STRICHARTZ ESTIMATES FOR THE SCHRÖDINGER EQUATION WITH TIME-PERIODIC $L^{n/2}$ POTENTIALS

MICHAEL GOLDBERG

Abstract. We prove Strichartz estimates for the Schrödinger operator $H = -\Delta + V(t,x)$ with time-periodic complex potentials V belonging to the scaling-critical space $L_x^{n/2}L_t^\infty$ in dimensions $n \geq 3$. This is done directly from estimates on the resolvent rather than using dispersive bounds, as the latter generally require a stronger regularity condition than what is stated above. In typical fashion, we project onto the continuous spectrum of the operator and must assume an absence of resonances. Eigenvalues are permissible at any location in the spectrum, including at threshold energies, provided that the associated eigenfunction decays sufficiently rapidly.

1. Introduction

The past decade has seen considerable progress in identifying classes of Schrödinger operators which retain the same dispersive properties as the Laplacian. In many cases these operators are described by a simple perturbation of the Laplacian, taking the form $H = -\Delta + L(t,x)$. Typically L is a linear self-adjoint differential operator of degree $d = 0, 1, 2$ representing electrostatic, magnetic, and/or geometric perturbations, respectively. In this paper we consider the Floquet-type potential $L(t,x) = V(t,x)$ satisfying $V(t+2\pi, x) = V(t,x)$ for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

We do not assume any self-adjointness in our main theorem, instead allowing V to be a complex-valued function. Further improvements for real and/or time-independent potentials are examined as corollaries and applications of the first result.

The propagator $e^{-it\Delta}$ of the free Schrödinger equation in $\mathbb{R}^n$ may be represented as a convolution operator with kernel $(4\pi i t)^{-n/2}e^{-i(|x|^2/4t)}$. From this formula it is clear that the free evolution satisfies the dispersive bound $\|e^{it\Delta}\|_{1\to\infty} \leq (4\pi|t|)^{-n/2}$ at all times $t \neq 0$. A TT^* argument combined with fractional integration bounds for the t variable then leads to the family of Strichartz inequalities

$$\|e^{-it\Delta}u_0\|_{L_t^pL_x^q} \leq C_p\|u_0\|_2, \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad p, q \in [2,\infty] \tag{1}$$

for all $u_0 \in L^2(\mathbb{R}^n)$. To be precise, the $p = 2$ endpoint requires a more detailed computation [12] and is false when $n = 2$. We will focus on dimensions $n \geq 3$ in order to take advantage of the full range of exponents $p \in [2,\infty]$ in (1).

The Schrödinger propagator of H generally fails to satisfy estimates like (1) due to the possible existence of bound states, quasiperiodic solutions obeying $u(t+2\pi, x) = e^{2\pi i\lambda}u(t,x)$ for all $t, x \in \mathbb{R}^{1+n}$ and possessing moderate spatial decay. These are best understood in terms of the Floquet Hamiltonian

$$K = i\partial_t - \Delta_x + V(t,x) \tag{2}$$

acting on 2π-periodic functions with domain $\mathbb{T} \times \mathbb{R}^n$. Precisely, each bound state $\phi(t,x)$ solves the distributional equation $(K - \lambda)e^{-i\lambda t}\phi = 0$. If $e^{-i\lambda t}\phi$ is time-periodic and belongs to the space

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$L^2(\mathbb{T}\times\mathbb{R}^n)$ then it is an eigenfunction of K with eigenvalue λ. We say that K has a *resonance* at λ if the resolvent $(K-(\lambda\pm i0))^{-1}$ is singular but the associated "eigenfunction" is not square-integrable. The precise definition is postponed until Section 3, where we attempt to estimate the resolvent of K in the neighborhood of singularities. The spectrum of K is invariant under integer shifts, as $(K-n)=e^{-int}Ke^{int}$ for any $n\in\mathbb{Z}$.

Because our assumptions do not imply that K is self-adjoint, the spectrum of K need not be confined to the real axis. Each eigenfunction ϕ_λ with $\lambda\notin\mathbb{R}$ illustrates the related lack of an L^2 conservation law for solutions of the Schrödinger equation. Since $|e^{2\pi i\lambda}|=e^{-2\pi\mathrm{Im}(\lambda)}\neq 1$, the L^2 norm of ϕ_λ decreases exponentially in one time direction and grows in the other.

For each $\lambda\in\mathbb{C}$ define N_λ to be the solution space

$$N_\lambda=\{\phi:(K-\lambda)e^{-i\lambda t}\phi=0,\ e^{-i\lambda t}\phi\in L^2(\mathbb{T}\times\mathbb{R}^n)\}\tag{3}$$

Local regularity theory dictates that every true eigenfunction also satisfies $e^{-i\lambda t}\phi\in C(\mathbb{T};L^2(\mathbb{R}^n))$. It is then permissible to discuss the initial value of an eigenfunction, $\Phi=\phi(0,\cdot)$. The projection of N_λ onto the space of initial data has as its image

$$X_\lambda=\{\Phi:\phi\in N_\lambda\}\subset L^2(\mathbb{R}^n).\tag{4}$$

We will show via a compactness argument that both N_λ and X_λ are always finite dimensional. Similarly define $\tilde N_\lambda$ and $\tilde X_\lambda$ to represent the eigenfunctions of $\overline{K}$ (the Floquet operator with potential $\overline{V(t,x)}$) that have eigenvalue $\bar\lambda$. These spaces are all invariant under real integer translations.

In this paper we prove that the Schrödinger evolution of $H=-\Delta+V(t,x)$ observes a space-time estimate identical to (1) once a finite-dimensional space of bound states are projected away. Our primary assumptions are that $V(t,x)$ be periodic and belong to the scaling-invariant space $L^{n/2}_xL^\infty_t$ and that each of the bound states is an eigenfunction of sufficient decay and/or regularity. If we further assume that V is real-valued with polynomial pointwise decay and some smoothness with respect to t, then only the bound states at $\lambda\in\mathbb{Z}$ are a concern, and only in dimensions $n\le 6$. Improvements of this type are discussed immediately following our statement of the main theorem.

Theorem 1. *Let $V(t,x)$ be a time-periodic function on $\mathbb{R}^{1+n}$, $n\geq 3$, satisfying $V(t+2\pi,x)=V(t,x)$ at almost every t,x and belonging to the class $L^{n/2}_xL^\infty_t$. Suppose that K and $\overline{K}$ have no resonances along the real axis, and that their behavior at each eigenvalue $\lambda\in\mathbb{C}$ satisfies the conditions*

- (C1) *$e^{-i\lambda t}N_\lambda$ and $e^{-i\bar\lambda t}\tilde N_\lambda$ are both contained in $L^{\frac{2n}{n+2}}(\mathbb{R}^n;L^2(\mathbb{T}))+\langle x\rangle^{-1}L^2(\mathbb{T}\times\mathbb{R}^n)$,*
- (C2) *X_λ and $\tilde X_\lambda$ are subspaces of $\langle x\rangle^{-1}L^2(\mathbb{R}^n)+W^{1,\frac{2n}{n+2}}(\mathbb{R}^n)$,*
- (C3) *The L^2-orthogonal projection of X_λ onto $\tilde X_\lambda$ is bijective.*

Under these assumptions, there exist at most finitely many eigenvalues of K, $\overline{K}$ in the strip $\mathbb{C}/\mathbb{Z}$, counted with multiplicity. Furthermore, the initial value problem for the Schrödinger equation

$$\begin{cases}(i\partial_t-\Delta_x+V(t,x))\quad u(t,x)=0, & x\in\mathbb{R}^n,t\in\mathbb{R}\\ \qquad\qquad\qquad\qquad u(0,x)=f(x), & x\in\mathbb{R}^n\end{cases}\tag{5}$$

possesses a unique weak solution in the Strichartz space $L^2_tL^{2n/(n-2)}_x$, satisfying

$$\|u\|_{L^2_tL^{2n/(n-2)}_x}\lesssim\|f\|_2\tag{6}$$

for all initial data f in the L^2-orthogonal complement of $\tilde X=\oplus_\lambda\tilde X_\lambda$.

Remark 1. If V is real-valued, then each eigenvalue λ is also real. Since $\overline{K}=K$, it also follows that $\tilde N_\lambda=N_\lambda$ and $\tilde X_\lambda=X_\lambda$, making the condition (C3) unnecessary.

Remark 2. The unweighted portion of condition (C2) is not sharp in terms of the number of derivatives required. Lemma 14 and its supporting propositions construct a family of lower-regularity spaces which may be used in place of $W^{1,2n/(n+2)}(\mathbb{R}^n)$.

Corollary 2. *Suppose that the time-periodic potential $V(t,x)$ is real valued and satisfies the bound*

$$\sup_{x\in\mathbb{R}^n} \langle x\rangle^\beta \|V(\,\cdot\,,x)\|_{H^s(\mathbb{T})} < \infty \tag{7}$$

for some $\beta > 2$ and $s > \frac{1}{2}$. The Strichartz estimates in Theorem 1 are valid provided that $\lambda \in \mathbb{Z}$ is not a resonance, and any eigenvectors at $\lambda \in \mathbb{Z}$ belong to $\langle x\rangle^{-1}L^2$.

In dimensions $n \geq 7$, Theorem 1 is valid for all real-valued potentials satisfying (7). *No further conditions are necessary.*

Proof. Due to the self-adjointness of K, there are no eigenvalues off of the real axis. Following the proof of Lemma 2.8 in [3], resonances can only exist at $\lambda \in \mathbb{Z}$, and if λ is not an integer then the eigenfunctions additionally satisfy $\phi_\lambda \in \langle x\rangle^{-N}H^s(\mathbb{T}; L^2(\mathbb{R}^n))$. The main ingredients are an Agmon-type bootstrapping argument (based on [1]) and the fact that multiplication by a function in $H^s(\mathbb{T})$ preserves the $H^{s-1/2}(\mathbb{T})$ norm.

When $\lambda \in \mathbb{Z}$, the bootstrapping process produces only as much spatial decay for ϕ_λ as is present in the Green's function of the Laplacian. In general, the Green's function belongs to $\langle x\rangle^\sigma L^2$ (aside from the local singularity) for all $\sigma > \frac{4-n}{2}$. For $n \geq 7$, the desired value $\sigma = -1$ is part of this range. □

Corollary 3. *Let $V(x) \in L^{\frac{n}{2}}(\mathbb{R}^n)$ be a complex valued time-independent potential. The Strichartz estimates in Theorem 1 are valid provided the equation*

$$(-\Delta + V - \lambda)\phi = 0$$

has no solutions $\phi \in L^{2n/(n-2)}(\mathbb{R}^n)$ for any $\lambda \in [0,\infty) \subset \mathbb{C}$, and condition (C3) is satisfied at every eigenvalue.

Proof. Similar to the preceding corollary, the point is that all of the permitted bound states $\phi_\lambda = e^{i\lambda t}\Phi(x)$ are necessarily eigenfunctions that decay rapidly enough to satisfy condition (C2). In this case the bootstrapping is based on the relation $\Phi = -(I + (-\Delta - \lambda)^{-1}V)\Phi$ Since $\lambda \notin [0,\infty)$, the resolvent of the Laplacian is bounded from every $L^p(\mathbb{R}^n)$ to itself, $1 \leq p \leq \infty$.

Starting with $\Phi \in L^{2n/(n-2)}$, one iteration brings the exponent down to $\Phi \in L^{2n/(n+2)}$. Furthermore it is quite easy to take two derivatives: $\Delta\Phi = V\Phi - \lambda\Phi \in L^{2n/(n+2)}$. Thus $\phi \in W^{1,2n/(n+2)}$ as is required by (C2). □

Corollary 4. *If $V \in L^{\frac{n}{2}}(\mathbb{R}^n)$ is a real-valued potential, then* (6) *holds provided the Schrödinger operator $H = -\Delta + V$ does not have a resonance or an eigenvalue at zero energy.*

Proof. In this case the spectrum of H is purely absolutely continuous on the interval $(0,\infty)$ due to the combined results of [4] and [6]. According to the previous corollary, the only remaining spectral point of concern is the behavior of H at $\lambda = 0$. The additional assumption ensures that zero is a regular point of the spectrum as well. □

Although Theorem 1 is presented as a perturbation of the Strichartz inequality (1), which in turn is based on dispersive estimates for the free Schrödinger evolution, we do not attempt to prove comparable dispersive estimates for H. This is partly a matter of convenience, as the study of time-asymptotics for Floquet operators (as in [3]) presents its own set of technical challenges. More importantly, the conditions for Theorem 1 include numerous potentials for which the corresponding dispersive estimate are known to fail.

