We paraphrase the relevant statement from [Gol].

**Proposition 9.** *Let  $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ ,  $p < \frac{3}{2} < q$ . Then there exist  $\alpha > 0$  and  $C < \infty$  such that*

$$(9) \quad \|(VR_0^+(\lambda^2))^k f\|_1 \lesssim (C\|V\|)^k (1 + \lambda)^{-(k-2)\alpha} \|f\|_1$$

*Sketch of Proof.* It is a trivial matter to prove a uniform version of this bound, without any decay in  $\lambda$ , as the operator  $VR_0^+(\lambda^2)$  is already known to map  $L^1(\mathbb{R}^3)$  to itself. For  $k = 0, 1, 2$ , this is sufficient. The challenge is to use oscillation in the integration kernel  $\frac{e^{i\lambda|x-y|}V(x)}{|x-y|}$  to strengthen the bounds for large  $\lambda$  and  $k > 2$ .

Choose a number  $r \in (1, \min(\frac{3q}{q+3}, \frac{3p}{5p-3}))$ . The free resolvent  $R_0^+(\lambda^2)$  is weak-type  $(1, 3)$ , and also maps  $L^{\frac{4}{3}}(\mathbb{R}^3)$  to  $L^4(\mathbb{R}^3)$  with norm proportional to  $\lambda^{-1/2}$  (This is a special case of Theorem 2.3 in [KRS]). By interpolation, we conclude that  $\|R_0^+(\lambda^2)\|_{r \rightarrow 3r} = C_r \lambda^{-2/r'}$ . Because  $\frac{3r}{2} \in [p, q]$ , it follows that

$$\|VR_0^+(\lambda^2)\|_{r \rightarrow r} \leq C_r \|V\| \lambda^{-2/r'}$$

Two additional mapping estimates on  $VR_0^+(\lambda^2)$  complete the proof. Since  $\left\| \frac{|V(\cdot)|}{|\cdot-y|} \right\|_r < \infty$  uniformly in  $y$ , it is bounded as a map from  $L^1(\mathbb{R}^3)$  to  $L^r(\mathbb{R}^3)$ . By the Hardy-Littlewood-Sobolev theorem, it is also a bounded map from  $L^r(\mathbb{R}^3)$  back to  $L^1(\mathbb{R}^3)$ .  $\square$

**Corollary 10.** *Let  $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ ,  $p < \frac{3}{2} < q$ . then*

$$\sup_{L \geq 1} \int_{\mathbb{R}^3} \left( \sup_{\rho \in \mathbb{R}} |(-\Delta_\rho + 1)^{\frac{s}{2}} [\chi(\cdot/L)(VR_0^+(\lambda^2))^k]^\wedge(\rho) f(x)| \right) dx \lesssim (C\|V\|)^k \|f\|_1$$

*for any complex number  $s$  with  $\Re(s) < (k-2)\alpha - 1$ . The inequality holds uniformly in  $(s, k)$  satisfying  $\Re(s) \leq (k-2)\alpha - 2$ .*

*Proof.* By the previous lemma,  $\langle \lambda \rangle^s (VR_0^+(\lambda^2))^k f$  will be an integrable family of functions in  $L^1(\mathbb{R}^3)$ . Multiplying by the cutoff  $\chi(\lambda/L)$  does not affect integrability. By Fubini's theorem, that makes  $(VR_0^+(\lambda^2))^k f(x)$  an integrable (over  $x \in \mathbb{R}^3$ ) family of functions in  $L^1(\langle \lambda \rangle^s d\lambda)$ . Taken pointwise in  $x$ , the Fourier transform in  $\lambda$  maps this space boundedly to  $W^{s, \infty}$ , as desired.  $\square$

## 2.2 Interpolation

We wish to interpolate between the estimates in Lemma 7 and Corollary 10 to conclude that the Fourier transform of  $\chi(\cdot/L)(VR_0^+(\lambda^2))^k f$  has a small number of derivatives in the space  $L_x^1 L_\rho^1$ . For technical reasons related to derivatives of imaginary order, it will be preferable to use  $L^{1+\varepsilon}$  with a polynomial weight as a proxy for  $L^1$ .

As a preliminary step, observe that the case  $s = 0$  in Corollary 10 provides an  $L_x^1 L_\rho^\infty$  estimate on the function  $[\chi(\cdot/L)(VR_0^+(\cdot^2))]^\wedge(\rho)f(x)$ , while Lemma 7 bounds its norm in  $L_x^1 L_\rho^1$ . Interpolate using Hölder's inequality to conclude that

$$\sup_{L \geq 1} \int_{\mathbb{R}^3} \left\| \langle \rho \rangle^{\varepsilon'} [\chi(\cdot/L)(VR_0^+(\cdot^2))]^\wedge(\rho)f(x) \right\|_{L^{1+\varepsilon/4}(d\rho)} dx \lesssim (C\|V\|)^k \|f\|_1$$

where  $\varepsilon' = \varepsilon/(1 + \frac{\varepsilon}{4})$ . The main step will be complex interpolation on the family of functions

$$F_\theta(x, \rho) = \langle \rho \rangle^{\varepsilon'(1-\theta)} (-\Delta_\rho + 1)^{\frac{\varepsilon}{2}\theta} [\chi(\cdot/L)(VR_0^+(\cdot^2))]^\wedge(\rho)f(x)$$

with  $s = (k-2)\alpha - 2$  and  $\theta \in \mathbb{C}$  ranging over the strip  $0 \leq \Im(\theta) \leq 1$ .

On the boundary of the strip with  $\theta = 1 + i\gamma$ , these functions are uniformly bounded in  $L_x^1 L_\rho^\infty$  by Corollary 10 and the fact that  $|\langle \rho \rangle^{-i\varepsilon'\gamma}| = 1$ . For the boundary with  $\theta = i\gamma$ , we use the fact that  $(-\Delta_\rho + 1)^{is\gamma/2}$  is a pseudo-differential operator of order zero, and can be represented by convolution with a singular kernel  $K_\gamma(\rho)$ . Following the calculations in [Ste], chapter 6, one obtains the bounds

$$(10) \quad |K_\gamma(\rho)| \lesssim \langle s\gamma \rangle^2 |\rho|^{-1}, \quad |K'_\gamma(\rho)| \lesssim \langle s\gamma \rangle^3 |\rho|^{-2}$$

for all  $x \in \mathbb{R} \setminus \{0\}$ . Additionally, since the second derivative of  $(1 + \lambda^2)^{is\gamma/2}$  is integrable,  $K_\gamma(x)$  satisfies the size bound  $|K_\gamma(x)| \lesssim \langle s\gamma \rangle^2 |x|^{-2}$ .

