RESTRICTIONS OF HIGHER DERIVATIVES OF THE FOURIER TRANSFORM

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ABSTRACT. We consider several problems related to the restriction of $(\nabla^k)\hat{f}$ to a surface $\Sigma \subset \mathbb{R}^d$ with nonvanishing Gauss curvature. While such restrictions clearly exist if f is a Schwartz function, there are few bounds available that enable one to take limits with respect to the $L_p(\mathbb{R}^d)$ norm of f. We establish three scenarios where it is possible to do so:

- When the restriction is measured according to a Sobolev space $H^{-s}(\Sigma)$ of negative index. We determine the complete range of indices (k, s, p) for which such a bound exists.
- Among functions where $\hat{f}$ vanishes on Σ to order $k-1$, the restriction of $(\nabla^k)\hat{f}$ defines a bounded operator from (this subspace of) $L_p(\mathbb{R}^d)$ to $L_2(\Sigma)$ provided $1 \leq p \leq \frac{2d+2}{d+3+4k}$.
- When there is *a priori* control of $\hat{f}|_\Sigma$ in a space $H^\ell(\Sigma)$, $\ell > 0$, this implies improved regularity for the restrictions of $(\nabla^k)\hat{f}$. If ℓ is large enough then even $\|\nabla \hat{f}\|_{L_2(\Sigma)}$ can be controlled in terms of $\|\hat{f}\|_{H^\ell(\Sigma)}$ and $\|f\|_{L_p(\mathbb{R}^d)}$ alone.

The proofs are based on three main tools: the spectral synthesis work Y. Domar, which provides a mechanism for L_p approximation by "convolving along surfaces in spectrum", a new bilinear oscillatory integral estimate valid for ordinary L_p functions, and a convexity-type property of the quantity $\|(\nabla^k)\hat{f}\|_{H^{-s}(\Sigma)}$ as a function of k and s that allows to employ the control of $\|\hat{f}\|_{H^\ell(\Sigma)}$.

CONTENTS

1. Introduction

1.1. **Overview of the Derivative Restriction Problem.** Questions regarding the fine properties of the Fourier transform of a function in $L_p(\mathbb{R}^d)$ have long played a central role in the development of classical harmonic analysis. While the Hausdorff–Young theorem guarantees that for $1 \leq p \leq 2$, the Fourier transform of $f \in L_p$ belongs to its dual space $L_{p/(p-1)}$, it does not provide guidance on whether $\hat{f}$ may be defined on a given measure-zero subset $\Sigma \subset \mathbb{R}^d$. The canonical question of this type, originating in the work of Stein circa 1967, is to find the complete range of pairs (p, q) for which the inequality

$$\text{(1)} \qquad \|\hat{f}|_{S^{d-1}}\|_{L_q(S^{d-1})} \lesssim \|f\|_{L_p(\mathbb{R}^d)}$$

holds true. The problem was solved in the case $d = 2$ in [8] and remains an active subject of research in higher dimensions (e.g. [5, 13, 14]).

In this paper we investigate the possibility of defining the surface trace of higher order gradients of the Fourier transform of an L_p function, with a focus on uniform estimates in the style of (1). Let Σ be a closed smooth embedded $(d-1)$-dimensional submanifold of $\mathbb{R}^d$. Assume that the principal curvatures of Σ are non-zero at any point. Let K be a compact subset of Σ and k be a natural number. We consider as a model problem the inequality

$$\text{(2)} \qquad \left\|(\nabla^k \hat{f})|_\Sigma\right\|_{L_2(K)} \lesssim_K \|f\|_{L_p(\mathbb{R}^d)}.$$

Here and in what follows the Fourier transform has priority over differentiation: we first compute the Fourier transform and then differentiate it. We choose the standard Hausdorff measure $d\sigma$ on Σ to define the L_2-space on the left hand-side. The notation "$\lesssim_K$" signifies that the constant in the inequality may depend on the choice of K, but should not depend on f. We restrict our study to the case of L_2 instead of L_q with arbitrary q on the left hand-side, because the Hilbert space properties of L_2 make this case more tractable. In fact, the range of all possible p in (1) when $q = 2$ is described by the classical Stein–Tomas Theorem (established in [25] and [21]).

Unfortunately, inequality (2) cannot hold true unless $k = 0$. To see that, consider the shifts of a function f, in other words $f_N(x) = f(x + Ny)$, where $y \neq 0$ is a fixed point in $\mathbb{R}^d$. If we plug f_N into (2) instead of f, the norm on the left hand-side will be of the order N^k, whereas the quantity on right will not depend on N.

The next question along these lines is: what modifications can be made so that (2) becomes a true statement for $k \geq 1$? Since the original inequality (1) is shift-invariant, we seek translation invariant conditions for f. This rules out natural candidates such as requiring $(1 + |x|)^k f \in L_p$.

One possibility is to relax the desired local regularity from $L_2(K)$ to a Sobolev space of negative order. Consider the inequality

$$\text{(3)} \qquad \left\|\phi(\nabla^k \hat{f})|_\Sigma\right\|_{H^{-s}(\Sigma)} \lesssim_\phi \|f\|_{L_p(\mathbb{R}^d)}.$$

Here $\phi \in C_0^\infty(\Sigma)$ is an arbitrary compactly supported smooth function (the constant in the inequality may depend on it). The parameter s is a non-negative real, and H^{-s} is the L_2-based Bessel potential space. Whenever (3) holds, there is a trace value for $\nabla^k \hat{f}$ in $H_{\mathrm{loc}}^{-s}(\Sigma)$ for all $f \in L_p(\mathbb{R}^d)$.

One might guess that the inequality (3) gets weaker as we increase s, opening the way to define the trace of $\nabla^k \hat{f}$ on Σ with an increasingly large range of p. This is indeed the case. The case $k = 0$ in (3) was considered by Cho, Guo, and Lee in [6]. They observed Sobolev-space trace values of $\hat{f}$ for $f \in L_p$ with p going up to the sharp exponent dictated by the Fourier transform of a surface measure.