The discrepancy is especially apparent in dimensions $n \geq 4$. No pointwise or L^p condition on the potential is sufficient by itself to imply an $L^1 \to L^\infty$ dispersive bound [5]. Either some extra regularity of V is needed, as in [10], or one must expect to suffer a loss of derivatives in the solution [22]. On the other hand, Strichartz estimates were proven in [17] for time-independent potentials satisfying $|V(x)| \lesssim \langle x\rangle^{-2-\varepsilon}$. In this work the authors used Kato smoothing estimates as the intermediary step in place of the nonexistent dispersive bounds. Corollary 4 represents a modest extension of this work.

We wish to emphasize one additional feature of Theorem 1 that appears to be unique in the literature: the treatment of eigenvalues depends only on the nature of the associated eigenfunction, not on its location relative to the spectrum of K. While it may be true in certain applications that threshold eigenvalues and/or resonances enjoy distinct properties from those embedded in the continuous spectrum or from isolated points, the criteria (C1)-(C3) apply equally in all these cases.

The proof of Theorem 1 is based on a direct application of Duhamel's formula. We consider the behavior of solutions when $t \geq 0$; the reasoning for $t \leq 0$ is identical. Let U^+ denote the forward propagator of the free Schrödinger equation, that is

$$U^+g(t,x) := \int_{s<t} e^{-i(t-s)\Delta} g(s,x)\,ds.$$

We will also allow U^+ to act on functions of x alone by the definition $U^+g(t,x) := \chi_{[0,\infty)}(t)e^{-it\Delta}g(x)$. The adjoint of U^+ in both cases is the backward propagator U^-. The full range of mapping properties of U^+ are established in [12]; of particular concern are the bounds

$$\begin{aligned} U^+ &: L^2_t L^{2n/(n+2)}_x \to L^2_t L^{2n/(n-2)}_x \cap C(\mathbb{R}; L^2_x) \\ U^+ &: \quad L^1_t L^2_x \quad\;\; \to L^2_t L^{2n/(n-2)}_x \cap C(\mathbb{R}; L^2_x) \\ U^+ &: \quad L^2_x \qquad\;\; \to L^2_t L^{2n/(n-2)}_x \cap C([0,\infty); L^2_x) \end{aligned} \tag{8}$$

Every weak solution of (5) on the time interval $[0,\infty)$ must solve the functional equation $u(t,x) = U^+f(t,x) + iU^+Vu(t,x)$. This leads to the formal solution

$$u = (I - iU^+V)^{-1}U^+f$$

where the inverse is taken among bounded operators on $L^2_t L^{2n/(n-2)}_x$. In order to work in the setting of $L^2_tL^2_x$, factorize $V = ZW$, with $Z, W \in L^\infty_t L^n_x$ and write

$$u = U^+f \; + \; iU^+Z(I - iWU^+Z)^{-1}WU^+f. \tag{9}$$

In the event that $I - iWU^+Z$ is invertible as an operator on $L^2([0,\infty); L^2(\mathbb{R}^n))$, one concludes that (6) holds for all $f \in L^2$ which implies an absence of bound states. This occurs for all $V \in L^\infty_t L^{n/2}_x$ of sufficiently small norm. In every other case, the challenge is to find a condition on f so that WU^+f belongs to the domain of the unbounded operator $(I - iWU^+Z)^{-1}$.

Much of our analysis is done with respect to the Fourier transform of the time variable, in deference to the fact that U^+ and V preserve the space of functions satisfying $g(t+2\pi, x) = e^{2\pi i\lambda}g(t,x)$ for each $\lambda \in [0,1]$. We show that $I - iWU^+Z$ is a compact perturbation of the identity on each of these spaces. The Fredholm Alternative then equates invertibility with the absence of eigenvalues or resonances at λ.

Common sense suggests that the singularities caused by a particular bound state ϕ can be avoided by requiring the initial data f to be orthogonal to Φ. Even in the time-independent case, however, eigenvalues at zero energy are known to disturb dispersive estimates after such a projection. This phenomenon is first identified in [9] and described in more detail in [2]. A full asymptotic expansion for Floquet solutions has recently been computed in three dimensions in [3]. We note that the intuitive suggestion above is also incorrect when the Schrödinger propagation is not unitary (i.e. when K has complex values). The projection employed in Theorem 1 is actually orthogonal to a function $\tilde{\Phi} \in \tilde{X}$ rather than Φ.

In order to determine the success of a projection, we closely examine the behavior of $(I - iWU^+Z)^{-1}$ for all λ in the neighborhood of an eigenvalue and assess whether it is compatible with the input WU^+f. The resulting eigenvalue condition appears in the form of a discrete-time Kato smoothing bound. This last computation, parts of which are adapted from [13] and [19], may be of independent interest.

2. Resolvents, Compactness, and Continuity

In this section we aim to find spaces on which $I - iWU^+Z$ is a compact perturbation of the identity. For each $\lambda \in \mathbb{C}$, define

$$Y_\lambda = \{g \in L_t^{2,loc}L_x^2 : g(t + 2\pi, x) = e^{2\pi i\lambda}g(t, x)\}.$$

Functions $g \in Y_\lambda$ are naturally associated with the periodic $e^{-it\lambda}g \in L^2(\mathbb{T} \times \mathbb{R}^n)$, and we use this identification to define a Hilbert space norm on Y_λ.

For each $\lambda \in \mathbb{R}/\mathbb{Z}$, there exists a "projection" P_λ from $L_t^2L_x^2$ onto Y_λ given by

$$P_\lambda g(t, x) = \sum_{m\in\mathbb{Z}} e^{-2\pi i\lambda m}g(t + 2\pi m, x)$$

The family of operators P_λ can be understood as a partial Fourier transform in the time variable, acting on the space $L_t^2L_x^2 \cong \ell_m^2(L^2([2\pi m, 2\pi(m + 1)]; L^2(\mathbb{R}^n)))$. For example the Plancherel identity is expressed as

$$\int_0^1 \|P_\lambda g\|_{Y_\lambda}^2 \, d\lambda = \|g\|_{L_t^2L_x^2}^2 \tag{10}$$

For functions g with support in the halfline $t \in [0, \infty)$, the definition of $P_\lambda g$ extends to the strip $\lambda = \lambda' + i\mu$, $\mu \le 0$, $\lambda' \in \mathbb{R}/\mathbb{Z}$ with the value $e^{-\mu t}P_{\lambda'}(e^{\mu t}g)$. The Plancherel identity in this case becomes

$$\int_0^1 \|P_{\lambda'+i\mu}g\|_{Y_{\lambda'+i\mu}}^2 \, d\lambda' = \int_0^1 \|P_{\lambda'}e^{\mu t}g\|_{Y_{\lambda'}}^2 \, d\lambda' = \|e^{\mu t}g\|_{L_t^2L_x^2}^2$$

On the Fourier side with respect to time, P_λ has a very clear interpretation. Let $\hat{g}(\tau, x)$ be the partial Fourier transform of g. Then $(P_\lambda g)\hat{}$ is the restriction of $\hat{g}$ to the planes $\{\tau \in \lambda + \mathbb{Z}\}$. If g is supported on $\{t \ge 0\}$ then $\hat{g}$ has an analytic extension to the lower halfplane, making the restrictions to $\{\tau \in \lambda' + i\mu + \mathbb{Z}\}$ well-defined. Clearly P_λ commutes with pointwise multiplication (in (t, x)) by any 2π-periodic function.

The action of U^+ in this setting is also easy to characterize. Since U^+ convolves functions in the time variable with the integral kernel $K(t) = \lim_{\varepsilon\downarrow 0} e^{-it\Delta-\varepsilon t}\chi_{t\ge 0}$, on the Fourier side it performs pointwise (in τ) "multiplication" by $\hat{K}(\tau) = \lim_{\varepsilon\downarrow 0} i(-\Delta - (\tau - i\varepsilon))^{-1}$. Using the notation of resolvents,

$$(U^+g)\hat{}(\tau, x) = iR^-(\tau)\hat{g}(\tau, x) \tag{11}$$

where $R^-(\tau)$ represents the branch of the resolvent of $-\Delta$ which continues analytically to $\{\mathrm{Im}(\tau) \le 0\}$. This shows that U^+ also commutes with each of the projections P_λ. Once again, if $\mathrm{supp}_t g \subset [0, \infty)$, the identity (11) remains valid for all τ in the lower halfplane, with the understanding that

$$\hat{g}(\tau, x) = (e^{\mathrm{Im}(\tau)}g)\hat{}(\mathrm{Re}(\tau), x).$$

Therefore the operator $I - iWU^+Z$ admits a restriction to each Y_λ, $\mathrm{Im}(\lambda) < 0$, and most importantly,

$$\|e^{\mu t}(I - iWU^+Z)^{-1}WU^+f\|^2_{L^2_tL^2_x} = \int_0^1 \|(I - iWU^+Z)^{-1}P_{\lambda'+i\mu}WU^+f\|^2_{Y_{\lambda'+i\mu}}\,d\lambda' \tag{12}$$

The proof of Theorem 1 will be complete once we bound this quantity in terms of the $L^2(\mathbb{R}^n)$ norm of f, uniformly over $\mu \le 0$.

The particular factorization we choose for $V(t,x)$ is to let $W(t,x) = w(x) = (\|V(\,\cdot\,,x)\|_\infty)^{\frac{1}{2}}$. By our assumptions, $w \in L^n(\mathbb{R}^n)$. The remaining factor can be decomposed as $w(x)z(t,x)$, with w the same function as above and $z(t,x)$ periodic and bounded almost everywhere by 1. Multiplication by z is a bounded operator of unit norm on Y_λ, so compactness of the operator wU^+wz follows directly from compactness of wU^+w.

Proposition 5. *Given any function $w \in L^n$, the collection $\{wR^-(\tau)w : \mathrm{Im}(\tau) \le 0\}$ forms a uniformly continuous family of compact operators on $L^2(\mathbb{R}^n)$ with norm decreasing to zero as $|\tau| \to \infty$.*

Proof. This is a compilation of well-known resolvent estimates, primarily the fact (proved in [13]) that $R^-(\tau)$ are uniformly bounded as operators from $L^{\frac{2n}{n+2}}$ to $L^{\frac{2n}{n-2}}$. All of the desired properties – compactness, continuity, and norm decay – are preserved if w is approximated in L^n by a sequence of bounded compactly supported functions w^ε.

For compactness, observe that $(-\Delta+1)R^-(\tau)w^\varepsilon g = w^\varepsilon g + (\tau+1)R^-(\tau)w^\varepsilon g \in L^2(\mathbb{R}^n)$. Within any ball of finite radius R, the Sobolev space H^2 embeds compactly inside $L^{\frac{2n}{n-2}}$. If this ball is much larger than the support of w^ε, then there is a pointwise bound

$$|R^-(\tau)w^\varepsilon g(x)| \lesssim |\tau|^{\frac{n-3}{4}}\|g\|_2\|w^\varepsilon\|_2|x|^{\frac{1-n}{2}}$$

outside of the ball. Allowing $R \to \infty$ expresses $w^\varepsilon R^-(\tau)w^\varepsilon$ as a norm-limit of compact operators on L^2.

For continuity, recall that the integration kernel of $R^-(\tau)$ is $|x-y|^{2-n}F(\tau^{\frac{1}{2}}|x|)$, where F can be expressed explicitly in terms of Hankel functions. In dimensions $n \ge 3$ it satisfies the pointwise bounds

$$|F(z)|, |F'(z)| \lesssim \langle z\rangle^{(n-3)/2}.$$

Using the mean value theorem, if $|\tau - \sigma| < \frac{1}{2}|\tau|$ then

$$|R^-(\tau) - R^-(\sigma)(x,y)| \lesssim \begin{cases} |\tau^{\frac{1}{2}} - \sigma^{\frac{1}{2}}|\,|x-y|^{3-n}, & \text{if } |x-y| < |\tau|^{-\frac{1}{2}} \\ |\tau|^{\frac{n-3}{4}}|\tau^{\frac{1}{2}} - \sigma^{\frac{1}{2}}||x-y|^{\frac{3-n}{2}}, & \text{if } |\tau|^{-\frac{1}{2}} < |x-y| < |\tau^{\frac{1}{2}} - \sigma^{\frac{1}{2}}|^{-1} \end{cases}$$

The case where $|x-y|$ is large is unimportant because w^ε has compact support. The Schur test then shows that $w^\varepsilon R^-(\tau)w^\varepsilon$ is continuous with respect to τ.