For each value of  $\gamma \in \mathbb{R}$ , let  $K_1(\rho) = \chi(\rho)K_\gamma(\rho)$  and  $K_2(\rho) = (1 - \chi(\rho))K_\gamma(\rho)$ . It is easy to verify that  $K_1(\rho)$  also satisfies the estimates in (10), and that its Fourier transform is a bounded function. Then convolution with  $K_1$  is a Calderón-Zygmund operator, hence it is bounded on  $L^{1+\varepsilon/4}(\mathbb{R})$ . Moreover, since  $K_1$  is supported on the interval  $[-2, 2]$ ,

$$\int_n^{n+1} \langle \rho \rangle^\varepsilon |g * K_1(\rho)|^{1+\varepsilon/4} d\rho \lesssim \langle s\gamma \rangle^{3(1+\varepsilon/4)} \int_{n-2}^{n+3} \langle \rho \rangle^\varepsilon |g(\rho)|^{1+\varepsilon/4} d\rho$$

It is permissible to include the weight  $\langle \rho \rangle^\varepsilon$  in this inequality because it has size comparable to  $\langle n \rangle^\varepsilon$  everywhere in both domains of integration. Summing over all  $n \in \mathbb{Z}$ ,

$$\begin{aligned} (\|\langle \rho \rangle^{\varepsilon'} g * K_1\|_{1+\varepsilon/4})^{1+\varepsilon/4} &\lesssim \langle s\gamma \rangle^{3(1+\varepsilon/4)} \sum_{n \in \mathbb{Z}} \int_{n-2}^{n+3} \langle \rho \rangle^\varepsilon |g(\rho)|^{1+\varepsilon/4} d\rho \\ &= 5 \langle s\gamma \rangle^{3(1+\varepsilon/4)} (\|\langle \rho \rangle^{\varepsilon'} g\|_{1+\varepsilon/4})^{1+\varepsilon/4} \end{aligned}$$

In other words, convolution with  $K_1$  preserves the weighted space  $L^{1+\varepsilon/4}(\langle \rho \rangle^\varepsilon d\rho)$ . The same is true of convolution with  $K_2$ . This is most readily seen by considering the action of the integral kernel  $\langle \rho \rangle^{\varepsilon'} K_2(\rho - \sigma) \langle \sigma \rangle^{-\varepsilon'}$  on unweighted  $L^{1+\varepsilon/4}(\mathbb{R})$ . Note that

$$\int_{|\rho - \sigma| > 1} \frac{\langle \rho \rangle^{\varepsilon'}}{|\rho - \sigma|^2} d\rho \lesssim \langle \sigma \rangle^{\varepsilon'}$$

If  $|\sigma| < 2$  this is immediate. for  $|\sigma| > 2$  break the domain into the segments  $\{|\rho| \leq 2|\sigma|\}$  and  $\{|\rho| > 2|\sigma|\}$ . Similarly, for any fixed  $\rho \in \mathbb{R}$  we have

$$\int_{|\rho-\sigma|>1} \frac{\langle \sigma \rangle^{\varepsilon'}}{|\rho-\sigma|^2} d\sigma \lesssim \langle \rho \rangle^{-\varepsilon'}$$

If  $|\rho| < 2$  this is also immediate. For  $|\rho| > 2$ , the domain of integration should be broken into three pieces:  $\{\sigma \in [\rho/2, 2\rho]\}$ ,  $\{\sigma \in [-2\rho, \rho/2]\}$ , and  $\{|\sigma| > 2|\rho|\}$ . Finally, one concludes from the Schur test that convolution with  $K_2$  is a bounded operator on the weighted space  $L^p(\langle \rho \rangle^{\varepsilon'/p} d\rho)$  for any exponent  $1 \leq p \leq \infty$ , with particular emphasis on the case  $p = 1 + \varepsilon/4$ . The operator norm is always less than  $\langle s\gamma \rangle^2$ , regardless of the choice of  $p$ , since  $|K_\gamma(x)| \lesssim \langle s\gamma \rangle^2 |x|^{-2}$ .

The end result of these calculations is that  $(-\Delta_\rho + 1)^{i\gamma}$  is bounded on the weighted space  $L^{1+\varepsilon/4}(\langle \rho \rangle^\varepsilon d\rho)$ , with operator norm growing at most polynomially in  $|\gamma|$  and  $s$ . It follows that

$$\|F_{i\gamma}\|_{L_x^1 L_\rho^{1+\varepsilon/4}} \lesssim \langle s\gamma \rangle^3 (C\|V\|_1)^k \|f\|_1 \quad \text{and} \quad \|F_{1+i\gamma}\|_{L_x^1 L_\rho^\infty} \lesssim (C\|V\|)^k \|f\|_1$$

for all  $\gamma \in \mathbb{R}$ . Apply complex interpolation and examine the case  $\theta = \frac{\varepsilon}{4+2\varepsilon}$ . The resulting bound is

$$\left\| \langle \rho \rangle^{2\varepsilon/(2+\varepsilon)} (-\Delta_\rho + 1)^{s\varepsilon/(8+4\varepsilon)} [\chi(\cdot/L)(VR_0^+(\cdot^2))^{k\wedge} (\rho)f(x)] \right\|_{L_x^1 L_\rho^{1+\varepsilon/2}} \lesssim \langle s \rangle^3 (C\|V\|)^k \|f\|_1$$

The parameter  $s$  was defined as a linear function of  $k$ , so the factor of  $\langle s \rangle^3$  may again be absorbed into the constant  $C$  as in Lemma 7. Observe that the reciprocal of  $\langle \rho \rangle^{2\varepsilon/(2+\varepsilon)}$  is a function in  $L^{(2+\varepsilon)/\varepsilon}(\mathbb{R})$ , the space dual to  $L^{1+\varepsilon/2}(\mathbb{R})$ . Hölder's inequality then leads to an estimate in  $L_x^1 L_\rho^1$ , which we formulate as a lemma.

**Lemma 11.** *Suppose  $V \in L^p(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$ ,  $p < \frac{3}{2} < q$ , and let  $k > \frac{2}{\alpha} + 2$ . Then*

$$(11) \quad \sup_{L \geq 1} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \left| (-\Delta_\rho + 1)^{s_k} [\chi(\cdot/L)(VR_0^+(\cdot^2))^{k\wedge} (\rho)f(x)] \right| d\rho dx \lesssim (C\|V\|)^k \|f\|_1$$

where  $s_k = \frac{\alpha\varepsilon}{8+4\varepsilon}k - \frac{(1+\alpha)\varepsilon}{4+2\varepsilon}$ .