There are two parameters that appear frequently as bounds in our arguments:

$$(4) \qquad \qquad \sigma_p = \frac{d}{p} - \frac{d+1}{2};$$

$$(5) \qquad \qquad \kappa_p = \frac{d+1}{p} - \frac{d+3}{2}.$$

Where it occurs later on, we also use the standard notation $p' = \frac{p}{p-1}$ for the dual exponent to L_p.

Proposition 1.1 (Corollary of Theorem 1.1 in [6]). *Let $p > 1$. The inequality* (3) *is true if and only if*

$$(6) \qquad \qquad k \leq s;$$

$$(7) \qquad \qquad k < \sigma_p;$$

$$(8) \qquad \qquad 2k - s \leq \kappa_p.$$

For fixed k and p with $k < \sigma_p$, that means $s \geq \max(k, 2k - \kappa_p)$. In the case $p = 1$, the case $k = \sigma_1$ is also permitted if $s > k$.

The parameter σ_p is related to the "surface measure extremizer." When condition (7) does not hold, Proposition 1.1 fails by testing its dual statement against a surface measure on Σ. The parameter κ_p, and its role in condition (8) are similarly associated with Knapp examples.

In odd dimensions there is an endpoint case $p = 1$, $k = \sigma_1 = \frac{d-1}{2} \in \mathbb{N}$ where inequality (3) is true for $s > \frac{d-1}{2}$. This is stated more precisely in Corollary 7.8 below. The proof of that bound is more direct than most of our other arguments (in fact it is nearly equivalent to the dispersive bound for the Schrödinger equation) and it is completely independent, see Proposition 7.7.

The paper contains two proofs of Proposition 1.1. First, it is a special case of the more extensive Theorem 1.16, whose proof is presented as Section 4. Then we also show in Subsection 7.3 how to derive Proposition 1.1 from the results of [6]. To be more specific, one can interpolate between the results of [6] for $k = 0$ and the Besov-space bound in Proposition 7.7 for $p = 1$, $k = \frac{d-1}{2}$ to obtain the full range of Theorem 1.1.

If one is determined not to weaken the $L_2(K)$ norm in (2), it is necessary to consider f belonging to an *a priori* narrower space than $L_p(\mathbb{R}^d)$. We introduce the main character.

Definition 1.2. *Let Σ be a closed smooth embedded $(d-1)$-dimensional submanifold of $\mathbb{R}^d$, $p \in [1, \infty)$, and $k \in \mathbb{N}$. Define the space $_\Sigma L_p^k$ by the formula*

$$_\Sigma L_p^k = \mathrm{clos}_{L_p}\left(\left\{f \in \mathcal{S}(\mathbb{R}^d) \,\middle|\, \forall l = 0, 1, 2, \ldots, k-1 \quad \nabla^l \hat{f} = 0 \quad \text{on} \quad \Sigma\right\}\right).$$

Define $_\Sigma L_p^0$ to be simply $L_p(\mathbb{R}^d)$. The first non-trivial space $_\Sigma L_p^1$ will often be denoted by $_\Sigma L_p$.

The symbol $\mathcal{S}$ denotes the Schwartz class of test functions. We note that in the definition above, we do not need any information about Σ. In fact, Σ may be an arbitrary closed set. The restriction $p < \infty$ is taken so that the Schwartz class is dense in L_p, though one could replace closure with weak closure in the case $p = \infty$ if needed. These generalities will not arise in the present paper. *From now on we assume that Σ is a closed smooth embedded $(d-1)$-dimensional submanifold of $\mathbb{R}^d$ with non-vanishing principal curvatures.*

It will turn out (See Theorem 1.6 below) that for a certain range of p and k, the space $_\Sigma L_p^k$ contains precisely the functions $f \in L_p$ whose Fourier transform vanishes on Σ to order $k-1$. We take advantage of the additional structure of the domain to formulate a second adaptation of inequality (2), this time with the trace of $\nabla^k \hat{f}$ still belonging to $L_{\text{loc}}^2(\Sigma)$:

$$(9) \qquad \left\| (\nabla^k \hat{f})|_\Sigma \right\|_{L_2(K)} \lesssim_K \|f\|_{L_p} \text{ for all } f \in {}_\Sigma L_p^k.$$

One might expect that a similar statement with the L_2 norm replaced by a weaker Sobolev norm will admit a larger range of p, that is:

$$(10) \qquad \left\| (\phi \nabla^k \hat{f})|_\Sigma \right\|_{H^{-s}(\Sigma)} \lesssim_\phi \|f\|_{L_p(\mathbb{R}^d)} \text{ for all } f \in {}_\Sigma L_p^k.$$

However at this point in the discussion it is not clear why (10) should be true outside the range established in Proposition 1.1, or why (9) should be true at all. Given a generic function $f \in L_p(\mathbb{R}^d)$, its Fourier transform $\hat{f}$ is not differentiable even to fractional order. We have reduced the obstruction somewhat by seeking derivatives of $\hat{f}$ only at the points $\xi \in \Sigma$, and by specifying a substantial number of its partial derivatives via the assumption $f \in {}_\Sigma L_p^k$. Never the less, values of $\hat{f}|_\Sigma$ alone do not uniquely determine $f \in L^p$, nor are they known to shed much light on the behavior of $\hat{f}$ in a neighborhood of Σ.

Theorem 1.4 below finds the complete range of p for which an L_2 gradient restriction (9) is true. In particular the range is nonempty when $d \geq 4k + 1$. The range of p permitted in (10) is also sharp in the same way as Proposition 1.1 and the results in [6]. The range of s we obtain here is much larger than what is true in the context of Proposition 1.1, but most likely not optimal due to some complications with linear programming over the integers.