Finally, decay as $|\tau| \to \infty$ is an immediate consequence of another resolvent bound from [13], namely that $|\tau|^{\frac{1}{n+1}}R^-(\tau)$ is a uniformly bounded family of maps from $L^{\frac{2n+2}{n+3}}$ to $L^{\frac{2n+2}{n-1}}$. The combination of continuity and decay at infinity immediately implies uniform continuity. □

Corollary 6. *Given any $w \in L^n(\mathbb{R}^n)$, the collection $\{e^{-i\lambda t}wU^+we^{i\lambda t} : \mathrm{Im}(\lambda) \le 0\}$ is a continuous family (with respect to λ) of compact operators on $L^2(\mathbb{T}\times\mathbb{R}^n)$, with norm decreasing to zero as $\mathrm{Im}(\lambda) \to -\infty$.*

The same is also true for the family of operators $e^{-i\lambda t}wU^+wze^{i\lambda t}$ for any bounded 2π-periodic function z.

Proof. For every choice of λ in the lower halfplane, the Fourier series coefficients of $e^{-i\lambda t}wU^+we^{+i\lambda t}g$ are precisely $\{wR^-(\lambda+k)w\hat{g}(k,x) : k \in \mathbb{Z}\}$. At each k this is a compact operator on $\mathbb{R}^n$, and the norms decrease as $|k| \to \infty$. It follows that their collective action on $\ell^2(k; L^2(\mathbb{R}^n))$ is a compact operator with norm $\sup_k \|wR^-(\lambda+k)w\|$. As $\mathrm{Im}(\lambda) \to -\infty$, the norm is bounded by

$$\sup_{|\tau|>|\mathrm{Im}(\lambda)|} \|wR^-(\tau)w\|$$

which decreases to zero.

Given two numbers λ_1 and λ_2, the norm difference of their associated operators is

$$\sup_k \|w(R^-(\lambda_1+k) - R^-(\lambda_2+k))w\|.$$

The uniform continuity assertion in Proposition 5 takes this to zero in the limit $\lambda_2 \to \lambda_1$.

Neither the compactness nor continuity properties of $e^{-i\lambda t}wU^+we^{i\lambda t}$ are affected by composition with the bounded operator $e^{-i\lambda t}ze^{i\lambda t}$. $\square$

3. Estimates for Inverse Operators

There are two main elements in the expression (12), an inverse operator $(I - iwU^+wz)^{-1}$ and a series of functions $P_\lambda wU^+f \in Y_\lambda$. In this section we prove uniform bounds for $(I - iwU^+wz)^{-1}$ on Y_λ where possible, and describe the singularities that occur as λ approaches the spectrum of K.

The spaces Y_λ are a natural setting for working with bound states, especially those bound states that grow exponentially over time. When we wish to vary λ as a parameter, however, a unified approach based on $L^2(\mathbb{T}\times\mathbb{R}^n)$ is preferred. Define the family of operators

$$T(\lambda) = I - ie^{-i\lambda t}wU^+wze^{i\lambda t} = I - iw(e^{-i\lambda t}U^+e^{i\lambda t})wz$$

acting on $L^2(\mathbb{T}\times\mathbb{R}^n)$. The kernel of $T(\lambda)$ provides valuable information about the spectrum of K, thanks to the intertwining relations

$$\begin{aligned}(K-\lambda)\big(e^{-i\lambda t}U^+e^{i\lambda t}wz\big) &= iwzT(\lambda),\\ \big(we^{-i\lambda t}U^+e^{i\lambda t}\big)(K-\lambda) &= iT(\lambda)w.\end{aligned}$$

Each element $g \in \ker T(\lambda)$ corresponds to a bound state $\phi = U^+wze^{i\lambda t}g$. Proposition 7 below shows that $e^{-i\lambda t}\phi$ is a true eigenfunction of K in $L^2(\mathbb{T}\times\mathbb{R}^n)$ if $\mathrm{Im}(\lambda) < 0$. Additional tools are available ([3], [23]) if V is real-valued and $\lambda \notin \mathbb{Z}$. In any of the remaining cases it is possible that the spatial decay of ϕ fails to be square-integrable. We say that K has a *resonance* at λ when this occurs; that is, when there exists some $g \in \ker T(\lambda)$ for which $\phi = U^+wze^{i\lambda t}g$ does not belong to $L^2(\mathbb{T}\times\mathbb{R}^n)$.

Note that $T(\lambda+1)$ is a unitary conjugate of $T(\lambda)$, so one only needs to check the invertibility of $T(\lambda)$ inside the strip

$$\Omega^- = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \in [0,1),\ \mathrm{Im}(\lambda) \le 0\}.$$

The set $\Omega^- \subset \mathbb{C}$ is a fundamental domain for the lower halfplane modulo the integers, and will always be given the quotient topology. We make some remarks about the size and differentiability properties of $e^{-\lambda t}U^+e^{i\lambda}$ for future reference.

Proposition 7. *For each λ with $\mathrm{Im}(\lambda) < 0$, the operator $e^{-i\lambda t}U^+e^{i\lambda t}$ is subject to the following estimates.*

$$\|e^{-i\lambda t}U^+e^{i\lambda t}g\|_{L^2(\mathbb{T}\times\mathbb{R}^n)} \lesssim |\mathrm{Im}(\lambda)|^{-1}\|g\|_{L^2(\mathbb{T}\times\mathbb{R}^n)} \tag{13}$$

$$\|e^{-i\lambda t}U^+e^{i\lambda t}g\|_{L^2(\mathbb{T}\times\mathbb{R}^n)} \lesssim |\mathrm{Im}(\lambda)|^{-\frac{1}{2}}\|g\|_{L^2(\mathbb{T};L^{2n/(n+2)}(\mathbb{R}^n))} \tag{14}$$

Given two values λ_1, λ_2, *the difference can be expressed as*

$$e^{-i\lambda_1 t}U^+e^{i\lambda_1 t} - e^{-i\lambda_2 t}U^+e^{i\lambda_2 t} = -i(\lambda_1-\lambda_2)\big(e^{-i\lambda_1 t}U^+e^{i\lambda_1 t}\big)\big(e^{-i\lambda_2 t}U^+e^{i\lambda_2 t}\big). \tag{15}$$

Therefore the family of operators $e^{-i\lambda t}U^+e^{i\lambda t}$ *possesses the holomorphic derivative*

$$\frac{d}{d\lambda}\big[e^{-i\lambda t}U^+e^{i\lambda t}\big] = -ie^{-i\lambda t}(U^+)^2e^{i\lambda t}. \tag{16}$$

over the domain $\mathrm{Im}(\lambda) < 0$.

Proof. The estimates (13) and (14) both exploit the facts that $U^+g(t,x)$ depends only on $\chi_{s<t}g(s,x)$, and that $e^{i\lambda t}$ decays exponentially as $t \to -\infty$. To be precise, if $g \in L^2(\mathbb{T}\times\mathbb{R}^n)$, then the $L^1_tL^2_x$ norm of $\chi_{(-\infty,t)}e^{i\lambda s}g$ is bounded by $|\mathrm{Im}(\lambda)|^{-1}e^{-\mathrm{Im}(\lambda)t}$. Similarly, if $g \in L^2(\mathbb{T}; L^{\frac{2n}{n+2}}(\mathbb{R}^n))$ then the $L^2_tL^{2n/(n+2)}_x$ norm of $\chi_{(-\infty,t)}e^{i\lambda s}g$ is bounded by $|\mathrm{Im}(\lambda)|^{-\frac12}e^{-\mathrm{Im}(\lambda)t}$. In either case the propagator estimates (8) complete the argument.

The difference and derivative formulas can be verified directly, or by expressing U^+ according to its Fourier representation (11). The equivalent identities for resolvents are $R^-(\lambda_1) - R^-(\lambda_2) = (\lambda_1-\lambda_2)R^-(\lambda_1)R^-(\lambda_2)$ and $\frac{d}{d\lambda}R^-(\lambda) = R^-(\lambda)^2$. □

Corollary 6 shows that each $T(\lambda)$, $\mathrm{Im}(\lambda) \le 0$, is a compact perturbation of the identity. Furthermore, $\|T(\lambda)^{-1}\|$ varies continuously over its domain of definition, is periodic with respect to translation by $\mathbb{Z}$, and is bounded by 2 once the imaginary part of λ is sufficiently negative. If $T(\lambda)^{-1}$ existed everywhere, this would suffice to bound its norm uniformly in λ. By the Fredholm Alternative, only an eigenvalue or resonance at λ can prevent $T(\lambda)$ from being invertible. We examine the structure of these singularities in the following two lemmas.

Lemma 8. *Let* $w \in L^n(\mathbb{R}^n)$ *and* $z \in L^\infty(\mathbb{T}\times\mathbb{R}^n)$. *Suppose the operator* $T(\lambda_0)$ *fails to be invertible for some* $\lambda_0 \in \mathbb{C}$ *with* $\mathrm{Im}(\lambda_0) < 0$. *Then the solution spaces* $N_{\lambda_0} \subset Y_{\lambda_0}$ *and* $\tilde N_{\lambda_0} \subset Y_{\bar\lambda_0}$ *are both nontrivial and finite dimensional. The set of their initial values,* X_{λ_0} *and* $\tilde X_{\lambda_0}$, *are well defined finite dimensional subspaces of* $L^2(\mathbb{R}^n)$.

If the orthogonal projection from X_{λ_0} *onto* $\tilde X_{\lambda_0}$ *is bijective, then* $T(\lambda)$ *is invertible for every other* λ *in a neighborhood of* λ_0. *More precisely,*

$$\|T(\lambda)^{-1}(h_1+h_2)\|_{L^2(\mathbb{T}\times\mathbb{R}^n)} \le C(w,z,\lambda_0)(|\lambda-\lambda_0|^{-1}\|h_1\| + \|h_2\|) \tag{17}$$

where $h_1 = e^{-i\bar\lambda_0 t}\overline{zw}\tilde\phi$, $\tilde\phi \in \tilde N_{\lambda_0}$, *and* h_2 *belongs to the* L^2*-orthogonal complement of* $e^{-i\bar\lambda_0 t}\overline{zw}\tilde N_{\lambda_0}$.

Proof. The operator $T(\lambda_0)$ is a compact perturbation of the identity, and by assumption it is not invertible. The Fredholm Alternative asserts that $T(\lambda_0)$ has a finite dimensional kernel, a cokernel of the same dimension, and that it is an invertible map between their respective orthogonal complements.

Every element $g \in L^2(\mathbb{T}\times\mathbb{R}^n)$ in the kernel of $T(\lambda_0)$ is associated to a prospective eigenfunction $e^{-i\lambda_0 t}\phi$ by the relations $\phi = U^+wze^{i\lambda_0 t}g$ and $g = ie^{-i\lambda_0 t}w\phi$. Note that $wzg \in L^2(\mathbb{T}; L^{\frac{2n}{n+2}}(\mathbb{R}^n))$, so the mapping estimate (14) implies that $e^{-\lambda_0 t}\phi$ belongs to $L^2(\mathbb{T}\times\mathbb{R}^n)$. That makes $e^{-i\lambda_0 t}\phi$ an eigenfunction of K, and $\phi \in N_{\lambda_0}$ by definition. It follows immediately that

$$\ker T(\lambda_0) = e^{-i\lambda_0 t}wN_{\lambda_0}.$$

In general, a function $\phi \in L^2(\mathbb{T}\times\mathbb{R}^n)$ should not have a meaningful initial value $\Phi(x) = \phi(0,x)$. On the other hand, ϕ solves the inhomogeneous Schrödinger equation

$$(i\partial_t - \Delta)\phi = -V\phi \in L^{2,loc}_tL^{2n/(n+2)}_x$$

from which Duhamel's formula (averaged over all starting times $s \in [-2\pi, 0]$) yields

$$\begin{aligned} \phi(0,x) &= (2\pi)^{-1} \int_{-2\pi}^{0} \left(e^{i\Delta s}\phi(s,x) + i\int_{s}^{0} e^{i\Delta r} V\phi(r,x)\,dr \right) ds \\ &= (2\pi)^{-1} \left(\int_{-2\pi}^{0} e^{i\Delta s}\phi(s,x)\,ds + i\int_{-2\pi}^{0} e^{i\Delta r}(r+2\pi)V\psi(r,x)\,dr \right) \end{aligned} \tag{18}$$

The first integral evaluates to a function in $L^2(\mathbb{R}^n)$ because $e^{i\Delta s}$ is unitary and $\phi \in L_t^{1,loc}L_x^2$. The second integral does likewise, via the dual statement of (8).