### 2.3 Proof of Theorem 4

Up to this point, our estimates have included the entire energy spectrum, and the bounds grow geometrically in  $k$  with ratio proportional to  $\|V\|$ . The next lemma suggests how introducing the high-energy cutoff  $(1 - \chi(\lambda/\lambda_1))$  can ensure convergence of the geometric series even if  $\|V\|$  is large.

**Lemma 12.** *Given  $\lambda_1 > 1$  and  $m > 0$ , define a function  $F(\lambda) = \langle \lambda \rangle^{-m} (1 - \chi(\lambda/\lambda_1))$ .*

$$\text{Then } \|\hat{F}\|_1 \lesssim \langle m \rangle \lambda_1^{-m}.$$

*Proof.* For  $|\rho| > \lambda_1^{-1}$ , use the identity

$$|\hat{F}(\rho)| = \left| \rho^{-2} \left( \frac{d^2 F}{d\lambda^2} \right)^\wedge (\rho) \right| \leq \rho^{-2} \left\| \frac{d^2 F}{d\lambda^2} \right\|_1$$



The second derivative can be computed using the product rule, and consists of three terms. Two of them are compactly supported on the intervals where  $|\lambda| \sim \lambda_1$ , and are no larger than  $\langle m \rangle \lambda_1^{-(m+2)}$  anywhere on this set. The last term, where both derivatives fall on  $\langle \lambda \rangle^{-m}$ , is supported where  $|\lambda| \gtrsim \lambda_1$  and is everywhere smaller than  $\langle m \rangle^2 \lambda^{-(m+2)}$ . The  $L^1$  norm of each piece is seen to be less than  $(1+m)\lambda_1^{-(m+1)}$ . We conclude that

$$|\hat{F}(\rho)| \lesssim \langle m \rangle \lambda_1^{-(m+1)} \rho^{-2}$$

for all  $|\rho| > \lambda_1^{-1}$ . This can contribute no more than  $\langle m \rangle \lambda_1^{-m}$  to the  $L^1$ -norm of  $\hat{F}$ .

If  $m \geq \frac{3}{4}$ , then  $\|F\|_2 \lesssim \lambda_1^{-(m-1/2)}$ . By Plancherel's identity, the  $L^2$ -norm of  $\hat{F}$  satisfies the same bound. Then by the Cauchy-Schwartz inequality,

$$\int_{-1/\lambda_1}^{1/\lambda_1} |\hat{F}(\rho)| d\rho \lesssim \lambda_1^{-m}$$

For  $m < \frac{3}{4}$ , write  $\langle \lambda \rangle^{-m}(1 - \chi(\lambda/\lambda_1)) = |\lambda|^{-m} + G(\lambda)$ . The remainder function  $G$  is dominated by  $|\lambda|^{-m}$  for all  $\lambda \leq 2\lambda_1$  and by  $m|\lambda|^{-(m+2)}$  for all  $\lambda > 2\lambda_1$ . Thus  $\|G\|_1 \lesssim \langle m \rangle \lambda_1^{1-m}$ .

The Fourier transform of  $|\lambda|^{-m}$  is exactly  $c_m |\rho|^{m-1}$ , where  $c_m \lesssim m$  for the range of  $m$  under consideration. The  $L^1$ -norm of this function on the interval  $[-\lambda_1^{-1}, \lambda_1^{-1}]$  is  $\frac{c_m}{m} \lambda_1^{-m}$ , and the constant  $\frac{c_m}{m}$  is bounded uniformly. Meanwhile, the Fourier transform of  $G$  is bounded above by  $\langle m \rangle \lambda_1^{1-m}$ , so its  $L^1$ -norm over the same interval is less than  $\langle m \rangle \lambda_1^{-m}$  as well. Adding the two pieces together proves the desired estimate.  $\square$

*Proof of Theorem 4.* Recall that we are trying to verify the inequality

$$(7) \quad \sup_{L \geq 1} \int_{\mathbb{R}} \left\| \left[ (1 - \chi(\cdot/\lambda_1)) T_L \right]^\wedge(\rho) f \right\|_1 d\rho \lesssim \|f\|_1$$

Fix  $L \geq 1$ , and assume that  $\lambda_1^\alpha > 2C\|V\|$ , where  $\alpha$  and  $C$  are the constants in Proposition 9. The power series

$$(12) \quad (1 - \chi(\lambda/\lambda_1)) T_L(\lambda) = \sum_{k=0}^{\infty} (1 - \chi(\lambda/\lambda_1)) \chi(\lambda/L) (VR_0^+(\lambda^2))^k$$

converges uniformly in  $\lambda$ , and is supported on the interval  $\lambda \in [\lambda_1, L]$ . The Fourier transform of the partial sums then converges in the sense of distributions.

For  $k \leq \frac{2}{\alpha} + 2$ , we use the estimate in Lemma 7 showing that  $[\chi(\cdot/L) (VR_0^+(\cdot^2))^k]^\wedge(\rho) f$  is an integrable family (indexed by  $x \in \mathbb{R}^3$ ) of functions in  $L^1(\mathbb{R})$ . The Fourier transform of  $(1 - \chi(\cdot/\lambda_1))$  is a measure whose total variation norm is finite and does not depend on  $\lambda_1$ . Each of these terms then contributes no more than  $(C\|V\|)^k \|f\|_1$  to the total on the right-hand side of (7).

For all  $k > \frac{2}{\alpha} + 2$ , multiply and divide the  $k^{\text{th}}$  term by a common factor to obtain

$$\left( \langle \lambda \rangle^{-2s_k} (1 - \chi(\lambda/\lambda_1)) \right) \left( \langle \lambda \rangle^{2s_k} \chi(\lambda/L) (VR_0^+(\lambda^2))^k \right)$$

where  $s_k$  is the same number as in (11). Consider the second factor in this product. By Lemma 11 its Fourier transform, acting on a fixed function  $f$ , also gives rise to an integrable family of  $L^1(\mathbb{R})$  functions indexed by  $x \in \mathbb{R}^3$ . The  $L^1$ -norm of this family is bounded by  $(C\|V\|)^k \|f\|_1$ .