The $k = 1$ case of Theorem 1.4 shows that an *a priori* assumption $\hat{f}|_\Sigma = 0$ leads to nontrivial bounds on $\nabla \hat{f}|_\Sigma$. In fact there is a larger family of bounds for trace values of $\nabla^k \hat{f}$, and one can begin the bootstrapping process with a much milder assumption $\hat{f}|_\Sigma \in H^\ell(\Sigma)$ instead of requiring it to vanish. We explore these generalizations in Proposition 1.11, Theorem 1.12, and the related discussion. The inequality which takes the place of (10) has the form

$$\left\| (\phi \nabla^k \hat{f})|_\Sigma \right\|_{H^{-s}(\Sigma)} \lesssim_\phi \left(\|f\|_{L_p(\mathbb{R}^d)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)} \right) \text{ for all } f \in L_p$$

(the right hand-side may be infinite). Remarkably, there are cases where this statement holds with only an $L_2(\Sigma)$ norm on the left side. In Corollary 1.13 we find a sizable range of indices (d, p, ℓ) that admit a local-L_2 bound on the gradient of $\hat{f}$,

$$\left\| \phi \nabla \hat{f} \right\|_{L_2(\Sigma)} \lesssim_\phi \left(\|f\|_{L_p(\mathbb{R}^d)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)} \right) \text{ for all } f \in L_p.$$

The spaces $_\Sigma L_p^k$ that arise in Definition 1.2 are not a new construction. They appeared in [12] (see Proposition 12 in that paper) and [11] where the authors investigated the action of Bochner–Riesz operators of negative order on these spaces. They arose in [24] in connection with Sobolev type embedding theorems. We describe this development in Section 2.

In fact, the spaces $_\Sigma L_1^k$ played the central role in the study of the spectral synthesis problem in 60s and 70s. We stress the work of Domar here (e.g. [7]) and will rely upon it in Section 3.

It is worth noting that the main inequality used to derive (9) and (10) is valid for all functions in L_p, not just those whose Fourier restriction vanishes on Σ. The formulation of this inequality, which may be of independent interest, is given in (29) below and the sharp range of p for which it holds is found in Theorem 1.16

1.2. **Statement of results.** It follows from Definition 1.2 that the spaces $_\Sigma L_p^k$ get more narrow as we increase k:

$$L_p \supseteq {}_\Sigma L_p^1 \supseteq {}_\Sigma L_p^2 \supseteq \ldots \supseteq {}_\Sigma L_p^k \supseteq \ldots \supseteq {}_\Sigma L_p^\infty.$$

The final space can be defined as the closure in L_p of the set of Schwartz functions whose Fourier transform vanishes in a neighborhood of Σ. We claim that $_\Sigma L_p^k = {}_\Sigma L_p^\infty$ when k is sufficiently large (i.e. the chain of spaces stabilizes). Here is the precise formulation.

Proposition 1.3. *We have $_\Sigma L_p^k = {}_\Sigma L_p^{k+1} = {}_\Sigma L_p^\infty$ provided $k \geq \sigma_p = \frac{d}{p} - \frac{d+1}{2}$ and $p > 1$. If $p = 1$, this is true provided $k > \frac{d-1}{2} = \sigma_1$.*

For the case $p = 1$, this theorem was proved in [7], and the proof works for arbitrary p (except for, possibly, $p = \infty$, which we do not consider here). The theorem is sharp in the sense that $_\Sigma L_p^k \neq {}_\Sigma L_p^{k+1}$ provided $k < \sigma_p$ (see Theorem 1.6 below).

Theorem 1.4. *The inequality (9) is true if and only if $p \in [1, \frac{2d+2}{d+3+4k}]$, or equivalently $2k \leq \kappa_p$.*

More generally, inequality (10) is true for $p \in [1, \frac{2d}{d+1+2k})$ and $s \geq \max(0, k + 1 - \lceil \sigma_p - k \rceil, 2k - \kappa_p)$, where the notation $\lceil \cdot \rceil$ indicates the smallest integer greater than or equal to the enclosed value. This covers the entire range $k < \sigma_p$. When $p = 1$ and $\sigma_1 = \kappa_1 = \frac{d-1}{2} \in \mathbb{N}$ the value $s = \max(0, 2k - \frac{d-1}{2})$ is also permitted.

Remark 1.5. The $p = 1$, $k = \sigma_1 = \frac{d-1}{2} \in \mathbb{N}$ endpoint case is handled in Corollary 7.8 below, with inequality (10) holding for all $s > k$.

The first claim in the theorem above is an "iff" statement. Usually, the "if" part is much more involved than the "only if" one. In fact, the "only if" part of Theorem 1.4 is proved with the standard Knapp example. Some of other theorems in the paper will have richer collection of "extremizers". Moreover, one and the same "extremizer" may prove sharpness of several related estimates. We collect the descriptions of such type "extremizers" (and thus, the proofs of the "only if" parts) in Section 6.

Theorem 1.4 says that the operator

$$\mathcal{R}_K^k \colon f \mapsto (\nabla^k f)\big|_\Sigma$$

acts continuously from the space $_\Sigma L_p^k$ to $L_2(K)$ when $p \in [1, \frac{2d+2}{d+3+4k}]$, or from $_\Sigma L_p^k$ to $H^{-s}(K)$ for some combinations of (p, s) with $p \in [1, \frac{2d}{d+1+2k})$. This allows us to define a new space

$$\text{(11)} \qquad\qquad \text{Ker}\, \mathcal{R}^k = \bigcap_{K \subset \Sigma} \text{Ker}\, \mathcal{R}_K^k,$$

which consists of all L_p functions for which the (L_2 or H^{-s}) traces of all partial derivatives of order k vanish on Σ. Note that $\mathcal{R}^{k-1}$ is formally defined on $_\Sigma L_p^{k-1} \supseteq {}_\Sigma L_p^k$, and so on, thus we have vanishing of lower order derivatives as well. We also note that in the case when Σ is compact, one does not need to use the intersection in (11) and may simply write $\text{Ker}\, \mathcal{R}^k = \text{Ker}\, \mathcal{R}_\Sigma^k$. It follows from definitions that $_\Sigma L_p^{k+1} \subset \text{Ker}\, \mathcal{R}^k$. In fact, the two spaces must coincide. This looks like a trivial approximation statement, however we do not know a straightforward proof.