Remark 3. Because $\ker T(\lambda_0)$ is a finite dimensional space, the norms of $g \in \ker T(\lambda_0) \subset L^2(\mathbb{T}\times\mathbb{R}^n)$ and $\phi \in N_{\lambda_0} \subset Y_{\lambda_0}$ are equivalent. These norms are also equivalent to the norm of $\Phi \in X_{\lambda_0} \subset L^2(\mathbb{R}^n)$ for the same reason.

The image of $T(\lambda_0)$ consists of all functions orthogonal to the kernel of its adjoint, namely

$$T(\lambda_0)^* = I + ie^{-i\bar{\lambda}_0 t}\overline{zw}U^-\overline{w}e^{i\bar{\lambda}_0 t}.$$

Every element $\tilde{g}$ in the kernel of $T(\lambda_0)^*$ is associated to an eigenfunction $e^{-i\bar{\lambda}_0 t}\tilde{\phi} \in \tilde{N}_{\lambda_0}$ of $\tilde{K}$ by the relations $\tilde{\phi} = U^-\overline{w}e^{i\bar{\lambda}_0 t}\tilde{g}$ and $\tilde{g} = -ie^{-i\bar{\lambda}_0 t}\overline{zw}\tilde{\phi}$. The argument which places $\tilde{\phi}$ in $\tilde{N}_{\lambda_0}$ and establishes the existence of $\tilde{\Phi}$ is the same as the one for ϕ above. We can now express the image of $T(\lambda_0)$ as

$$\operatorname{image} T(\lambda_0) = \{g \in L^2(\mathbb{T}\times\mathbb{R}_n) : \langle g,\, e^{-i\bar{\lambda}_0 t}\overline{zw}\tilde{\phi}\rangle = 0,\ \tilde{\phi} \in \tilde{N}_{\lambda_0}\}. \tag{19}$$

and the cokernel of $T(\lambda_0)$ as the subspace $e^{-i\bar{\lambda}_0 t}\overline{zw}\tilde{N}_{\lambda_0}$. Our next goal is to find an inverse image for each $h_1 \in \operatorname{coker} T(\lambda_0)$ with respect to the map $T(\lambda)$, $\lambda \neq \lambda_0$.

At first, let g and h be any two functions in $L^2(\mathbb{T}\times\mathbb{R}^n)$. By Proposition 7, the scalar restriction of $T(\lambda)$ described by

$$a_{g,h}(\lambda) = \langle T(\lambda)g, h\rangle$$

is a holomorphic function in the lower halfplane, with derivative

$$|a'_{g,h}(\lambda)| = |\langle we^{-i\lambda t}(U^+)^2 e^{i\lambda t}wzg, h\rangle| \lesssim |\mathrm{Im}(\lambda)|^{-1}\|g\|\|h\|. \tag{20}$$

Now fix a particular $h_1 = e^{-i\bar{\lambda}_0 t}\overline{zw}\tilde{\phi}_1$ with $\tilde{\phi}_1 \in \tilde{N}_{\lambda_0}$ of approximately unit norm, and suppose that $g = e^{-i\lambda_0 t}w\phi$, $\phi \in N_{\lambda_0}$. By construction $a_{g,h_1}(\lambda_0) = 0$ and

$$\begin{aligned} a'_{g,h_1}(\lambda_0) = -i\langle U^+V\phi, U^-\overline{V}\tilde{\phi}_1\rangle &= i\langle \phi, \tilde{\phi}_1\rangle \\ &= i\int_0^{2\pi} \langle \phi(t,\cdot), \tilde{\phi}_1(t,\cdot)\rangle_{L_x^2}\,dt \\ &= 2\pi i\langle \Phi, \tilde{\Phi}_1\rangle_{L_x^2} \end{aligned}$$

The last line in this chain of equations is a non-selfadjoint version of the unitarity of propagation. The key property is that $\tilde{\phi}_1$ solves a Schrödinger equation with the potential $\overline{V}$.

If the orthogonal projection of X_{λ_0} onto $\tilde{X}_{\lambda_0}$ is bijective, then there exists a unique unit vector $\Phi_1 \in X_{\lambda_0}$ such that

$$|\langle \Phi_1, \tilde{\Phi}_1\rangle| \gtrsim 1$$

while $\langle \Phi_1, \tilde{\Phi}'\rangle = 0$ for all $\tilde{\Phi}' \in \tilde{X}_{\lambda_0}$ orthogonal to $\tilde{\Phi}_1$.

For the associated function $g_1 = e^{-i\lambda_0 t}w\phi_1$, this provides the lower bound

$$|a_{g_1,h_1}(\lambda)| \gtrsim |\lambda - \lambda_0|$$

while at the same time

$$|a_{g_1,h'}(\lambda)| \lesssim |\lambda - \lambda_0|^2$$

for all unit vectors $h' \in \operatorname{coker} T(\lambda_0)$ orthogonal to h_1.

Returning to the derivative estimate (20), we observe that

$$\Big\| T(\lambda) g_1 \big|_{\operatorname{image} T(\lambda_0)} \Big\| \lesssim |\lambda - \lambda_0|.$$

Switching the roles of g and h gives the bound

$$|\langle T(\lambda) g, h_1 + h'\rangle| \lesssim |\lambda - \lambda_0| \|g\|$$

for every $g \in L^2(\mathbb{T} \times \mathbb{R}^n)$ and any unit vector $h_1 + h' \in \operatorname{coker} T(\lambda_0)$.

Recall that $T(\lambda_0)$ is an invertible map between its co-image and image. By continuity, the restrictions of $T(\lambda)$ to these spaces are uniformly invertible within a small neighborhood of λ_0. Therefore, given g_1 as constructed above there exists a unique element $g'(\lambda) \in \operatorname{coimage} T(\lambda_0)$ so that $T(\lambda)(g_1 + g'(\lambda)) \in \operatorname{coker} T(\lambda_0)$. The norm of $g'(\lambda)$ is of order $|\lambda - \lambda_0|$.

Let $g_{h_1}(\lambda) = g_1 + g'(\lambda)$. This is a vector of approximately unit norm that satisfies both

$$T(\lambda) g_{h_1}(\lambda) = C_{h_1}(\lambda - \lambda_0) h_1 + \mathcal{O}(|\lambda - \lambda_0|^2)$$

and also $T(\lambda) g_{h_1}(\lambda) \in \operatorname{coker} T(\lambda_0)$. Choose any basis $\{h_j\}$ for $\operatorname{coker} T(\lambda_0)$. The desired inverse image $T(\lambda)^{-1} h_1$ will be a linear combination (with bounded coefficients) of the functions $(\lambda - \lambda_0)^{-1} g_{h_j}(\lambda)$.

For any unit vector $h_2 \in \operatorname{image} T(\lambda_0)$, there exists a unique $g_{h_2}(\lambda)$ in the co-image of $T(\lambda_0)$ so that

$$T(\lambda) g_{h_2}(\lambda) - h_2 = h' \in \operatorname{coker} T(\lambda_0).$$

The norms of g_{h_2} and h' will be of order 1 and $|\lambda - \lambda_0|$, respectively. Thus $T(\lambda)^{-1} h'$, and finally $T(\lambda)^{-1} h_2 = g_{h_2} + T(\lambda)^{-1} h'$ will both be of bounded norm. □

The fact that $\operatorname{Im}(\lambda_0) < 0$ only played a role to the extent that we relied upon the propagator estimates of Proposition 7. If $\lambda_0 \in \mathbb{R}$ instead, these can be replaced with a weaker set of bounds based on the mapping properties of $R^-(\lambda)$ along the Real axis.

Proposition 9. *For each $\lambda \in \mathbb{C}$, $\operatorname{Im}(\lambda) \le 0$, the operator $e^{-i\lambda t} U^+ e^{i\lambda t}$ is subject to the following estimates.*

$$\|e^{-i\lambda t} U^+ e^{i\lambda t} g\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n; L^2(\mathbb{T}))} \lesssim \|g\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n; L^2(\mathbb{T}))} \tag{21}$$

$$\|\langle x\rangle^{-1} e^{-i\lambda t} U^+ e^{i\lambda t} g\|_{L^2(\mathbb{R}^n \times \mathbb{T})} \lesssim \|\langle x\rangle g\|_{L^2(\mathbb{R}^n \times \mathbb{T})} \tag{21'}$$

$$\|e^{-i\lambda t} U^+ e^{i\lambda t} g\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n; L^2(\mathbb{T}))} \lesssim \|\langle x\rangle g\|_{L^2(\mathbb{R}^n \times \mathbb{T})} \tag{21''}$$

Given two values $\lambda_1 \neq \lambda_2$, the difference can still be expressed as

$$e^{-i\lambda_1 t} U^+ e^{i\lambda_1 t} - e^{-i\lambda_2 t} U^+ e^{i\lambda_2 t} = -i(\lambda_1 - \lambda_2)\big(e^{-i\lambda_1 t} U^+ e^{i\lambda_1 t}\big)\big(e^{-i\lambda_2 t} U^+ e^{i\lambda_2 t}\big). \tag{15}$$

Proof. The order of variables is interchanged from Proposition 7 so that we may work entirely on the Fourier side with respect to t. By Minkowski's inequality for mixed norms [14] and Plancherel's identity,

$$\|\hat{g}\|_{\ell^2_n L^{2n/(n+2)}_x} \le \|\hat{g}\|_{L^{2n/(n+2)}_x \ell^2_n} = \|g\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n; L^2(\mathbb{T}))}$$

Following the Fourier characterization of U^+ given in (11) leads to the statement of (21),

$$\begin{aligned}\|e^{-i\lambda t}U^+e^{i\lambda t}g\|_{L^{\frac{2n}{n-2}}(\mathbb{R}^n;L^2(\mathbb{T}))} &= \|(e^{-i\lambda t}U^+e^{i\lambda t}g)\hat{}\,\|_{L_x^{2n/(n-2)}\ell_n^2} \\ &\le \|(e^{-i\lambda t}U^+e^{i\lambda t}g)\hat{}\,\|_{\ell_n^2 L_x^{2n/(n-2)}} \lesssim \|\hat{g}\|_{\ell_n^2 L_x^{2n/(n+2)}} \\ &\le \|g\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n;L^2(\mathbb{T}))}\end{aligned}$$

where the second to last inequality is the uniform $L^{\frac{2n}{n+2}} \to L^{\frac{2n}{n-2}}$ bound for $R^-(\lambda+n), n \in \mathbb{Z}$ proved in [13].

A proof of (21') which captures the sharp constant is given in [19]. The basic argument is the same as the one above, however the Hilbert space structure of $\langle x\rangle L^2(\mathbb{R}^n)$ and the Plancherel identity permit precise computation of the various norms. Finally, the statement (21") is equivalent to the resolvent bound

$$\|R^-(\tau)\psi\|_{\frac{2n}{n-2}} \lesssim \|\langle x\rangle\psi\|_2 \tag{22}$$

uniformly over all $\mathrm{Im}(\tau) \le 0$. It is conceivable that (22) can be derived directly from the resolvent estimates in [19] and [13] by factorizing $R^-(\tau)$ through unweighted L^2. Theorem 3.1 of [18] is another closely related statement, differing only in the weights and regularity of the domain ($\langle x\rangle^{-\frac{1}{2}-\varepsilon}\dot{H}^{-\frac{1}{2}}(\mathbb{R}^n)$ versus $\langle x\rangle^{-1}L^2(\mathbb{R}^n)$). We present a complete proof as Lemma 12, in the section devoted to Fourier analysis. ☐

Lemma 10. *Let $w \in L^2(\mathbb{R}^n)$ and $z \in L^\infty(\mathbb{T}\times\mathbb{R}^n)$. Suppose the operator $T(\lambda_0)$ fails to be invertible at $\lambda_0 \in \mathbb{R}$ and that neither K nor $\overline{K}$ has a resonance at λ_0. The solution spaces $N_{\lambda_0} \subset Y_{\lambda_0}$ and $\tilde{N}_{\lambda_0} \subset Y_{\lambda_0}$ are nontrivial and finite dimensional, and their initial values form finite dimensional subspaces $X_{\lambda_0}, \tilde{X}_{\lambda_0} \subset L^2(\mathbb{R}^n)$.*

If the orthogonal projection from X_{λ_0} onto $\tilde{X}_{\lambda_0}$ is bijective, and if the spaces $e^{-i\lambda_0 t}N_{\lambda_0}$ and $e^{-i\lambda_0 t}\tilde{N}_{\lambda_0}$ are both contained in $L^{\frac{2n}{n+2}}(\mathbb{R}^n;L^2(\mathbb{T})) + \langle x\rangle^{-1}L^2(\mathbb{R}^n\times\mathbb{T})$, then $T(\lambda)$ is invertible for every other λ in the lower halfplane sufficiently close to λ_0, with the norm estimate

$$\|T(\lambda)^{-1}(h_1+h_2)\|_{L^2(\mathbb{T}\times\mathbb{R}^n)} \le C(w,z,\lambda_0)(|\lambda-\lambda_0|^{-1}\|h_1\| + \|h_2\|). \tag{23}$$

In this expression $h_1 \in e^{-i\lambda_0 t}\overline{zw}\tilde{N}_{\lambda_0}$, and h_2 belongs to the L^2-orthogonal complement of $e^{-i\lambda_0 t}\overline{zw}\tilde{N}_{\lambda_0}$.