The Fourier transform of the first factor is an integrable function of  $\rho$ , with norm less than  $\langle s_k \rangle \lambda_1^{-2s_k}$ . When this is convolved against the expression from the second factor, the result is again an integrable family of  $L^1(\mathbb{R})$  functions with the norm bound

$$(13) \quad \int_{\mathbb{R}^3} \int_{\mathbb{R}} \left| [(1 - \chi(\cdot/\lambda_1))\chi(\cdot/L)(VR_0^+(\cdot^2))^k]^\wedge(\rho)f(x) \right| d\rho dx \lesssim \langle s_k \rangle (C\|V\|)^k \lambda_1^{-2s_k} \|f\|_1$$

holding uniformly in  $L$ . Recall that  $s_k = \frac{\alpha\varepsilon}{8+4\varepsilon}k - \frac{(1+\alpha)\varepsilon}{4+2\varepsilon}$  by definition, so  $s_k$  is a linear function of  $k$ . The bound shown above is then geometric in  $k$ , and its ratio is moderated by a negative power of  $\lambda_1$ . If  $\lambda_1$  is chosen so that  $\lambda_1^{(\alpha\varepsilon)/(4+2\varepsilon)} > 2C\|V\|$ , the geometric series converges, therefore

$$\sup_{L \geq 1} \sum_{k=0}^{\infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \left| [(1 - \chi(\cdot/\lambda_1))\chi(\cdot/L)(VR_0^+(\cdot^2))^k]^\wedge(\rho)f(x) \right| d\rho dx \lesssim (C\|V\|)^{2+2/\alpha} \|f\|_1$$

by comparing the entire series to its largest term.

The Fourier transform of the series (12) converges in  $L^1$  as well as in the distributional sense, and its limit has norm controlled by  $A(V)\|f\|_1$ .  $\square$

### 3 The Low-Energy Case

In this section we prove the complementary statement to Theorem 4, namely

**Theorem 13.** *Let  $V$  satisfy the conditions of Theorem 1, and fix any  $0 < \lambda_1 < \infty$*

$$(14) \quad \sup_{L \geq 1} \int_{\mathbb{R}^3} \int_{\mathbb{R}} \left| [\chi(\cdot/\lambda_1)T_L]^\wedge(\rho)f(x) \right| d\rho dx \lesssim \|f\|_1$$

*holds for all  $f \in L^1(\mathbb{R}^3)$ .*

There are two low-energy cutoffs present in the statement of this theorem, since  $T_L$  is shorthand for  $\chi(\lambda/L)(I + VR_0^+(\lambda^2))^{-1}$ . We will relegate both of these to the background by introducing a third cutoff function which localizes to much smaller intervals in  $\lambda$ . The theorem is then proved by adding up a finite number of local results.

At low energies the Neumann series expansion of  $(I + VR_0^+(\lambda^2))^{-1}$  will typically diverge unless  $\|V\|$  is small. The existence of inverses must instead be demonstrated by a Fredholm alternative argument. For  $\lambda = 0$  this requires that zero energy is neither an eigenvalue nor a resonance, as defined below.

**Definition 14.** *We say that a resonance occurs at zero energy if the equation  $(I + VR_0^+(0))g = 0$  admits a distributional solution  $g \notin L^2(\mathbb{R}^3)$  such that  $\langle x \rangle^{-\beta}g \in L^2(\mathbb{R}^3)$  for every  $\beta > \frac{1}{2}$ .*

The Fredholm alternative does not construct inverses explicitly, which limits our ability to perform subsequent calculations. We therefore avoid its use, except in a finite number of instances, by the following scheme:

Fix a ‘‘benchmark’’ energy  $\lambda_0 \in \mathbb{R}$  and let  $S_0 = (I + VR_0^+(\lambda_0^2))^{-1}$ . For all values of  $\lambda$  sufficiently close to  $\lambda_0$ , we may regard  $R_0^+(\lambda^2)$  as a perturbation of  $R_0^+(\lambda_0^2)$  and treat the corresponding inverse as a perturbation of  $S_0$ . The underlying perturbation is a difference of free resolvents, hence it can

be represented explicitly by an integration kernel. The role of  $S_0$  is limited to its existence as a (fixed) bounded operator on  $L^1(\mathbb{R}^3)$ . In this manner the entire interval of energies  $|\lambda - \lambda_0| < \delta$  may be considered with only one application of the Fredholm theory. The perturbation radius  $\delta$  can be chosen independent of  $\lambda_0$ , so the low-energy spectrum  $\lambda \in [-2\lambda_1, 2\lambda_1]$  is covered by a finite collection of such intervals.

The details of the proof are clearly foreshadowed by the low-energy discussion in [GS1]. Two technical modifications allow us to work with a larger class of potentials while reducing the burden of computation. One is the use of  $L^1(\mathbb{R}^3)$  as the natural setting instead of weighted  $L^2$  spaces. The other is, for a family of operators  $T(\rho)$ , estimating the quantity  $\int_{\mathbb{R}} \|T(\rho)f\| d\rho$  rather than  $\int_{\mathbb{R}} \|T(\rho)\| d\rho$ .

### 3.1 Invertibility of $I + VR_0^+(\lambda^2)$

Here we show that  $(I + VR_0^+(\lambda^2))^{-1}$  exists as a bounded operator on  $L^1(\mathbb{R}^3)$  for each  $\lambda \in \mathbb{R}$ , and that the operator norm of these inverses can be controlled uniformly in  $\lambda$ . Essentially identical arguments have been made in various function spaces, and with varying assumptions on  $V$ , for example in [DanPie] and [GS2], and can be traced back to Agmon's work on the limiting absorption principle [Ag].

**Lemma 15.** *Suppose  $V$  satisfies the conditions of Theorem 1. Then*

$$(15) \quad \sup_{\lambda \in \mathbb{R}} \|(I + VR_0^+(\lambda^2))^{-1}\|_{1 \rightarrow 1} < \infty$$

*Sketch of Proof.* Observe that if  $V \in C_c^\infty(\mathbb{R}^3)$ , then  $VR_0^+(\lambda^2)$  maps  $L^1(\mathbb{R}^3)$  to  $W^{2,1}(\text{supp}V)$ , hence it is a compact operator on  $L^1(\mathbb{R}^3)$  by Rellich's theorem. For general potentials, compactness of  $VR_0^+(\lambda^2)$  is seen by writing  $V$  as a limit of functions in  $C_c^\infty(\mathbb{R}^3)$ .