Theorem 1.6. *For any $p \in [1, \frac{2d+2}{d+3+4k}]$, the spaces $_\Sigma L_p^{k+1}$ and $\text{Ker}\, \mathcal{R}^k$ coincide with $\mathcal{R}^k$ being regarded as a map from $_\Sigma L_p^k$ to $L_{\text{loc}}^2(\Sigma)$. This occurs when $2k \leq \kappa_p$.*

For any $p \in [1, \frac{2d}{d+1+2k})$, the spaces $_\Sigma L_p^{k+1}$ and $\text{Ker}\, \mathcal{R}^k$ coincide with $\mathcal{R}^k$ being regarded as a map from $_\Sigma L_p^k$ to $H_{\text{loc}}^{-s}(\Sigma)$ for the same range of s as in Theorem 1.4. This occurs when $k < \sigma_p$, or $k \leq \sigma_1$ when $p = 1$.

Remark 1.7. Since $\mathcal{R}^k$ acts nontrivially on the Schwartz functions contained in $_\Sigma L_p^k$, it follows that $_\Sigma L_p^k \supsetneq {}_\Sigma L_p^{k+1}$ in this range of k. Thus, Proposition 1.3 and Theorem 1.6 completely classify the spaces $_\Sigma L_p^k$, modulo some details about the optimal target space for $\mathcal{R}^k$.

Remark 1.8. In the papers [11] and [12], the condition "$\hat{f} = 0$ on the unit sphere" was understood in the sense of L_2 traces.

Theorem 1.4 covers many combinations $k > s \geq 0$ that are forbidden in Proposition 1.1 by demanding that $\hat{f}$ vanishes to order $k-1$ on Σ. We now introduce a family of statements which assume only smoothness of $\hat{f}|_\Sigma$ instead of vanishing. Bessel spaces already appear on the left hand-side of inequality (3), so it is reasonable to use the same scale to describe the smoothness of $\hat{f}|_\Sigma$.

Definition 1.9. *Let k be a natural number, let ℓ and s be non-negative reals, let $p \in [1, \infty)$. We say that the higher derivative restriction property $\mathrm{HDR}(\Sigma, k, s, \ell, p)$ holds true if for any smooth compactly supported function ϕ in d variables, the estimate*

$$(12) \qquad \left\| (\phi \nabla^k \hat{f})\big|_\Sigma \right\|_{H^{-s}(\Sigma)} \lesssim_\phi \left(\|f\|_{L_p(\mathbb{R}^d)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)} \right)$$

holds true for any Schwartz function f.

The HDR property resembles a Dirichlet-to-Neumann bound for Fourier transforms in the sense that regularity of $\hat{f}$ along the surface Σ implies a certain degree of improved regularity in the transverse direction. It is notable that the restriction $\hat{f}|_\Sigma$ does not uniquely determine $f \in L_p$ or the values of $\hat{f}(\xi)$ anywhere else in $\mathbb{R}^d$, so the inequality (12) must hold uniformly for all functions whose Fourier transforms coincide with $\hat{f}$ on Σ.

Remark 1.10. A complete generalization of Theorem 1.4 would include *a priori* estimates on $\|\phi \nabla^j \hat{f}\|_{H^{\ell_j}(\Sigma)}$ for $j = 0, 1, \ldots, J \leq k-1$. We consider only the $J = 0$ case above for relative simplicity of notation.

Proposition 1.11. *If $\mathrm{HDR}(\Sigma, k, s, \ell, p)$ holds true and $p > 1$, then*

$$(13) \qquad k \leq s + \ell;$$

$$(14) \qquad k \leq s + 1;$$

$$(15) \qquad k < \sigma_p;$$

$$(16) \qquad \frac{k\ell}{s + \ell - k} \leq \kappa_p \quad \text{when } k > s;$$

$$(17) \qquad 2k - s \leq \kappa_p,$$

where the numbers σ_p and κ_p are defined by (4) and (5) respectively. In the case $p = 1$, equality in (15) may also occur.

The sufficient conditions we are able to provide for the HDR inequalities do not always match the necessary ones listed above. Roughly speaking, they get close to necessary conditions when ℓ or k is relatively small and there is a gap if ℓ and k are both large. By "getting close to necessary conditions" we mean that the non-sharpness comes only from our limitation of working with integer k. The sufficient conditions we are able to obtain are rather bulky (this is again due to "integer arithmetic"). They are formulated in terms of certain convex hulls of finite collections of points in the plane. Since we need to introduce more notation before formulating the strongest available statement, we refer the reader to Theorem 5.21 in Section 5 for the details and state a representative subset of the results here.

Theorem 1.12. *Let $p > 1$ and $\kappa_p \in \mathbb{N}$. If*

$$(18) \qquad 2\left\lceil \frac{\ell - 1}{\ell} \kappa_p \right\rceil \leq \kappa_p \quad or \quad k \leq \left\lceil \frac{\kappa_p}{2} \right\rceil,$$

then $\mathrm{HDR}(\Sigma, k, s, \ell, p)$ holds true provided (13), (14), (15), (16), and (17) are satisfied. If (18) does not hold, then $\mathrm{HDR}(\Sigma, k, s, \ell, p)$ holds true provided (13) − (17) are satisfied as well as the inequality

$$(19) \qquad s \geq k - \frac{\kappa_p - k}{\kappa_p - \left\lceil \frac{\kappa_p}{2} \right\rceil}.$$

Here and in what follows, $\lceil \cdot \rceil$ is the upper integer part of a number, i.e. the smallest integer that is greater or equal to the number; the notation $\lfloor \cdot \rfloor$ denotes the lower integer part of a number, i.e. the largest integer that does not exceed the number:

$$(20) \qquad \lfloor x \rfloor = \sup\{z \in \mathbb{Z} \mid z \leq x\}; \quad \lceil x \rceil = \inf\{z \in \mathbb{Z} \mid z \geq x\}.$$

The $s = 0$, $k = 1$ cases of Theorem 1.12 illustrate its ability to extract derivatives of $\hat{f}$ in all directions when only regularity along Σ is assumed.