Proof. As in Lemma 8, one determines that each $g \in \ker T(\lambda_0)$ is associated with an eigenfunction $\phi \in N_{\lambda_0}$ by the relations $\phi = U^+e^{i\lambda_0 t}wg$ and $g = ie^{-i\lambda_0 t}zw\phi$. Because the available estimate (21) for U^+ does not map into $L^2(\mathbb{R}^n\times\mathbb{T})$, the extra assumption that λ_0 is not a resonance is required in order to place $\phi \in N_{\lambda_0}$. It then follows that $\ker T(\lambda_0) = e^{i\lambda_0 t}wN_{\lambda_0}$ and $\operatorname{coker} T(\lambda_0) = e^{-i\lambda_0 t}\overline{zw}\tilde{N}_{\lambda_0}$.

The next step is again to evaluate $T(\lambda)^{-1}h_1$ for $h_1 \in \operatorname{coker} T(\lambda_0)$ using the function $a_{g,h}(\lambda) = \langle T(\lambda)g, h\rangle$ as a guide. While $a_{g,h}(\lambda)$ is holomorphic inside the lower halfplane, in general one expects it to be merely continuous at the boundary, based on Corollary 6.

Better behavior occurs locally if $h \in \operatorname{coker} T(\lambda_0)$. Choose any $h_1 = e^{-i\lambda_0 t}\overline{zw}\tilde{\phi}_1$, $\tilde{\phi}_1 \in \tilde{N}_{\lambda_0}$. By construction, $a_{g,h_1}(\lambda_0) = 0$, and the statements in Proposition 9 imply the local Lipschitz bound

$$\begin{aligned}|a_{g,h_1}(\lambda)| &= |\lambda-\lambda_0||\langle wzg, e^{-i\lambda t}U^-e^{i(\lambda-\lambda_0)t}\tilde{\phi}_1\rangle| \\ &\lesssim |\lambda-\lambda_0|\|g\|_{L^2(\mathbb{R}^n\times\mathbb{T})}\|e^{-\lambda_0 t}\tilde{\phi}_1\|_{L^{\frac{2n}{n+2}}(R^n;L^2(\mathbb{T}))+\langle x\rangle^{-1}L^2(\mathbb{R}^n\times\mathbb{T})}\end{aligned}$$

for all λ in the lower halfplane. A similar bound holds for $a_{g_1,h}(\lambda)$, where $g_1 \in \ker T(\lambda_0)$ and h is any vector in $L^2(\mathbb{R}^n\times\mathbb{T})$. We do not claim any differentiability unless both $g = e^{-i\lambda_0}w\phi \in \ker T(\lambda_0)$ and

$h_1 \in \operatorname{coker} T(\lambda_0)$. In that case,

$$\begin{aligned}
a_{g,h_1}(\lambda) &= (\lambda-\lambda_0)\langle e^{-i\lambda_0 t}\phi, e^{-i\lambda t}U^- e^{i\lambda t}\overline{w}h_1\rangle \\
&= i(\lambda-\lambda_0)\langle \phi, \tilde{\phi}_1\rangle \;+\; (\lambda-\lambda_0)^2\langle e^{-i\lambda_0 t}\phi, e^{-i\lambda t}U^- e^{i(\lambda-\lambda_0)t}\tilde{\phi}_1\rangle \\
&= 2\pi i(\lambda-\lambda_0)\langle \Phi, \tilde{\Phi}_1\rangle_{L^2_x} \\
&\quad + \mathcal{O}\big(|\lambda-\lambda_0|^2\|e^{-i\lambda_0 t}\phi\|\|e^{-i\lambda_0 t}\tilde{\phi}_1\|\big).
\end{aligned}$$

The norms in the last line can be taken with respect to $L^{\frac{2n}{n+2}}(\mathbb{R}^n; L^2(\mathbb{T})) + \langle x\rangle^{-1}L^2(\mathbb{R}^n\times\mathbb{T})$, since $e^{-i\lambda t}U^- e^{i\lambda t}$ maps this space to its dual (see Proposition 9). Once again the finite dimensionality of N_{λ_0} and $\tilde{N}_{\lambda_0}$ makes every norm space for $e^{-i\lambda_0 t}\phi$ equivalent to $\|g\|_{L^2(\mathbb{R}^n\times\mathbb{T})}$ and similarly for $\tilde{\phi}_1$ and h_1.

If the projection of X_{λ_0} onto $\tilde{X}_{\lambda_0}$ is bijective, then for a fixed unit vector $h_1 \in \operatorname{coker} T(\lambda_0)$ there exists a unique unit vector $g_1 \in \ker T(\lambda_0)$ with the properties

$$\begin{aligned}
|\langle T(\lambda)g_1, h_1\rangle| &\gtrsim |\lambda-\lambda_0| \\
\Big\| T(\lambda)g_1\big|_{\operatorname{image} T(\lambda_0)}\Big\| &\lesssim |\lambda-\lambda_0| \\
|\langle T(\lambda)g_1, h'\rangle| &\lesssim |\lambda-\lambda_0|^2
\end{aligned}$$

for all λ in a small neighborhood of λ_0 in the lower halfplane, and all unit vectors $h' \in \operatorname{coker} T(\lambda_0)$ orthogonal to h_1.

From this point onward one can follow the proof of Lemma 8 exactly. By continuity, $T(\lambda)$ is an invertible map between the co-image and image of $T(\lambda_0)$. Given g_1 with the properties above there exists a unique $g'(\lambda) \in \operatorname{coimage} T(\lambda_0)$ with $\|g'(\lambda)\| \lesssim |\lambda-\lambda_0|$ so that $T(\lambda)(g_1+g'(\lambda)) \in \operatorname{coker} T(\lambda_0)$. The combined vector $g_{h_1}(\lambda) = g_1 + g'(\lambda)$ is still of approximately unit norm and satisfies

$$T(\lambda)g_{h_1}(\lambda) = C_{h_1}(\lambda-\lambda_0)h_1 + \mathcal{O}(|\lambda-\lambda_0|^2)$$

with the error lying entirely in $\operatorname{coker} T(\lambda_0)$. After choosing a (finite) basis for $\operatorname{coker} T(\lambda_0)$, the true inverse $T(\lambda)^{-1}h_1$ is a linear combination of $(\lambda-\lambda_0)^{-1}g_{h_j}(\lambda)$.

The inverse image of $h_2 \in \operatorname{image} T(\lambda_0)$ is first approximated by considering the restricted operator $T(\lambda) : \operatorname{coimage} T(\lambda_0) \to \operatorname{image} T(\lambda_0)$. This may produce an error $h' \in \operatorname{coker} T(\lambda_0)$ which can be removed via a correction of size proportional to that of h_2. $\square$

Corollary 11. *Let $V = w^2 z$ be a complex potential in $L^{n/2}_x L^\infty_t$. Suppose the associated Floquet operators K and $\overline{K}$ have no resonances on the real axis, that condition (C1) is satisfied at every real eigenvalue, and condition (C3) at every eigenvalue.*

Then K has finitely many eigenvalues λ_j, counted with multiplicity, inside the strip $\lambda \in \Omega^-$. Similarly, $\overline{K}$ has only the eigenvalues $\bar{\lambda}_j$ in the reflected strip $\Omega^+ = \{\bar{\lambda} : \lambda \in \Omega^-\}$.

For all $\lambda \in \Omega^-$, the action of $T(\lambda)^{-1}$ is governed by the bound

$$\begin{aligned}
\|T(\lambda)^{-1}g\|_{L^2(\mathbb{T}\times\mathbb{R}^n)} &\lesssim \|g\|_{L^2(\mathbb{T}\times\mathbb{R}^n)} + \sum_j |1 + i\cot\pi(\lambda-\lambda_j)| \left| \int_0^{2\pi} \langle g, e^{-i\bar{\lambda}t}\overline{zw}\tilde{\phi}_j\rangle_{L^2_x}\, dt \right| \\
&= \|g\|_{L^2(\mathbb{T}\times\mathbb{R}^n)} + \sum_j |1 + i\cot\pi(\lambda-\lambda_j)| \left| \Big\langle g, e^{-i\bar{\lambda}t}\overline{zw}\tilde{\phi}_j\big|_{t\in[0,2\pi]}\Big\rangle_{L^2_t L^2_x} \right| \qquad (24)
\end{aligned}$$

where $\tilde{\phi}_j \in \tilde{N}_{\lambda_j}$ enumerate the linearly independent eigenvectors of $\overline{K}$ with eigenvalues in Ω^+.

Proof. The continuity and norm-decay properties of Corollary 6 imply that $T(\lambda)^{-1}$ is invertible for all λ in an open subset of Ω^-, with uniform bounds once $\mathrm{Im}(\lambda)$ is sufficiently large. Its complement is therefore compact in Ω^-. If conditions (C1) and (C3) are satisfied, then Lemmas 8 and 10 show that the complement is discrete as well, making it a finite set. At each point where $T(\lambda)^{-1}$ fails to exist, the corresponding eigenvalues of K and $\overline{K}$ have finite multiplicity as a consequence of the Fredholm Alternative.

For the quantitative statement, first recall that $T(\lambda+1) = e^{-it}T(\lambda)e^{it}$. This makes $\|T(\lambda)^{-1}\|$ periodic with respect to integer translations. A finite number of local statements such as (17) and (23) is sufficient to completely categorize the singularities of $T(\lambda)^{-1}$ in the entire lower halfplane.

The conclusion (24) rewrites these local bounds to make them periodic and gathers them into a finite sum. For example the single pole $(\lambda-\lambda_j)^{-1}$ is replaced with a cotangent function. The alterations to the inner product are designed to express projection onto the cokernel of $T(\lambda)$ as a periodic operation. Note that $\operatorname{coker} T(\lambda+1) = e^{-it}\operatorname{coker} T(\lambda)$ for every λ, and $N_{\lambda+1} = N_\lambda$ exactly. In the neighborhood of λ_j we have the estimate

$$\int_0^{2\pi} \langle g, e^{-i(\bar\lambda-\bar\lambda_j)t}h\rangle_{L^2_x}\,dt = \langle g,h\rangle_{L^2(\mathbb{T}\times\mathbb{R}^n)} + \mathcal{O}(|\lambda-\lambda_j|)\|g\|_2\|h\|_2$$

and it is bounded everywhere by $(1+e^{2\pi\mathrm{Im}(\lambda_j-\lambda)})\|g\|_2\|h\|_2$. Choosing a specific unit vector h_j gives us

$$\begin{aligned}|1+i\cot\pi(\lambda-\lambda_j)|&\left|\int_0^{2\pi}\langle g, e^{-i(\bar\lambda-\bar\lambda_j)t}h_j(t,\,\cdot)\rangle_{L^2_x}\,dt\right|\\ &= \sup_{m\in\mathbb{Z}}\frac{1}{\pi|\lambda-(\lambda_j+m)|}\left|\langle g, e^{-imt}h_j\rangle_{L^2(\mathbb{T}\times\mathbb{R}^n)}\right| + \mathcal{O}(\|g\|)\end{aligned}$$

in each neighborhood of $\lambda_j+\mathbb{Z}$ and it is bounded by $\|g\|$ over the remainder of Ω^-. (To construct the global bound we have used the fact that $|1+i\cot\pi(\lambda)| \sim e^{2\pi\mathrm{Im}(\lambda)}$ as $\mathrm{Im}(\lambda)\to-\infty$.) Taking $h_j = e^{-i\bar\lambda_j t}\overline{zw}\tilde\phi_j$, the expression in (24) is seen to possess the same poles as (17) and (23) near each point $\lambda_j+\mathbb{Z}$ and the appropriate global bound away from these singularities. $\square$

4. Proof of Theorem 1

Based on the solution formula (9), it suffices to show that $(I-iwU^+wz)^{-1}wU^+f \in L^2_tL^2_x$, with support on the time halfline $t\in[0,\infty)$. The method of choice is suggested by (12), namely to demonstrate the finiteness of

$$\begin{aligned}\sup_{\mu\le 0}\|e^{\mu t}(I-iwU^+wz)^{-1}wU^+f\|^2_{L^2_tL^2_x} &= \sup_{\mu\le0}\int_0^1 \|T(\lambda'+i\mu)^{-1}e^{-i(\lambda'+i\mu)t}P_{\lambda'+i\mu}wU^+f\|^2_{L^2(\mathbb{T}\times\mathbb{R}^n)}\,d\lambda'\\ &= \sup_{\mu\le0}\int_0^1 \|T(\lambda'+i\mu)^{-1}e^{-i\lambda' t}P_{\lambda'}e^{\mu t}wU^+f\|^2_{L^2(\mathbb{T}\times\mathbb{R}^n)}\,d\lambda'.\end{aligned}$$

Using the inequality (24) to control the behavior of $T(\lambda'+i\mu)^{-1}$, we are left to show that

$$\begin{aligned}&\int_0^1 \|P_{\lambda'}e^{\mu t}wU^+f\|_2^2\,d\lambda'\\ &+\sum_j\int_0^1 |1+i\cot\pi(\lambda'-\lambda'_j+i(\mu-\mu_j))|^2\left|\left\langle e^{\mu t}U^+f, P_{\lambda'}\left(e^{-\mu t}\overline{V}\tilde\phi_j\big|_{t\in[0,2\pi]}\right)\right\rangle_{L^2_tL^2_x}\right|^2 d\lambda' \lesssim \|f\|_2^2\end{aligned}\tag{25}$$

uniformly in $\mu \le 0$. To write things in this form we have taken advantage of the facts that $P_{\lambda'}$ is self-adjoint on $L^2_tL^2_x$ and commutes with pointwise multiplication by $w(x)$.