The Fredholm Alternative Theorem then dictates that either  $(I + VR_0^+(\lambda^2))^{-1}$  is bounded on  $L^1(\mathbb{R}^3)$  or else it has a nonempty null-space, that is there exists  $g_\lambda \in L^1(\mathbb{R}^3)$  solving  $(I + VR_0^+(\lambda^2))g_\lambda = 0$ . By bootstrapping the identity  $g_\lambda = -VR_0^+(\lambda^2)g_\lambda$  with the Hardy-Littlewood-Sobolev inequality, we see that  $g_\lambda \in L^1(\mathbb{R}^3) \cap L^q(\mathbb{R}^3)$  and  $R_0^+(\lambda^2)g_\lambda \in L^\infty(\mathbb{R}^3)$ . Since

$$0 = \Im \langle R_0^+(\lambda^2)g_\lambda, VR_0^+(\lambda^2)g_\lambda \rangle = -\Im \langle R_0^+(\lambda^2)g_\lambda, g_\lambda \rangle = c\lambda \int_{S^2} |\hat{g}_\lambda(\lambda\omega)|^2 d\omega$$

it follows that for any  $\lambda \in \mathbb{R} \setminus \{0\}$ , the Fourier transform of  $g_\lambda$  vanishes (in the  $L^2$  trace sense) on the sphere of radius  $\lambda$ . By Proposition 12 in [GS2],  $R_0^+(\lambda^2)g_\lambda \in L^2(\mathbb{R}^3)$ . On the other hand, the definition of  $g_\lambda$  implies that  $(-\Delta + V - \lambda^2)(R_0^+(\lambda^2)g_\lambda) = 0$  in the sense of distributions. Finally, a theorem of Ionescu and Jerison [IonJer] states that  $(-\Delta + V)$  has no nontrivial eigenfunctions with positive energy, so  $R_0^+(\lambda^2)g_\lambda = 0$ . It follows immediately that  $g_\lambda = 0$  as well.

In the case  $\lambda = 0$ , the distributional equation  $(-\Delta + V)R_0^+(0)g_0 = 0$  is still valid. For any  $\beta > \frac{1}{2}$ ,  $\sup_{y \in \mathbb{R}^3} \|\langle x \rangle^{-2\beta} |x - y|^{-1}\|_2 < \infty$ , therefore  $R_0^+(0)$  is a bounded map from  $L^1(\mathbb{R}^3)$  to the weighted space  $L^{2,-\beta}(\mathbb{R}^3)$ . By our assumption that zero energy is neither an eigenvalue nor a resonance, we exclude the possibility that a nontrivial function  $R_0^+(0)g_0$  can belong to this class, leaving only the solution  $g_0 = 0$ .

So far we have established that  $(I + VR_0^+(\lambda^2))^{-1}$  exists at each  $\lambda \in \mathbb{R}$ , but have not shown uniformity. By Proposition 9,  $\|(VR_0^+(\lambda^2))^3\|_{1 \rightarrow 1} < \frac{1}{2}$  for sufficiently large  $\lambda$ . For these values of  $\lambda$ ,

$$\|(I + VR_0^+(\lambda^2))^{-1}\|_{1 \rightarrow 1} \leq \|I - VR_0^+(\lambda^2) + (VR_0^+(\lambda^2))^2\|_{1 \rightarrow 1} \|(I + (VR_0^+(\lambda^2))^3)^{-1}\|_{1 \rightarrow 1} \lesssim 1 + \|V\|^2$$

which provides a uniform bound. For small  $\lambda$ , observe that the family of operators  $I + VR_0^+(\lambda^2)$  vary continuously in  $\lambda$  (In fact, the variation is Hölder continuous because  $V \in L^p(\mathbb{R}^3)$  for some  $p < \frac{3}{2}$ ). Since inverses exist at every  $\lambda \in \mathbb{R}$ , they also form a continuous family of operators, and are uniformly bounded on any compact set.  $\square$

### 3.2 Proof of Theorem 13

Fix  $\lambda_0 \in \mathbb{R}$  and let  $S_0 = (I + VR_0^+(\lambda_0^2))^{-1}$ . Then  $I + VR_0^+(\lambda^2) = S_0^{-1} + VB^+(\lambda)$ , where  $B^+(\lambda)$  denotes the difference  $R_0^+(\lambda^2) - R_0^+(\lambda_0^2)$ . Taking inverses,

$$(16) \quad (I + R_0^+(\lambda^2))^{-1} = (I + S_0VB^+(\lambda))^{-1}S_0$$

We remarked above that  $VR_0^+(\lambda^2)$  varies continuously in  $\lambda$ , which suggests that  $VB^+(\lambda)$  should vanish in the limit  $\lambda \rightarrow \lambda_0$ . More precisely,  $VB^+(\lambda)$  is an integral operator with associated kernel

$$|VB^+(\lambda, x, y)| = \left| \frac{V(x)(e^{i\lambda|x-y|} - e^{i\lambda_0|x-y|})}{|x-y|} \right| \leq \begin{cases} |\lambda - \lambda_0| |V(x)|, & \text{if } |x-y| \leq |\lambda - \lambda_0| \\ \frac{|V(x)|}{|x-y|}, & \text{if } |x-y| > |\lambda - \lambda_0| \end{cases}$$

For fixed  $y \in \mathbb{R}^3$ ,  $\lambda \in \mathbb{R}$ , the  $L^1$ -norm of this kernel in the  $x$  variable is controlled by  $|\lambda - \lambda_0|^{1-3/p'} \|V\|$ , by applying Hölder's inequality with  $V \in L^p(\mathbb{R}^3)$  and the remaining factors in  $L^{p'}(\mathbb{R}^3)$ . In other words,  $\|VB^+(\lambda)\|_{1 \rightarrow 1} \leq C|\lambda - \lambda_0|^{1-3/p'} \|V\|$ , with the constant  $C < \infty$  independent of the choice of  $\lambda_0$ . Since  $\|S_0\|_{1 \rightarrow 1}$  is bounded above by (15), there exists  $r > 0$  so that  $\|S_0VB^+(\lambda)\|_{1 \rightarrow 1} < \frac{1}{2}$  whenever  $|\lambda - \lambda_0| \leq 4r$ .