Corollary 1.13. *Suppose* $p = \frac{2d+2}{d+3+2m}$ *for some integer* $2 \leq m < \frac{d-1}{2}$, *and let* $\ell \geq \frac{2d+2-p(d+3)}{2d+2-p(d+5)} = \frac{m}{m-1}$. *Then*

$$(21) \qquad \left\| \nabla \hat{f} \right\|_{L_2(K)} \lesssim_{K,\phi} \left(\|f\|_{L_p(\mathbb{R}^d)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)} \right).$$

for any Schwartz function f, *compact subset* $K \subset \Sigma$, *and smooth cutoff* ϕ *that is identically 1 on* K.

In Section 6.4 we construct a translated Knapp example to show that the lower bound for ℓ is sharp.

The property $\mathrm{HDR}(\Sigma, k, s, \ell, p)$ has a dual formulation in terms of the Fourier extension operator. We denote the Lebesgue measure on Σ by $d\sigma$.

Corollary 1.14. *Suppose* $\mathrm{HDR}(\Sigma, k, s, \ell, p)$ *holds true and* $p < \frac{2d}{d+1}$ *(e.g. if the conditions of Theorem 1.12 or Theorem 5.21 are satisfied). Then for each* $g \in H^s(\Sigma)$, *multi-index* α *with* $|\alpha| \leq k$, *and smooth compactly supported* ϕ, *there exist* $F_\alpha \in L_{p'}(\mathbb{R}^d)$ *and* $g_\alpha \in H^{-\ell}(\Sigma)$ *such that*

$$(22) \qquad F_\alpha + (\phi g_\alpha \, d\sigma)^\vee = x^\alpha (\phi g \, d\sigma)^\vee,$$

and furthermore

$$(23) \qquad \|F_\alpha\|_{L_{p'}(\mathbb{R}^d)} + \|g_\alpha\|_{H^{-\ell}(\Sigma)} \lesssim_\phi \|g\|_{H^s(\Sigma)}.$$

Conversely, if for any compactly supported smooth function ϕ, *for any* g, *and for any* α *there exist* F_α *and* g_α *such that* (22) *and* (23), *then* $\mathrm{HDR}(\Sigma, k, s, \ell, p)$ *holds true (we still assume* $p < \frac{2d}{d+1}$).

When Σ is the paraboloid $\xi_d = \frac{1}{2}|\xi|^2$, the Fourier extension operator doubles as the linear propagator of the Schrödinger equation on $\mathbb{R}^n$ with $n = d - 1$. In this context Corollary 1.13 implies a time-weighted scattering property for solutions of the Schrödinger equation.

Corollary 1.15. *Let* m *be an integer* $2 \leq m < \frac{n}{2}$. *Given* $g \in L_2(\mathbb{R}^n)$ *with Fourier support in the unit ball, there exists a function* $\tilde{g}$, *also with Fourier support in the unit ball, such that*

$$(24) \qquad \left\| (1 + |x|)^{-\frac{m}{m-1}} \tilde{g} \right\|_{L_2} \lesssim \|g\|_{L_2}$$

and

$$(25) \qquad \left\| t e^{it\Delta} g - e^{it\Delta} \tilde{g} \right\|_{L_{\frac{2n+4}{n-2m}}(\mathbb{R}^{n+1})} \lesssim \|g\|_{L_2(\mathbb{R}^n)}.$$

If n *is even, the result holds for* $m = \frac{n}{2}$ *provided the exponent of* $(1 + |x|)$ *in* (24) *is strictly less than* $-\frac{n}{n-2}$.

Finally, we present the main analytic tool used in our proofs of HDR inequalities. We will formulate it in local form: now Σ is a graph of a function on $\mathbb{R}^{d-1}$ rather than an arbitrary submanifold.

Let U be a neighborhood of the origin in $\mathbb{R}^{d-1}$. Let h be a C^∞-smooth function on U such that $h(0) = 0$ and $\nabla h(0) = 0$. We also assume that the Hessian of h at zero does not vanish,

$$\det \frac{\partial^2 h}{\partial \zeta^2}(0) \neq 0.$$

Moreover, we assume that the gradient of h is sufficiently close to zero and the second differential is sufficiently close to $\frac{\partial^2 h}{\partial \zeta^2}(0)$:

$$(26) \qquad \forall \zeta \in U \quad \left\| \frac{\partial h}{\partial \zeta}(\zeta) \right\| \leq \frac{1}{10d}, \quad \left\| \frac{\partial^2 h}{\partial \zeta^2}(\zeta) - \frac{\partial^2 h}{\partial \zeta^2}(0) \right\| \leq \frac{1}{10} \left| \det \frac{\partial^2 h}{\partial \zeta^2}(0) \right|.$$

The function h naturally defines the family of surfaces

$$\Sigma_r = \big\{ (\zeta, h(\zeta) + r) \,\big|\, \zeta \in U \big\}, \qquad r \in (-\infty, \infty).$$

We also take some small number $\varepsilon > 0$ and consider the set $V = U \times (-\varepsilon, \varepsilon)$.

We will be using Bessel potential spaces adjusted to these surfaces. Now we will need the precise quantity defining the Bessel norm. It is convenient to parametrize everything with U. For $\gamma \in \mathbb{R}$ and a compactly supported function ϕ on Σ_r (for some fixed r), define its $H^{-\gamma}$-norm by the formula

$$(27) \qquad \|\phi\|_{H^{-\gamma}(\Sigma_r)}^2 = \int\limits_{\mathbb{R}^{d-1}} \left| \mathcal{F}_{\zeta \to z}\big[\phi(\zeta, h_r(\zeta)) \big](z) \right|^2 \big(1 + |z| \big)^{-2\gamma} \, dz.$$