The first integral above is exactly $\|e^{\mu t}wU^+f\|^2_{L^2_tL^2_x} \lesssim \|f\|_2$ as a result of the Plancherel identity (10) and the free Strichartz inequality (8). The second integral appears more complicated, but it is also evaluated (separately for each j) using Plancherel's identity in the λ' variable. Designate by $b_{j,\mu}(\lambda')$ the function

$$b_{j,\mu}(\lambda') = \big[1 + i\cot\pi(\lambda' - \lambda'_j + i(\mu-\mu_j))\big]\Big\langle e^{\mu t}U^+f, P_{\lambda'}\big(e^{-\mu t}\overline{V}\tilde{\phi}_j\big|_{t\in[0,2\pi]}\big)\Big\rangle_{L^2_xL^2_t}. \tag{26}$$

The desired bound (25) is achieved by showing that

$$\|b_{j,\mu}\|_{L^2([0,1])} \le C_j\|f\|_2$$

for each j and all $\mu \le 0$.

Let $k \in \mathbb{Z}$ be the Fourier variable dual to λ'. Given any function $g \in L^2_tL^2_x$ and a multiplier $M(\lambda')$, the inverse Fourier transform of $M(\lambda')P_{\lambda'}g$ has the form

$$(MP_{\lambda'})\check{\ }g(k,t,x) = \sum_{m\in\mathbb{Z}} \check{M}(k-m)g(t+2\pi m, x).$$

Integration inside the infinite sum is justified in the same manner as the Fourier inversion formula. The fact that $P_{\lambda'}$ resides in the conjugate-linear half of an inner product creates some minor bookkeeping issues. When we wish to find the inverse Fourier transform of a function $B(\lambda') = M(\lambda')\langle F, P_{\lambda'}g\rangle$, the end result is instead

$$\check{B}(k) = \sum_{m\in\mathbb{Z}} \langle F, \overline{\check{M}(k+m)}g(t+2\pi m, x)\rangle.$$

The multiplier of interest, $M(\lambda') = 1 + i\cot\pi(\lambda' - \lambda'_j + i(\mu-\mu_j))$, has as its inverse Fourier transform

$$\check{M}(k) = \big(e^{2\pi i\lambda'_j}e^{2\pi(\mu-\mu_j)}\big)^k \times \begin{cases} -2\big|_{k\ge 1} & \text{if } \mu \le \mu_j \\ 2\big|_{k\le 0} & \text{if } \mu > \mu_j \end{cases} \tag{27}$$

We have chosen to handle the case $\mu = \mu_j$ by analytic continuation from $\mu < \mu_j$ rather than as a principal value. For our purposes the distinction is irrelevant, as the inner product in (25) will be made to vanish wherever there is a singularity of the cotangent function.

We are now prepared to evaluate $\|b_{j,\mu}\|_2$. First consider the case $\mu \le \mu_j$. Applying the top line from (27) to the function $g(t,x) = e^{-\mu t}\overline{V}\tilde{\phi}_j\chi_{t\in[0,2\pi]}$ and recalling the periodicity relation for $\tilde{\phi}_j$ yields

$$\check{B}(k) = -2\big(e^{-2\pi i\lambda'_j}e^{2\pi(\mu-\mu_j)}\big)^k\big\langle F, e^{-\mu t}\overline{V}\tilde{\phi}_j\big|_{t\le 2\pi k}\big\rangle.$$

After substituting $F(t,x) = e^{\mu t}U^+f$ into this expression, Plancherel's identity tells us that

$$\|b_{j,\mu}\|^2_{L^2([0,1])} = \sum_{k\in\mathbb{Z}} 4e^{4\pi(\mu-\mu_j)k}\Big|\Big\langle f, U^-\big(\overline{V}\tilde{\phi}_j\big|_{t\le 2\pi k}\big)(0,\,\cdot\,)\Big\rangle_{L^2(\mathbb{R}^n)}\Big|^2. \tag{28}$$

The support of $U^-(\overline{V}\tilde{\phi}_j\big|_{t\le 2\pi k})$ is contained within the time interval $t \in (-\infty, 2\pi k]$, therefore the inner product vanishes for each $k \le 0$ (It vanishes when $k = 0$ because of local L^2 continuity). For each $k \ge 1$ we use the eigenvector property $\tilde{\phi}_j = U^-\overline{V}\tilde{\phi}_j$ and the periodicity of $e^{-i\bar{\lambda}_j t}\tilde{\phi}_j$ to assert that

$$\begin{aligned} U^-\big(\overline{V}\tilde{\phi}_j\big|_{t\le 2\pi k}\big)(0,\,\cdot\,) &= \Big[U^-(\overline{V}\tilde{\phi}_j) - U^-\big(\overline{V}\tilde{\phi}_j\big|_{t>2\pi k}\big)\Big](0,\,\cdot\,) \\ &= \tilde{\Phi}_j - e^{2\pi i\bar{\lambda}_j k}e^{2\pi ik\Delta}\tilde{\Phi}_j \end{aligned}$$

with the conclusion

$$\begin{aligned}\|b_{j,\mu}\|^2_{L^2([0,1])} &\leq 8\sum_{k\geq 1} e^{4\pi(\mu-\mu_j)k}|\langle f,\tilde{\Phi}_j\rangle|^2 + 8\sum_{k\geq 1} e^{4\pi\mu k}|\langle f, e^{2\pi ik\Delta}\tilde{\Phi}_j\rangle|^2\\ &\lesssim |\mu-\mu_j|^{-1}|\langle f,\tilde{\Phi}_j\rangle|^2 + |\mu|^{-1}\|f\|_2^2\|\tilde{\Phi}_j\|_2^2.\end{aligned}$$

If $\mu_j < 0$, then we have shown that $\|b_{j,\mu}\| \lesssim |\mu_j|^{-1/2}\|f\|$ for all $f \in L^2(\mathbb{R}^n)$ orthogonal to $\tilde{\Phi}_j$ and all $\mu \leq \mu_j$. The extra assumption (C2) is unnecessary in this case.

The calculations are more delicate when $\mu_j = 0$ because the unitarity of $e^{2\pi ik\Delta}$ on L^2 does not provide a satisfactory estimate of the inner product. In its place we use the bound

$$\sum_{k\in\mathbb{Z}} |\langle e^{-2\pi ik\Delta}f,\psi\rangle|^2 \lesssim \|f\|_2^2\|\psi\|^2_{\langle x\rangle^{-1}L^2+W^{1,2n/(n+2)}}, \tag{29}$$

which is proved as Lemma 14 in the last section. This is essentially a discrete-time version of more familiar Kato smoothing estimates

$$\int_{\mathbb{R}} |\langle e^{-it\Delta}f,\psi\rangle|^2\,dt \lesssim \|f\|_2^2\|\psi\|^2_{\langle x\rangle^{-1}L^2+L^{2n/(n+2)}}$$

gathered from [19] and [18]. It is worth re-iterating that $\tilde{\Phi}_j$ has approximately unit norm in any space that contains the finite-dimensional subspace $\tilde{X}_{\lambda_j}$.

The remaining case $\mu_j < \mu \leq 0$ is treated similarly. The same sequence of computations using the appropriate case of (27) leads to the identity

$$\|b_{j,\mu}\|^2_{L^2([0,1])} = \sum_{k\in\mathbb{Z}} 4e^{4\pi(\mu-\mu_j)k}\left|\left\langle f, U^-\big(\overline{V}\tilde{\phi}_j\big|_{t\geq 2\pi k}\big)(0,\,\cdot\,)\right\rangle_{L^2(\mathbb{R}^n)}\right|^2.$$

This time the properties of $\tilde{\phi}_j$ simplify the inner product so that

$$\begin{aligned}\|b_{j,\mu}\|^2_{L^2([0,1])} &= 4\sum_{k\leq 0} e^{4\pi(\mu-\mu_j)k}|\langle f,\tilde{\Phi}_j\rangle|^2 + 4\sum_{k\geq 1} e^{4\pi\mu k}|\langle f, e^{2\pi ik\Delta}\tilde{\Phi}_j\rangle|^2\\ &\lesssim |\mu-\mu_j|^{-1}|\langle f,\tilde{\Phi}_j\rangle|^2 + \|f\|_2^2\|\tilde{\Phi}_j\|^2_{\langle x\rangle^{-1}L^2+W^{1,2n/(n+2)}}.\end{aligned}$$

This concludes the proof of Theorem 1, with the exception of the technical lemmas whose proofs are postponed until the last section.

5. Fourier Analysis

In this section we prove the various technical estimates employed during the proof of Theorem 1. A recurring theme will be the use of Fourier restriction theorems, with particular emphasis on whether the restriction to a sphere varies smoothly with respect to changes in radius.

Lemma 12. *The resolvents $R^-(\tau)$ observe the following inequality*

$$\|R^-(\tau)\psi\|_{\frac{2n}{n-2}} \lesssim \|\langle x\rangle\psi\|_2 \tag{22}$$

with a constant that is uniform over the closed halfplane $\mathrm{Im}(\tau) \leq 0$.

Proof. Let $\hat{\psi}_r(\omega) = \hat{\psi}(r,\omega)$ indicate the restriction of $\hat{\psi}$ to the sphere with radius r. Since we have assumed that $\langle x\rangle\psi \in L^2(\mathbb{R}^n)$, the radial derivative $\partial_r\hat{\psi}_r(\omega) = \nabla\hat{\psi}(x)\cdot\frac{x}{|x|}$ is square-integrable with respect to spherical coordinates. Combined with the convexity of norms, this means

$$\int_0^\infty r^{n-1}\Big(\frac{d}{dr}\big[\|\hat\psi_r\|_{L^2(S^{n-1})}\big]\Big)^2\,dr \le \int_0^\infty r^{n-1}\|\partial_r\hat\psi_r(\omega)\|^2_{L^2(S^{n-1})}\,dr \lesssim \|\langle x\rangle\psi\|_2^2. \tag{30}$$

The left-hand side is a weighted L^2 norm of the derivative of $\|\hat\psi_r\|$. Hardy's inequality (or the Schur test when $n\ge 4$) then gives a weighted L^2 estimate for $\|\hat\psi\|$ itself,

$$\int_0^\infty r^{n-3}\|\hat\psi_r\|^2_{L^2(S^{n-1})}\,dr \lesssim \|\langle x\rangle\psi\|_2^2,$$

which is in effect a bound on $\|(-\Delta)^{-\frac12}\psi\|_2$. Applying the L^p fractional integration bound for $(-\Delta)^{-\frac12}$ on top of this leads to the conclusion

$$\|R^-(0)\psi\|_{\frac{2n}{n-2}} = C_n\|\psi * |x|^{2-n}\|_{\frac{2n}{n-2}} \lesssim \|\langle x\rangle\psi\|_2. \tag{31}$$

Applying the Cauchy-Schwartz inequality to (30) gives a pointwise bound for $\|\hat\psi_r\|$ instead.

$$\|\hat\psi_r\|_{L^2(S^{n-1})} \lesssim r^{1-\frac n2}\|\langle x\rangle\psi\|_2. \tag{32}$$

The resolvent $R^-(\lambda)$ multiplies Fourier transforms by $(|\xi|^2-\lambda)^{-1}$. If $\mathrm{Re}(\lambda) < |\mathrm{Im}(\lambda)|$ then standard estimates show that the convolution kernel of $R^-(\lambda)$ is bounded pointwise by $|x|^{2-n}$, uniformly in λ over this range. The conclusion of the lemma is verified by taking absolute values and applying (31).