In that case, the Neumann series

$$(17) \quad \chi\left(\frac{\lambda - \lambda_0}{r}\right)(I + VR_0^+(\lambda^2))^{-1} = \sum_{k=0}^{\infty} (-1)^k \left( \chi\left(\frac{\lambda - \lambda_0}{2r}\right)(S_0VB^+(\lambda)) \right)^k \chi\left(\frac{\lambda - \lambda_0}{r}\right) S_0$$

converges uniformly over all  $\lambda \in \mathbb{R}$ . Recall here that  $\chi\left(\frac{\lambda - \lambda_0}{2r}\right) = 1$  everywhere on the support of  $\chi\left(\frac{\lambda - \lambda_0}{r}\right)$ . The Fourier transforms of the partial sums converge in the sense of distributions.

**Lemma 16.** *The Fourier transform of  $\chi\left(\frac{\lambda - \lambda_0}{2r}\right)S_0VB^+(\lambda)$  satisfies the bound*

$$(18) \quad \int_{\mathbb{R}} \left\| \left[ \chi\left(\frac{\cdot - \lambda_0}{2r}\right)S_0VB^+(\cdot) \right]^\wedge(\rho) f \right\|_1 d\rho \leq Cr^{1-3/p'} \|f\|_1$$

for all functions  $f \in L^1(\mathbb{R}^3)$ .

*Proof.* The Fourier transform of  $\chi\left(\frac{\lambda - \lambda_0}{2r}\right)VB^+(\lambda)$  can be represented by the integration kernel

$$K_r(\rho, x, y) = e^{-i\lambda_0(\rho - |x-y|)} V(x) \left( 2r \frac{\hat{\chi}(2r(\rho - |x-y|)) - \hat{\chi}(2r\rho)}{|x-y|} \right)$$

This leads to an immediate estimate  $\int_{\mathbb{R}} |K_r(\rho, x, y)| d\rho \leq 2|x-y|^{-1} |V(x)| \|\hat{\chi}\|_1$ , by assuming no cancellation between the two evaluations of  $\hat{\chi}$ . For small values of  $|x-y|$  a better estimate is possible. By the Mean Value theorem,

$$|K_r(\rho, x, y)| \leq |V(x)| \frac{2r}{|x-y|} \int_{2r(\rho - |x-y|)}^{2r\rho} |\hat{\chi}'(\tau)| d\tau$$

Fubini's theorem permits integrations to be carried out in any order, so that  $\int_{\mathbb{R}} |K_r(\rho, x, y)| d\rho \leq 2r|V(x)| \|\hat{\chi}'\|_1$ . Putting the two estimates together,

$$\int_{\mathbb{R}} |K_r(\rho, x, y)| d\rho \lesssim |V(x)| \min(r, |x - y|^{-1})$$

which leads to the further integral estimate

$$\iiint_{\mathbb{R}^{1+3+3}} |K_r(\rho, x, y)| f(y) d\rho dx dy \lesssim r^{1-3/p'} \|V\| \|f\|_1.$$

Once again, Fubini's theorem allows for the integration to take place in the reverse order. This means that  $\int_{\mathbb{R}} \|\chi((\cdot - \lambda_0)/2r)VB^+(\cdot)^\wedge(\rho)f\|_1 d\rho \lesssim r^{1-3/p'} \|f\|_1$ . Applying the bounded operator  $S_0$  pointwise at each  $\rho \in \mathbb{R}$  only increases the estimate by a finite factor.  $\square$

**Corollary 17.** *If  $f_\rho$  is a family of functions in  $L^1(\mathbb{R}^3)$  indexed by  $\rho \in \mathbb{R}$ , then*

$$(19) \quad \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} \left[ \chi\left(\frac{\cdot - \lambda_0}{2r}\right) S_0 B^+(\cdot) \right]^\wedge(\sigma - \rho) f_\rho d\rho \right\|_{L^1(\mathbb{R}^3)} d\sigma \leq Cr^{1-3/p'} \int_{\mathbb{R}} \|f_\rho\|_1 d\rho$$

*Proof.* The expression on the left-hand side is dominated by

$$\iint_{\mathbb{R}^2} \left\| \left[ \chi\left(\frac{\cdot - \lambda_0}{2r}\right) S_0 VB^+(\cdot) \right]^\wedge(\sigma - \rho) f_\rho \right\|_1 d\rho d\sigma$$

which, after applying Fubini's theorem and the previous lemma, is seen to be less than the expression on the right-hand side.  $\square$

A pointwise product of functions in the  $\lambda$  variable corresponds to convolution in the  $\rho$  variable when Fourier transforms are taken. The previous two statements can be combined iteratively to prove Fourier bounds for  $(S_0VB^+(\lambda))^k$ .

**Corollary 18.** *The Fourier transform of  $(\chi(\frac{\lambda - \lambda_0}{2r})S_0VB^+(\lambda))^k$  satisfies the bound*

$$(20) \quad \int_{\mathbb{R}} \left\| \left[ \left( \chi\left(\frac{\cdot - \lambda_0}{2r}\right) S_0 VB^+(\cdot) \right)^k \right]^\wedge(\rho) f \right\|_1 d\rho \leq (Cr^{(1-3/p')})^k \|f\|_1$$

for all functions  $f \in L^1(\mathbb{R}^3)$ .

*Proof of Theorem 13.* Apply the corollary above to  $S_0f \in L^1(\mathbb{R}^3)$ , then convolve in  $\rho$  with the function  $r\hat{\chi}(r\rho)$ . This has  $L^1$ -norm  $\|\hat{\chi}\|_1 < \infty$ , and  $S_0$  is a bounded map, so the previous estimates are multiplied by a fixed constant. If  $r > 0$  is chosen small enough so that  $Cr^{1-3/p'} < \frac{1}{2}$ , then the Neumann series for  $\chi(\frac{\cdot - \lambda_0}{r})(I + VR_0^+(\lambda^2))^{-1}$  given in (17) converges in  $L^1$ -norm (as well as in distributions) on the Fourier transform side, and has norm bounded by  $\|f\|_1$ . Note that the chosen value of  $r$  does not depend on  $\lambda_0$ .