The symbol $\mathcal{F}$ denotes the Fourier transform in $(d-1)$ variables, and we have used the notation $h_r(\zeta) = h(\zeta) + r$. We will also use the homogeneous norm

$$\|\phi\|_{\dot{H}^{-\gamma}(\Sigma_r)}^2 = \int\limits_{\mathbb{R}^{d-1}} \left| \mathcal{F}_{\zeta \to z}\big[\phi(\zeta, h_r(\zeta)) \big](z) \right|^2 |z|^{-2\gamma} \, dz, \quad \gamma \in \left(0, \frac{d-1}{2} \right).$$

Since all our functions are supported on U, this norm is equivalent to the inhomogeneous norm (27) when $\gamma \in (0, \frac{d-1}{2})$. We will often use another formula for the homogeneous norm:

$$(28) \qquad \|\phi\|_{\dot{H}^{-\gamma}(\Sigma_r)}^2 = C_{d,\gamma} \iint\limits_{U \times U} \phi(\zeta, h_r(\zeta)) \overline{\phi(\eta, h_r(\eta))} |\zeta - \eta|^{2\gamma - d + 1} \, d\zeta \, d\eta, \quad \gamma \in \left(0, \frac{d-1}{2} \right).$$

The constant $C_{d,\gamma}$ may be computed explicitly, however, we do not need the sharp expression for it.

Let α and β be integers between 0 and $\frac{d-1}{2}$, let $\gamma \in [0, \frac{d-1}{2})$ be real, let $p \in [1, \infty]$. Let also ψ be an arbitrary C_0^∞ function supported in U. We are interested in the differentiated restriction inequality

$$(29) \qquad \left| \left(\frac{\partial}{\partial r} \right)^\beta \left\| \frac{\partial^\alpha \hat{f}}{\partial \xi_d^\alpha} \psi \right\|_{\dot{H}^{-\gamma}(\Sigma_r)}^2 \bigg|_{r=0} \right| \lesssim \|f\|_{L_p(\mathbb{R}^d)}^2.$$

So we compute the Fourier transform of an L_p function, calculate its derivative with respect to d-th coordinate, compute the $\dot{H}^{-\gamma}$-norms of traces of this derivative on the surfaces Σ_r, and then differentiate β times with respect to r. We use the variable ξ for points in $\mathbb{R}^d$ on the spectral side and ζ for points in $\mathbb{R}^{d-1}$ decoding points on Σ_r (for example, ξ is quite often equal to $(\zeta, h(\zeta))$).

The crucial statement which unlocks most results in this paper is a sharp characterization of when (29) is valid.

Theorem 1.16. *Let $h \in C^\infty(U)$ satisfy the assumptions above, and let $\gamma \in [0, \frac{d-1}{2})$. Inequality (29) is true for the combination of $(\alpha, \beta, \gamma, p)$ if and only if*

$$(30) \qquad\qquad\qquad \alpha \leq \gamma,$$

$$(31) \qquad\qquad\qquad \alpha + \beta \leq \sigma_p,$$

$$(32) \qquad\qquad\qquad 2\alpha - \gamma + \beta \leq \kappa_p,$$

and the inequality (31) is strict if $p > 1$.

In the case $\alpha = \beta = \gamma = 0$, the estimate (29) reduces to the classical Stein–Tomas bound. Though our proof will follow the scheme of the fractional integration method (see e.g. [18], 11.2.2), both the $p = 1$ case and the interpolation of operators in the proof of (29) require significant new efforts.

Note that Proposition 1.1, except for the endpoint case $p = 1$, $k = \frac{d-1}{2}$, follows from choosing $\beta = 0$ in Theorem 1.16 and applying the localization argument given in Subsection 7.1 below. It is not clear to the authors whether one can derive the full statement of Theorem 1.16 from the results of [6] or from Proposition 1.1 (which correspond to the cases $\alpha = \beta = 0$ and $\beta = 0$ respectively).

Our approach using Theorem 1.16 seems to be a different strategy from the one in [6]. It makes possible the extensive family of conditional restriction estimates proved in Theorem 1.12 and its corollary. The method also allows us to work with Strichartz estimates, i.e. consider the larger scale of mixed-norm Lebesgue spaces on $\mathbb{R}^{d-1} \times \mathbb{R}$ on the right hand-side of (29).

The organization of the paper is as follows. Section 2 is a brief statement of some problems in the literature that provided motivation for the current work. Section 3 tackles the functional analysis of spaces $_\Sigma L_p^k$. Proposition 1.3 is proved here, and Theorems 1.4 and 1.6 are reduced to corollaries of Theorem 1.16. The proof of Theorem 1.16 takes up the entirety of Section 4. In Section 5 we present the argument deriving an expanded version of Theorem 1.12 from Theorem 1.16. Section 6 contains examples demonstrating the necessity of conditions in Theorems 1.4 and 1.16 as well as Proposition 1.11. The final section contains miscellaneous technical results: an argument for working locally on Σ, a Stein–Weiss inequality, and statements related to the $p = 1$, $k = \frac{d-1}{2}$ endpoint case of Proposition 1.1 in odd dimensions.

2. Precursors to the current work

Fredholm conditions. Functions whose Fourier transform vanish on a compact surface in $\mathbb{R}^d$, and in particular on a sphere, arise in the study of spectral theory of Schrödinger operators $H = -\Delta + V$. It is well known that the Laplacian has absolutely continuous spectrum on the positive halfline $[0, \infty)$, and no eigenvalues or singular continuous spectrum. If $V(x)$ can be approximated by bounded, compactly supported functions in a suitable norm (for example $V \in L_{d/2}(\mathbb{R}^d)$ suffices when $d \geq 3$), then H is a relatively compact perturbation of the Laplacian and may have countably many eigenvalues with a possible accumulation point at zero. If V is real-valued, then H is a self-adjoint operator whose eigenvalues must all be real numbers as well.