The case $\mathrm{Re}(\lambda) > |\mathrm{Im}(\lambda)|$ requires more care. Let χ be a smooth function identically equal to 1 on $[\frac12, 2]$ and supported on $[\frac14, 4]$. Decompose the resolvent into two pieces,

$$\begin{aligned}(R_1\psi)\hat{}(\xi) &= \big(1-\chi\big(|\mathrm{Re}(\lambda)|^{-\frac12}|\xi|\big)\big)(|\xi|^2-\lambda)^{-1}\hat\psi(\xi)\\ (R_2\psi)\hat{}(\xi) &= \chi\big(|\mathrm{Re}(\lambda)^{-\frac12}|\xi|\big)(|\xi|^2-\lambda)^{-1}\hat\psi(\xi)\end{aligned}$$

The convolution kernel associated to R_1 is again controlled pointwise by $|x|^{2-n}$, making it subject to the same bound as in (31).

Each restriction of $\hat\phi$ to the sphere radius r makes the contribution

$$(\hat\psi_r)\check{}(x) = (2\pi)^{-n}r^{n-1}\int_{S^{n-1}} e^{irx\cdot\omega}\hat\psi_r(\omega)\,d\omega$$

toward the original function ψ. Once the normalization is taken into account, the Stein-Tomas theorem [21] indicates that

$$\|(\hat\psi_r)\check{}\|_{\frac{2n}{n-2}} \lesssim r^{\frac n2}\|\hat\psi_r\|_{L^2(S^{n-1})} \tag{33}$$

Set $r_0 = |\mathrm{Re}(\lambda)|^{\frac12}$ and write out $\hat\psi_r = (\hat\psi_r - \hat\psi_{r_0}) + \hat\psi_{r_0}$. This splits $R_2\psi$ into the sum of two pieces.

$$\begin{aligned}R_2\psi(x) = &\int_{\frac{r_0}{4}}^{4r_0} \chi\big(\tfrac{r}{r_0}\big)(r^2-\lambda)^{-1}\big(\hat\psi_r - \hat\psi_{r_0}\big)\check{}(x)\,dr\\ &+ \int_{\frac{r_0}{4}}^{4r_0} \big(\tfrac{r}{r_0}\big)^{n-1}\chi\big(\tfrac{r}{r_0}\big)(r^2-\lambda)^{-1}(\hat\psi_{r_0})\check{}\big(\tfrac{r_0}{r}x\big)\,dr\\ =&I_1 + I_2.\end{aligned}$$

For the first integral, (30) shows that $r^{(n-1)/2}\hat{\psi}_r$, viewed as a $L^2(S^{n-1})$-valued function of r, has a square-integrable weak derivative. Therefore $\hat{\psi}_r$ is Hölder-continuous of order $1/2$ in the interval $r \in [\frac{r_0}{4}, 4r_0]$, with constant $r_0^{(1-n)/2}\|\langle x\rangle\psi\|_2$. Combined with (33) this shows

$$\begin{aligned}\|I_1\|_{\frac{2n}{n-2}} &\lesssim \Big(\int_{\frac{r_0}{4}}^{4r_0} r^{\frac{n}{2}} r_0^{\frac{1-n}{2}} \frac{|r-r_0|^{1/2}}{|r^2-\lambda|}\,dr\Big)\|\langle x\rangle\psi\|_2 \\ &\lesssim r_0^{1/2}\Big(\int_{\frac{r_0}{4}}^{4r_0} \frac{|r-r_0|^{1/2}}{|r^2-r_0^2|}\,dr\Big)\|\langle x\rangle\psi\|_2 \;\lesssim\; \|\langle x\rangle\psi\|_2.\end{aligned}$$

The primary estimate for I_2 is that

$$\|(\hat{\psi}_{r_0})\check{\ }\|_{\frac{2n}{n-2}} \lesssim r_0\|\langle x\rangle\psi\|_2$$

by virtue of (32) and (33). After a suitable change of variables, this function can be transformed into I_2 via a singular integral operator that preserves L^p norms. The proposition below completes the proof. $\square$

Proposition 13. *Given the cutoff χ as defined above and any $\lambda = r_0^2 + i\mu$ with $|\mu| \le r_0^2$, the operator*

$$Sg = \int_{\frac{r_0}{4}}^{4r_0} \big(\tfrac{r}{r_0}\big)^{n-1}\chi\big(\tfrac{r}{r_0}\big)(r^2-\lambda)^{-1} g\big(\tfrac{r_0}{r}x\big)\,dr$$

satisfies the bounds $\|Sg\|_p \le C_p r_0^{-1}\|g\|_p$ for every $1 < p < \infty$.

Proof. Consider the logarithmic spherical coordinates $(s,\omega) \in \mathbb{R}\times S^{n-1}$ defined by $s = \log|x|$ and $\omega = \frac{x}{|x|}$. The Jacobian factor transforms the L^p norms according to the rule

$$\|g\|_p^p = \int_{S^{n-1}}\int_{\mathbb{R}} |g(s,\omega)|^p e^{ns}\,ds d\omega = \int_{S^{n-1}} \|g(\,\cdot\,,\omega)\|^p_{L^p(e^{ns}\,ds)}\,d\omega.$$

In these coordinates the action of S takes place entirely along the s variable. Let $\rho = \log(\frac{r}{r_0})$. Then

$$\begin{aligned}Sg(s,\omega) &= r_0^{-1}\int_{-\log 4}^{\log 4} e^{n\rho}\chi(e^\rho)\frac{g(s-\rho,\omega)}{e^{2\rho}-(1+i\mu/r_0^2)}\,d\rho \\ &= r_0^{-1}\, g * \left[\frac{e^{n\rho}\chi(e^\rho)}{e^{2\rho}-(1+\mu/r_0^2)}\right](s,\omega)\end{aligned}$$

where the convolution takes place in the s variable only. This is a Calderón-Zygmund singular integral which can be controlled by the Hilbert transform independently of the value of μ. The unweighted bounds for the Hilbert transform apply here (despite the fact that e^{ns} belongs to no A_p class) because the convolution kernel is supported in $[-2,2]$ and the exponential function is essentially constant over any interval of similar length. $\square$

Lemma 14. *The propagator of the free Schrödinger equation obeys the sampling estimates*

$$\begin{aligned}&\sum_{k\in\mathbb{Z}} |\langle e^{-2\pi ik\Delta}f,\psi\rangle|^2 \lesssim \|f\|_2^2\,\|\psi\|^2_{\langle x\rangle^{-1}L^2} \\ &\sum_{k\in\mathbb{Z}} |\langle e^{-2\pi ik\Delta}f,\psi\rangle|^2 \lesssim \|f\|_2^2\,\|\psi\|^2_{L^2\cap L^{2n/(n+2)}\cap \dot{W}^{\alpha,2\gamma/(\gamma+2)}},\end{aligned}\tag{34}$$

provided $\gamma \in [\frac{n+1}{2}, n+1]$ and $\alpha + \frac{1}{\gamma} > 1$.

Proof. For each k the inner product $\langle e^{-2\pi i k\Delta} f, \psi\rangle$ represents the integral

$$(2\pi)^{-\frac{n}{2}} \int_R^n \hat{f}(\xi)\overline{\hat{\psi}(\xi)} e^{2\pi i k|\xi|^2}\, d\xi \;=\; (2\pi)^{-\frac{n}{2}} \int_0^\infty s^{\frac{n-2}{2}} \int_{S^{n-1}} \hat{f}(s,\omega)\overline{\hat{\psi}(s,\omega)} e^{2\pi i k s}\, d\omega ds$$

where (s,ω) are the spherical coordinates $s = |\xi|^2$, $\omega = \frac{x}{|x|}$. This in turn describes (up to constants) the kth Fourier coefficient of the periodic function $\sum_m F(s+m)$, where

$$F(s) = (s)^{\frac{n-2}{2}} \int_{S^{n-1}} \hat{f}(s,\omega)\overline{\hat{\psi}(s,\omega)}\, d\omega.$$

We are therefore concerned with finding conditions on ψ that lead to $\sum_m F(s+m)$ belonging to $L^2([0,1])$. It would be sufficient to show instead that $F \in \ell^1_m(L^2([m,m+1]))$.

Plancherel's identity dictates that $s^{(n-2)/4}\hat{f}(s,\omega)$ is an element of $\ell^2_m L^2([m,m+1]; L^2(S^{n-1}))$. Bounds of the type (34) will follow provided that $s^{(n-2)/4}\hat{\psi}$ belongs to $\ell^2_m L^\infty([m,m+1]; L^2(S^{n-1}))$. Taking $\hat{\psi}_s$ to be the restriction of $\hat{\psi}$ to the sphere $|\xi|^2 = s$, we wish to show that

$$\sum_{m\geq 0} \sup_{s\in[m,m+1]} s^{\frac{n-2}{2}} \|\hat{\psi}_s\|^2_{L^2(S^{n-1})}$$

is controlled by the norm of ψ in a space of our choosing.

Suppose $\psi \in \langle x\rangle^{-1}L^2$. Changing variables from r to s in (30) leads to the derivative estimates

$$\int_0^\infty s^{\frac{n}{2}} \Big(\frac{d}{ds}\big[\|\hat{\psi}_s\|_{L^2(S^{n-1})}\big]\Big)^2\, ds \lesssim \|\langle x\rangle\psi\|_2^2. \tag{35}$$

Local differences are estimated by the mean value theorem and Cauchy-Schwartz. For any pair of points $s_1, s_2 \in [m,m+1]$,

$$\begin{aligned}\Big|\|s_1^{(n-2)/4}\hat{\psi}_{s_2}\| - \|s_2^{(n-2)/4}\hat{\psi}_{s_1}\|\Big|^2 &\leq \int_m^{m+1} \Big(\frac{d}{ds}\big[s^{(n-2)/4}\|\hat{\psi}_s\|_{L^2(S^{n-1})}\big]\Big)^2\, ds \\ &\leq 2\int_m^{m+1} s^{\frac{n-6}{2}}\|\hat{\psi}_s\|^2 + s^{\frac{n-2}{2}}\Big(\frac{d}{ds}\|\hat{\psi}_s\|\Big)^2\, ds\end{aligned}$$

The L^∞ norm of a positive function over a unit interval is controlled by its integral and the variation of its values, hence

$$\begin{aligned}\sum_{m\geq 1} \sup_{s\in[m,m+1]} s^{\frac{n-2}{2}}\|\hat{\psi}_s\|^2 &\leq \sum_{m\geq 1}\int_m^{m+1} \big(s^{\frac{n-2}{2}} + 2s^{\frac{n-6}{2}}\|\hat{\psi}_s\|^2 + 2s^{\frac{n-2}{2}}\Big(\frac{d}{ds}\|\hat{\psi}_s\|\Big)^2\, ds \\ &\lesssim \int_1^\infty s^{\frac{n-2}{2}}\|\hat{\psi}_s\|^2\, ds + \int_1^\infty s^{\frac{n}{2}}\Big(\frac{d}{ds}\big[\|\hat{\psi}_s\|\big]\Big)^2\, ds \\ &\lesssim \|\psi\|_2^2 + \|\langle x\rangle\psi\|_2^2\end{aligned} \tag{36}$$

by Plancherel and (35), respectively. The supremum over the interval $s \in [0,1]$ is controlled separately by the estimate

$$\begin{aligned}s^{\frac{n-2}{4}}\|\hat{\psi}_s\| \leq s^{\frac{n-2}{4}}\int_s^\infty \Big|\frac{d}{ds}\big[\|\hat{\psi}_s\|\big]\Big|\, ds &\lesssim \Big(\int_s^\infty s^{\frac{n}{2}}\Big|\frac{d}{ds}\big[\|\hat{\psi}_s\|\big]\Big|^2 ds\Big)^{1/2} \\ &\lesssim \|\langle x\rangle\psi\|_2\end{aligned}$$

which is a combination of Cauchy-Schwartz and (35).