Further convolutions in  $\rho$  with the functions  $\lambda_1\hat{\chi}(\lambda_1\rho)$  and  $L\hat{\chi}(L\rho)$ , each of which also has  $L^1$ -norm  $\|\hat{\chi}\|_1$ , yields a similar estimate for  $\left[ \chi\left(\frac{\lambda - \lambda_0}{r}\right) \chi(\lambda/\lambda_1) T_L \right]^\wedge$ . Since translations of  $\chi$  form a partition of unity, the localization caused by  $\chi\left(\frac{\lambda - \lambda_0}{r}\right)$  may be removed by obtaining separate bounds for each choice of  $\lambda_0 = 3nr$ ,  $n \in \left[-\frac{2\lambda_1}{r}, \frac{2\lambda_1}{r}\right]$ , and adding these together.  $\square$

## 4 Proof of Theorem 1

We now return to the goal of proving

$$(4) \quad \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda^2} \lambda \chi^3(\lambda/L) \langle [R_V^+(\lambda^2) - R_V^-(\lambda^2)]f, g \rangle \frac{d\lambda}{\pi i} \right| \lesssim |t|^{-\frac{3}{2}} \|f\|_1 \|g\|_1$$

Integrate the left-hand expression by parts once to obtain

$$(21) \quad \frac{1}{2\pi|t|} \sup_{L \geq 1} \left| \int_0^\infty e^{it\lambda^2} \left( \chi^3(\lambda/L) \langle [\frac{d}{d\lambda} R_V^+(\lambda^2) - \frac{d}{d\lambda} R_V^-(\lambda^2)]f, g \rangle \right. \right. \\ \left. \left. + \frac{3}{L} \chi'(\lambda/L) \chi^2(\lambda/L) \langle [R_V^+(\lambda^2) - R_V^-(\lambda^2)]f, g \rangle \right) d\lambda \right|$$

The two terms are considered separately. For the first one, use the identity  $R_V^+((-\lambda)^2) = R_V^-(\lambda^2)$  to rewrite it as

$$\frac{1}{2\pi|t|} \sup_{L \geq 1} \left| \int_{-\infty}^\infty e^{it\lambda^2} \chi^3(\lambda/L) \langle \frac{d}{d\lambda} R_V^+(\lambda^2) f, g \rangle d\lambda \right|$$

compare this to (4'). The operator  $\frac{d}{d\lambda} R_V^+(\lambda^2)$  can be written in terms of free resolvents by differentiating the identity (2). There are several algebraically equivalent expressions to choose from, one of which is  $(I + R_0^+(\lambda^2)V)^{-1} \frac{d}{d\lambda} [R_0^+(\lambda^2)] (I + VR_0^+(\lambda^2))^{-1}$ . Substituting this into the integral yields

$$\frac{1}{2\pi|t|} \sup_{L \geq 1} \left| \int_{\mathbb{R}} e^{it\lambda^2} \langle \chi(\lambda/L) \frac{d}{d\lambda} [R_0^+(\lambda^2)] T_L(\lambda) f, T_L^-(\lambda) g \rangle d\lambda \right|$$

We wish to apply Parseval's theorem, separating the integrand into the product  $e^{it\lambda^2} \cdot A(\lambda)$ . The factor denoted by  $A(\lambda)$  is bounded with compact support, since every operator  $T_L(\lambda)$  and  $T_L^-(\lambda)$  is bounded on  $L^1(\mathbb{R}^3)$  and  $\frac{d}{d\lambda} R_0^+(\lambda^2)$  maps  $L^1(\mathbb{R}^3)$  to  $L^\infty(\mathbb{R}^3)$ . More precisely, the differentiated resolvent has integral kernel  $(-4\pi i)^{-1} e^{i\lambda|x-y|}$ , so the Fourier transform of  $\chi(\lambda/L) \frac{d}{d\lambda} R_0^+(\lambda^2)$  is a family of integral operators with kernel

$$K(\rho, x, y) = \frac{iL}{2} \hat{\chi}(L(\rho - |x - y|)).$$

The Fourier transform of  $e^{it\lambda^2}$  is well known to be  $\sqrt{\pi/(2|t|)}(1 + i \operatorname{sign}(t))e^{-i\rho^2/4t}$ . Thus Parseval's theorem leads us to evaluate

$$(22) \quad |t|^{-\frac{3}{2}} \sup_{L \geq 1} \left| L \int_{\mathbb{R}} e^{(-i\rho^2/4t)} \iiint \iiint_{\mathbb{R}^8} [\widehat{T}_L(\sigma) f(x)] [\overline{\widehat{T}_L^-(\tau) g(y)}] \hat{\chi}(L(\rho - \sigma - \tau - |x - y|)) dx d\sigma dy d\tau d\rho \right|$$

modulo constants. The fact that  $A(\lambda)$  is a product of three terms means that  $\hat{A}(\rho)$  is an iterated convolution, hence the presence of auxiliary variables  $\sigma$  and  $\tau$ . Take the absolute value inside all the integrals, so that we may evaluate them in a more convenient order.

The integral  $\int_{\mathbb{R}} L |\hat{\chi}(L(\rho - \sigma - \tau - |x - y|))| d\rho$  contributes  $\|\hat{\chi}\|_1$  for any fixed value of the other variables. Then, since  $\widehat{T}_L(\sigma) f$  and  $\widehat{T}_L^-(\tau) g$  are both functions in  $L^1(\mathbb{R}^{1+3})$  by Theorem 2, the entire expression is controlled by  $|t|^{-3/2} \|f\|_1 \|g\|_1$ .

The second term in (21) is treated similarly.  $R_V^+(\lambda^2) - R_V^-(\lambda^2)$  becomes an odd function when both pieces are extended to all of  $\lambda \in \mathbb{R}$ , as is  $\chi'(\lambda/L)$ , so the entire integrand is even. We can then evaluate

$$\frac{3}{4\pi|t|} \sup_{L \geq 1} \frac{1}{L} \left| \int_{-\infty}^{\infty} \chi'(\lambda/L) \chi^2(\lambda/L) \langle [R_V^+(\lambda^2) - R_V^-(\lambda^2)]f, g \rangle d\lambda \right|$$

Cancellation between the two resolvents plays a much greater role here, as we need  $R_V^+(\lambda^2) - R_V^-(\lambda^2)$  to map  $L^1(\mathbb{R}^3)$  into  $L^\infty(\mathbb{R}^3)$  for the inner product to be well-defined. Starting with the relations  $\mathbb{R}_V^\pm(\lambda^2) = R_0^\pm(\lambda^2)(I + VR_0^\pm(\lambda^2))^{-1}$  and performing some algebra, we obtain the identity

$$R_V^+(\lambda^2) - R_V^-(\lambda^2) = (I + R_0^-(\lambda^2)V)^{-1} (R_0^+(\lambda^2) - R_0^-(\lambda^2)) (I + VR_0^+(\lambda^2))^{-1}$$