It is not immediately obvious how the continuous spectrum of H relates to that of the Laplacian, and whether any eigenvalues are embedded within it. An argument due to Agmon [1] proceeds as follows. Suppose ψ is a formal solution of the eigenvalue equation $(-\Delta - \lambda)\psi = -V\psi$ for some $\lambda > 0$. Then

$$(33) \qquad \psi = - \lim_{\epsilon \to 0^+} (-\Delta - (\lambda + i\epsilon))^{-1} V\psi$$

from which it follows that the imaginary parts of $\langle V\psi, \psi \rangle$ and $- \lim_{\epsilon \to 0} \langle V\psi, (-\Delta - (\lambda + i\epsilon))^{-1} V\psi \rangle$ must agree. The former is clearly zero since $V(x)$ is real-valued. The latter turns out to be a multiple of $\|(V\psi)\hat{\ }|_\Sigma\|_{L_2(\Sigma)}^2$, where Σ is the sphere $\{\xi \in \mathbb{R}^d \mid |\xi|^2 = \lambda\}$. Hence $(V\psi)\hat{\ }$ vanishes on the sphere of radius $\sqrt{\lambda}$.

It is not surprising that the Fourier multiplication operator $m_\epsilon(\xi) = (|\xi|^2 - (\lambda + i\epsilon))^{-1}$ might have favorable mapping properties when applied specifically to $V\psi$, whose Fourier transform vanishes where $m_\epsilon(\xi)$ is greatest. Bootstrapping arguments using (33) show that $\psi \in L_2(\mathbb{R}^d)$, even if it was not assumed *a priori* to belong to that space.

Viewed another way, the eigenvalue problem $(-\Delta - \lambda)\psi = -V\psi$ is an inhomogeneous partial differential equation were the principal symbol is elliptic. The Fredholm condition for existence of solutions is that $-V\psi$ should be orthogonal to the null-space of the adjoint operator $(-\Delta - \lambda)^*$. As we will argue later

in Subsection 3.1, this nullspace consists of all distributions whose Fourier transform acts as a linear functional on $C^\infty(\Sigma)$. Thus $V\psi$ satisfies the Fredholm condition precisely if $(V\psi)\widehat{\,}|_\Sigma = 0$.

The analysis in [1] is carried out in polynomially weighted $L_2(\mathbb{R}^d)$ and applies to a wide family of elliptic differential operators $H = P(i\nabla) + V$. The main non-degeneracy condition is that the gradient of P does not vanish on the level set $\{\xi \mid P(\xi) = \lambda\}$. Similar arguments in [12], [15] and [11] are carried out (for $P(\xi) = |\xi|^2$) in $L_q(\mathbb{R}^d)$ and related Sobolev spaces for various ranges of q. Curvature of the level sets of P is a crucial feature in these works, as it is in the present paper.

Sobolev type inequalities. We start with the classical Sobolev Embedding Theorem

$$\|f\|_{L_q} \lesssim \|\nabla f\|_{L_p}, \quad f \in C_0^\infty(\mathbb{R}^d), \quad q = \frac{dp}{d-p}, \quad 1 \le p < d, d \ge 2.$$

For $p > 1$, it follows from the Hardy–Littlewood–Sobolev inequality. In the limiting case $p = 1$, the Hardy–Littlewood–Sobolev inequality fails, however, as it was proved by Gagliardo and Nirenberg, the Sobolev Embedding holds. This happens because the space

$$\dot{W}_1^1(\mathbb{R}^d) = \mathrm{clos}_{L_1}\left(\left\{\nabla f \,\Big|\, f \in C_0^\infty(\mathbb{R}^d)\right\}\right)$$

is strictly narrower than L_1 (they are even non-isomorphic as Banach spaces). Later, it was observed that there are many similar inequalities where ∇f may be replaced with a more complicated differential vector-valued expression (see [4], [19], and the survey [20]).

In [24], the second named author studied the anisotropic bilinear inequality

$$(34) \qquad \left|\langle f, g\rangle_{\dot{W}_2^{\alpha,\beta}(\mathbb{R}^2)}\right| \lesssim \left\|(\partial_1^k - \tau\partial_2^l)f\right\|_{L_1(\mathbb{R}^2)} \left\|(\partial_1^k - \sigma\partial_2^l)g\right\|_{L_1(\mathbb{R}^2)}, \quad f, g \in C_0^\infty(\mathbb{R}^2)$$

(such type inequalities were used in [17] for purposes of Banach space theory). Here $\dot{W}_2^{\alpha,\beta}(\mathbb{R}^2)$ is the anisotropic Bessel potential space equipped with the norm

$$\|f\|_{\dot{W}_q^{\alpha,\beta}} = \left\|\mathcal{F}^{-1}\Big(\hat{f}(\xi,\eta)|\xi|^\alpha|\eta|^\beta\Big)\right\|_{L_q},$$

the symbols σ and τ denote complex scalars, and ∂_1 and ∂_2 are partial derivatives with respect to the first and the second coordinates correspondingly. It appeared that (34) holds even in the cases where the differential polynomials on the right hand-side are not elliptic, however, this may happen only in the anisotropic case $k \ne l$. This lead to natural conjecture that the inequality

$$(35) \qquad \|f\|_{\dot{W}_q^{\alpha,\beta}} \lesssim \|(\partial_1^k - \sigma\partial_2^l)f\|_{L_p}, \quad \frac{\alpha}{k} + \frac{\beta}{l} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)\left(\frac{1}{k} + \frac{1}{l}\right), k \ne l, p > 1, q < \infty,$$

might hold true. We are especially interested in the case where the operator on the right hand-side is non-elliptic, that is $i^{l-k}\sigma \in \mathbb{R}$. Assume this is so. Similar to the classical proof of the Sobolev Embedding Theorem, one may express f in terms of $(\partial_1^k - \sigma\partial_2^l)f$ using a certain integral operator. This will be a Bochner–Riesz type operator of order -1 with the singularity on the curve

$$\Gamma_{k,l} = \left\{(\xi,\eta) \in \mathbb{R}^2 \,\Big|\, (2\pi i\xi)^k = \sigma(2\pi i\eta)^l\right\}.$$

Note that this curve is convex outside the origin. Application of the Littlewood–Paley inequality and homogeneity considerations (see [24]) reduce (35) to the case where the spectrum of f lies in a small neighborhood of a point on $\Gamma_{k,l}$. So, by the results of [2], the inequality (35) is true if $\frac{1}{p} - \frac{1}{q} \ge \frac{2}{3}$, $p < \frac{4}{3}$, and $q > 4$. Moreover, one may construct examples to show that the conditions $\frac{1}{p} - \frac{1}{q} \ge \frac{2}{3}$ and $p < \frac{4}{3}$ are necessary.