For the second statement, the condition $\psi \in L^{2n/(n+2)}$ is most important in the interval $s \in [0,1]$ and the Sobolev regularity condition plays a major role as $s \to \infty$. It is clearly necessary to have $\psi \in L^2$, otherwise the inner product in (34) could be undefined for one or more values of k.

The dual statement to (33), when normalized with the correct factor of r^{n-1} indicates that $s^{(n-2)/4}\|\hat{\psi}_s\| \lesssim \|\psi\|_{\frac{2n}{n+2}}$ for all $s > 0$. In particular, the supremum over $s \in [0,1]$ is bounded in this manner.

The fact that $\psi \in L^2$ implies that $s^{(n-2)/4}\|\hat{\psi}_x\|^2$ is integrable. Controlling its L^∞ norm on a unit interval in terms of its L^1 norm generally requires some degree of continuity. In the previous case we were able to infer differentiability of $\hat{\psi}_s$ from the polynomial weighted decay of ψ. With ψ merely belonging to an L^p space, it may still be true that $\hat{\psi}$ is continuous, but the modulus of continuity is not determined by $\|\psi\|$ alone. We exploit the observation (also used in [4]), that the norm of $\hat{\psi}_s$ varies smoothly even when the restrictions themselves do not.

Proposition 15. *Let $\gamma \in [\frac{n+1}{2}, n+1]$. The Fourier restrictions of $\psi \in L^{\frac{2\gamma}{\gamma+2}}(\mathbb{R}^n)$ satisfy the continuity bound*

$$m^{\frac{n-2}{2}}\big(\|\hat{\psi}_{s_1}\|^2 - \|\hat{\psi}_{s_2}\|^2\big) \lesssim \Big(\frac{|s_1 - s_2|^{n+1-\gamma}}{m}\Big)^{1/\gamma}\|\psi\|^2_{\frac{2\gamma}{\gamma+2}} \tag{37}$$

for every pair $s_1, s_2 \in [m, m+1]$, $m \geq 1$.

The power of $|s_1 - s_2|$ does not matter much so long as it is nonnegative. Of considerably greater interest is the factor of $m^{-1/\gamma}$, as it contributes meaningfully to the bound

$$\Big|\|s_1^{(n-2)/4}\hat{\psi}_{s_1}\|^2 - \|s_2^{(n-2)/4}\hat{\psi}_{s_2}\|^2\Big| \lesssim \big(m^{-(\alpha+\frac{1}{\gamma})} + m^{-(\alpha+2-\frac{n}{\gamma})}\big)\|\psi\|^2_{\dot{W}^{\alpha,2\gamma/(\gamma+2)}}$$

for each pair $s_1, s_2 \in [m, m+1]$, $m \geq 1$. The first term is derived from (37), and the second (which is dominated by the first) from the Stein-Tomas theorem. As before, the L^∞ norm of a function on a unit interval is controlled by the its L^1 norm and the diameter of its image. Consequently,

$$\begin{aligned}\sum_{m\geq 0}\sup_{s\in[m,m+1]}\|s^{(n-2)/4}\hat{\psi}_s\|^2 &\lesssim \|\psi\|^2_{\frac{2n}{n+2}} + \sum_{m\geq 1}\Big(\int_m^{m+1} s^{(n-2)/2}\|\hat{\psi}_s\|^2\,ds + m^{-(\alpha+\frac{1}{\gamma})}\|\psi\|^2_{\dot{W}^{\alpha,2\gamma/(\gamma+2)}}\Big)\\ &\lesssim \|\psi\|^2_{\frac{2n}{n+2}} + \|\psi\|^2_{\dot{W}^{\alpha,2\gamma/(\gamma+2)}} + \int_1^\infty s^{(n-2)/2}\|\hat{\psi}_s\|^2\,ds\\ &= \|\psi\|^2_{\frac{2n}{n+2}} + \|\psi\|^2_{\dot{W}^{\alpha,2\gamma/(\gamma+2)}} + \|\psi\|_2^2\end{aligned}$$

provided the sum of $m^{-(\alpha+1/\gamma)}$ is convergent. □

Proof of Proposition 15. On each interval $[m, m+1]$ the function $s^{(n-2)/4}$ can be replaced by the constant $m^{(n-2)/4}$. Recalling the proof of the Stein-Tomas theorem, Fourier restriction to the sphere is described by a convolution operator, with the TT^* estimate

$$\|\hat{\psi}_s\|^2 = C_n \int_{\mathbb{R}^{2n}} f(x)K\big(s^{\frac{1}{2}}(x-y)\big)\overline{f(y)}\,dxdy. \tag{38}$$

The kernel is an oscillatory function bounded pointwise by $|K(z)| \lesssim \langle z\rangle^{-(n-1)/2}$. The related function $\tilde{K}(z) = zK'(z)$ is also oscillatory, and bounded pointwise by $\langle z\rangle^{-(n-3)/2}$. If f is a Schwartz function it is permissible to differentiate (38) with respect to s, obtaining

$$\frac{d}{ds}\Big(\|\hat{\psi}_s\|^2\Big) = C_n s^{-1}\int_{\mathbb{R}^{2n}} f(x)\tilde{K}\big(s^{\frac{1}{2}}(x-y)\big)\overline{f(y)}\,dxdy.$$

The same interpolation argument that proves the Stein-Tomas theorem also suffices to show that convolution with $\tilde{K}$ is a bounded operator from $L^{(2n+2)/(n+5)}(\mathbb{R}^n)$ to its dual space $L^{(2n+2)/(n-3)}(\mathbb{R}^n)$. Combining this with the usual restriction estimate and scaling appropriately,

$$m^{(n-2)/2}\Big|\|\hat{\psi}_{s_1}\|^2 - \|\hat{\psi}_{s_2}\|^2\Big| \lesssim \max\Big(m^{1/(n+1)}\|f\|^2_{\frac{2n+2}{n+3}},\ m^{2/(n+1)}|s_1 - s_2|\|f\|^2_{\frac{2n+2}{n+5}}\Big).$$

These represent the cases $\gamma = n+1$ and $\gamma = \frac{n+1}{2}$, respectively. The intermediate cases follow from Riesz-Thorin interpolation, noting that the norm of a self-adjoint linear operator agrees with the extremal value of its quadratic form. $\square$

Remark 4. The proof of Lemma 14 hinges on placing the spherical restrictions of $\hat{\psi}$ inside a mixed-norm space $\ell^2(L^\infty)$ with respect to the radial variable. This consists of three essentially independent requirements.

1. Because of the embedding $\ell^2(L^\infty) \subset \ell^2(L^2)$ and the Plancherel identity, we must have $\psi \in L^2$. This is the only way to produce ℓ^2 decay as $m \to \infty$.
2. Since $\ell^2(L^\infty)$ also embeds into $\ell^\infty(L^\infty)$, the normalized restrictions $s^{(n-2)/4}\hat{\psi}_s$ must be uniformly bounded, in particular as $s \to 0$. This is achieved so long as ψ belongs to either of the spaces $\langle x\rangle^{-1}L^2$ or $L^{2n/(n+2)}$.
3. The norm of the restrictions must also be sufficiently continuous so that the $\ell^2(L^2)$ bound implied by the first item can be improved into $\ell^2(L^\infty)$.

   Proposition 15 provides one estimate for the modulus of continuity of $\|\hat{\psi}_s\|^2$ based on the Stein-Tomas restriction theorem. Another estimate, based on L^2 trace properties, is available when $\langle x\rangle^\beta\psi \in L^2$ for some $\beta > \frac{1}{2}$. The latter bounds are well known from the proof of the limiting absorption principle [16] and spectral theory of Schrödinger operators.

The norm spaces in the statement of Proposition 15 were chosen to meet these requirements entirely with weights, or entirely with homogeneous L^p conditions, respectively. To create a more comprehensive list, one can mix and match the two approaches in any combination. A precise but unwieldy formulation is presented below.

Proposition 16. *The propagator of the free Schrödinger equation obeys the sampling estimates*

$$\sum_{k\in\mathbb{Z}} |\langle e^{-2\pi ik\Delta}f, \psi\rangle|^2 \lesssim \|f\|_2\|\psi\|$$

where the norm of ψ is taken in the interpolation space

$$\psi \in L^2 \cap \big(\langle x\rangle^{-1}L^2 + L^{\frac{2n}{n+2}}\big) \cap \big(\langle x\rangle^{-\frac{1}{2}-\varepsilon}L^2 + \dot{W}^{\frac{n}{n+1}+\varepsilon,\frac{2n+2}{n+3}} + \dot{W}^{\frac{n-1}{n+1}+\varepsilon,\frac{2n+2}{n+5}}\big).$$

References

1. Agmon, S. *Spectral properties of Schrödinger operators and scattering theory.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 2, 151–218.
2. Erdoğan, M. B., Schlag, W. *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: I*, Dynamics of PDE, **1** no. 4 (2004), 359–379.
3. Galtbayar, A., Jensen, A., Yajima, K. *Local time-decay of solutions to Schrödinger equations with time-periodic potentials*, J. Statist. Phys. **116** (2004), 231-282.
4. Goldberg, M., Schlag, W. *A limiting absorption principle for the three–dimensional Schrödinger equation with L^p potentials*, Intl. Math. Res. Not. **2004:75** (2004), 4049–4071.
5. Goldberg, M., Visan, M. *A Counterexample to dispersive estimates for Schrödinger operators in higher dimensions*, Comm. Math. Phys. bf 266 (2006), 211–238.
6. Ionescu, A., Jerison, D. *On the absence of positive eigenvalues of Schrödinger operators with rough potentials.* Geom. and Func. Anal. **13** (2003), 1029–1081.

7. Jensen, A. *Spectral properties of Schrödinger operators and time-decay of the wave functions results in* $L^2(R^m)$, $m \geq 5$. Duke Math. J. **47** no. 1 (1980), 57–80.
8. Jensen, A. *Spectral properties of Schrödinger operators and time-decay of the wave functions. Results in* $L^2(R^4)$. J. Math. Anal. Appl. **101** no. 2 (1984), 397–422.
9. Jensen, A., Kato, T. *Spectral properties of Schrödinger operators and time-decay of the wave functions.* Duke Math. J. **46** no. 3 (1979), 583–611.
10. Journé, J.-L., Soffer, A., Sogge, C. D. *Decay estimates for Schrödinger operators.* Comm. Pure Appl. Math. **44** no. 5 (1991), 573–604.
11. Kato, T. *Wave operators and similarity for some non-selfadjoint operators.* Math. Ann. **162** (1965/1966), 258–279.
12. Keel, M., Tao, T. *Endpoint Strichartz inequalities*, Amer. J. Math. **120** (1998), 955–980.
13. Kenig, C. E. , Ruiz, A., Sogge, C. D. *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators*, Duke Math. J. **55** (1987), no. 2, 329–347.
14. Lieb, E., Loss, M., *Analysis.* Second edition. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, 2001.
15. Reed, M., Simon, B. *Methods of Modern Mathematical Physics. II. Fourier Analysis, Self Adjointness.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London, 1975.
16. Reed, M., Simon, B. *Methods of Modern Mathematical Physics. IV. Analysis of Operators.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York–London, 1978.
17. Rodnianski, I., Schlag, W. *Time decay for solutions of Schrödinger equations with rough and time-dependent potentials.* Invent. Math. **155** no. 3 (2004), 451–513.
18. Ruiz, A., Vega, L., *on local regularity of schrödinger equations*, Intl. Math. Res. Not. **1** (1993), 13–27.
19. Simon, B., *Best constants in some operator smoothness estimates*, J. Funct. Anal. **107** no. 1 (1992), 66–71.
20. Stein, E. *Harmonic Analysis: Real–Variable Methods, Orthogonality, and Oscillatory Integrals.* Princeton University Press, Princeton, NJ, 1993.
21. Tomas, P., *Restriction theorems for the Fourier transform*, Proc. Symp. Pure Math. **35** (1979), 111–114.
22. Vodev, G. *Dispersive estimates of solutions to the Schrd̈inger equation in dimensions* $n \geq 4$, Asymptot. Anal. **49** (2006), 61–86.
23. Yajima, K. *Exponential decay of quasi-stationary states of time periodic Schrödinger equations with short range potentials*, Sci. Papers Collega Arts Sci. Univ. Tokyo **40** (1990), 27–36.
24. Yajima, K. *The* $W^{k,p}$*-continuity of wave operators for Schrödinger operators.* J. Math. Soc. Japan **47** no. 3 (1995), 551–581.

Department of Mathematics, Johns Hopkins University, 3400 N. Charles St., Baltimore, MD 21218
E-mail address: mikeg@math.jhu.edu