The middle factor is precisely convolution with the kernel  $\frac{-\sin(\lambda|x|)}{2\pi|x|}$ , which indeed maps  $L^1(\mathbb{R}^3)$  to  $L^\infty(\mathbb{R}^3)$ , and the outer factors are each bounded on their respective spaces. Written another way, the expression in question is

$$\frac{3}{4\pi|t|} \sup_{L \geq 1} \frac{1}{L} \left| \int_{\mathbb{R}} \langle \chi'(\lambda/L) (R_0^+(\lambda^2) - R_0^-(\lambda^2)) T_L(\lambda) f, T_L(\lambda) g \rangle d\lambda \right|$$

After applying Plancherel's theorem and discarding fixed constants, this is equivalent to

$$(23) \quad |t|^{-\frac{3}{2}} \sup_{L \geq 1} \left| \int_{\mathbb{R}} e^{-i\rho^2/4t} \iiint_{\mathbb{R}^8} [\widehat{T}_L(\sigma)f(x)] [\overline{\widehat{T}_L(\tau)g(y)}] \right. \\ \left. \times \frac{\widehat{\chi}'(L(\rho - \sigma - \tau - |x - y|)) - \widehat{\chi}'(L(\rho - \sigma - \tau + |x - y|))}{|x - y|} dx d\sigma dy d\tau d\rho \right|$$

Take the absolute value inside all integrals, and perform integration first with respect to  $d\rho$ . The complex exponential function disappears, and we are left with

$$\frac{1}{|x - y|} \int_{\mathbb{R}} \left| \int_{L(\rho - \sigma - \tau - |x - y|)}^{L(\rho - \sigma - \tau + |x - y|)} \frac{d}{ds} \widehat{\chi}'(s) ds \right| d\rho \leq 2 \|\widehat{\chi}'\|_1$$

by taking the absolute value inside again and using Fubini's theorem. This bound is independent of all other variables, including  $L$ , so by Theorem 2 the size of (23) is controlled by  $|t|^{-3/2} \|f\|_1 \|g\|_1$ . The proof of Theorem 1 is complete.

**Remark.** The second part of (21) gives the impression of being a boundary term, so it would be satisfying to see it vanish as  $L \rightarrow \infty$ . An additional estimate for (23) shows that this occurs. After absolute values are brought inside, the integral in  $\rho$  can also be bounded above by  $2\|\widehat{\chi}'\|_1/(L|x - y|)$  by assuming no cancellation between the evaluations of  $\widehat{\chi}'$ . This provides pointwise (in  $(x, \sigma, y, \tau)$ ) convergence to zero as  $L \rightarrow \infty$ , and the bound used above lets us apply dominated convergence.

## References

- [Ag] Agmon, S. *Spectral properties of Schrödinger operators and scattering theory*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 2, 151–218.

- [BPST] Burq, N., Planchon, F., Stalker, J., Tahvildar-Zadeh, A., *Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential*. J. Funct. Anal. 203 (2003), no. 2, 519–549.
- [DanPie] D’Ancona, P., Pierfelice, V. *On the wave equation with a large rough potential*. preprint, 2003.
- [Duy] Duyckaerts, T. *A singular critical potential for the Schrödinger operator*. preprint, 2003.
- [Gol] Goldberg, M. *Dispersive estimates for the three-dimensional Schrödinger equation with rough potentials*, preprint, 2004.
- [GS1] Goldberg, M., Schlag, W. *Dispersive estimates for the Schrödinger operator in dimensions one and three*, Comm. Math. Phys. 251 (2004), no. 3, 157–178.
- [GS2] Goldberg, M., Schlag, W. *A limiting absorption principle for the three-dimensional Schrödinger equation with  $L^p$  potentials*, Intl. Math. Res. Not. 2004:75 (2004), 4049–4071.
- [IonJer] Ionescu, A., Jerison, D. *On the absence of positive eigenvalues of Schrödinger operators with rough potentials*. Geom. and Func. Anal. 13 (2003), 1029–1081.
- [Jen1] Jensen, A. *Spectral properties of Schrödinger operators and time-decay of the wave functions results in  $L^2(\mathbb{R}^m)$ ,  $m \geq 5$* . Duke Math. J. 47 (1980), no. 1, 57–80.
- [Jen2] Jensen, A. *Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in  $L^2(\mathbb{R}^4)$* . J. Math. Anal. Appl. 101 (1984), no. 2, 397–422.
- [JenKat] Jensen, A., Kato, T. *Spectral properties of Schrödinger operators and time-decay of the wave functions*. Duke Math. J. 46 (1979), no. 3, 583–611.
- [JSS] Journé, J.-L., Soffer, A., Sogge, C. D. *Decay estimates for Schrödinger operators*. Comm. Pure Appl. Math. 44 (1991), no. 5, 573–604.
- [Kato] Kato, T. *Wave operators and similarity for some non-selfadjoint operators*. Math. Ann. 162 (1965/1966), 258–279.
- [KRS] Kenig, C. E. , Ruiz, A., Sogge, C. D. *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators*, Duke Math. J. 55 (1987), no. 2, 329–347.
- [KocTat] Koch, H., Tataru, D., *Sharp counterexamples in unique continuation for second order elliptic equations*. J. Reine Angew. Math. 542 (2002), 133–146.
- [LieLos] Lieb, E., Loss, M., *Analysis*. Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, 2001.
- [PST] Planchon, F., Stalker, J., Tahvildar-Zadeh, A., *Dispersive estimate for the wave equation with the inverse-square potential*. Disc. Cont. Dyn. Sys. 9 (2003), no. 6, pp. 1387–1400.
- [Rau] Rauch, J. *Local decay of scattering solutions to Schrödinger’s equation*. Comm. Math. Phys. 61 (1978), no. 2, 149–168.



- [ReedSim] Reed, M., Simon, B. *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self Adjointness*. Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London, 1975.
- [RodSch] Rodnianski, I., Schlag, W. *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials*. Invent. Math. 155 (2004), no. 3, 451–513.
- [Ste] Stein, E. *Harmonic Analysis: Real–Variable Methods, Orthogonality, and Oscillatory Integrals*. Princeton University Press, Princeton, NJ, 1993.
- [Yaj] Yajima, K. *The  $W^{k,p}$ -continuity of wave operators for Schrödinger operators*. J. Math. Soc. Japan 47 (1995), no. 3, 551–581.

DIVISION OF ASTRONOMY, MATHEMATICS, AND PHYSICS, 253-37 CALTECH, PASADENA, CA 91125, U.S.A.

*email:* **mikeg@its.caltech.edu**