Note that the Fourier transform of the function $(\partial_1^k - \sigma\partial_2^l)f$ vanishes on $\Gamma_{k,l}$, which is a smooth convex curve in the plane (with, possibly, a singularity at zero). Thus, we need to analyze the action of Bochner–Riesz-type operator on the space $_\Sigma L_p$ with $\Sigma = \Gamma_{k,l}$. It appears that passing to a narrower space allows to get rid of the condition $q > 4$. This work was done half year later in [11].

Theorem 2.1 ([24]+[11]). *The inequality* (35) *holds true if* $\frac{1}{p} - \frac{1}{q} \geq \frac{2}{3}$, $1 < p < \frac{4}{3}$, *and* $q < \infty$.

3. STUDY OF THE SPACES $_\Sigma L_p^k$

3.1. Description of the annihilator and Domar's theory.
Let ∂_Σ^l denote the operator of normal derivative of order l with respect to Σ, $\partial_\Sigma : \mathcal{S}(\mathbb{R}^d) \to C^\infty(\Sigma)$,

$$\partial_\Sigma^l[\Phi](\xi) = \frac{\partial^l \Phi}{\partial \mathrm{n}_\Sigma^l(\xi)}(\xi), \quad \Phi \in \mathcal{S}(\mathbb{R}^d), \xi \in \Sigma.$$

The symbol $\mathrm{n}_\Sigma(\xi)$ denotes the normal vector to Σ at the point ξ. In particular, $\partial_\Sigma^0[\Phi]$ is simply the restriction of Φ to Σ.

There are conjugate operators $(\partial_\Sigma^l)^* : (C^\infty(\Sigma))' \to \mathcal{S}'(\mathbb{R}^d)$. We can also form an operator $\mathrm{J}_\Sigma^l : \mathcal{S}(\mathbb{R}^d) \to \bigoplus_{0 \leq s \leq l} C^\infty(\Sigma)$ composed of pure normal derivatives:

$$\mathcal{S}(\mathbb{R}^d) \ni \Phi \mapsto \mathrm{J}_\Sigma^l[\Phi] = \left(\partial_\Sigma^0[\Phi], \partial_\Sigma^1[\Phi], \ldots, \partial_\Sigma^l[\Phi] \right) \in \bigoplus_{0 \leq s \leq l} C^\infty(\Sigma).$$

This operator also has an adjoint, which maps a vector-valued distribution $\Lambda = \{\Lambda_s\}_{0 \leq s \leq l}$ with compact support on Σ to a Schwartz distribution on $\mathbb{R}^d$.

Lemma 3.1. *Let* $p \in [1, \infty)$. *The annihilator of* $_\Sigma L_p^k$ *in* $L_{p'}$ *can be described as*

$$(36) \qquad \mathrm{Ann}_{L_{p'}}(_\Sigma L_p^k) = \mathrm{clos}_{L_{p'}} \left(\left\{ g \in L_{p'} \,\Big|\, \exists \Lambda \in \bigoplus_{0 \leq s \leq k-1} \left(C^\infty(\Sigma)\right)' \quad \textit{such that} \quad \hat{g} = \left(\mathrm{J}_\Sigma^{k-1}\right)^*[\Lambda] \right\} \right).$$

Remark 3.2. Since the distribution Λ has compact support, g has bounded spectrum. Clearly, if Σ is not compact, one may construct a function g in the annihilator of $_\Sigma L_p$ whose spectrum is not bounded. That is why we need to add closure on the right hand-side of (36). In the case where Σ is compact, this is not needed:

$$\mathrm{Ann}_{L_{p'}}(_\Sigma L_p^k) = \left\{ g \in L_{p'} \,\Big|\, \exists \Lambda \in \bigoplus_{0 \leq s \leq k-1} \left(C^\infty(\Sigma)\right)' \quad \text{such that} \quad \hat{g} = \left(\mathrm{J}_\Sigma^{k-1}\right)^*[\Lambda] \right\}, \quad \Sigma \text{ is compact.}$$

The proof of Lemma 3.1 presented below also simplifies in the case where Σ is compact. The functions ψ and Ψ may be omitted in this case.

We will need a technical fact to prove Lemma 3.1. It is standard, so we omit its proof.

Lemma 3.3. *For any bounded domain* Ω, *consider the subspace* $C^\infty(\Sigma, \Omega, l)$ *of vector-valued functions in* $\bigoplus_{0 \leq s \leq l} C^\infty(\Sigma)$ *supported in* $\Omega \cap \Sigma$. *There exists a linear operator* $\mathrm{Ext}_{\Omega, l} : C^\infty(\Sigma, \Omega, l) \to \mathcal{S}(\mathbb{R}^d)$, *which is inverse to* J_Σ^l *in the sense*

$$\forall \varphi \in C^\infty(\Sigma, \Omega, l) \quad \mathrm{J}_\Sigma^l \big[\mathrm{Ext}_{\Omega, l}[\varphi] \big] = \varphi.$$

Proof of Lemma 3.1. First, we note that since $_\Sigma L_p^k$ is a translation invariant space, the set of functions g with compact spectrum is dense in $\mathrm{Ann}_{L_{p'}}(_\Sigma L_p^k)$. Consider such a function g. It suffices to construct

$$\Lambda \in \bigoplus_{0 \leq s \leq k-1} \left(C^\infty(\Sigma)\right)'$$

such that $\hat{g} = \left(\mathrm{J}_\Sigma^{k-1}\right)^*[\Lambda]$.

Let Ω be a bounded domain containing the spectrum of g. Consider the operator $\mathrm{Ext}_{\Omega, k-1}$ constructed in Lemma 3.3 and define Λ (as a functional on $\bigoplus_{0 \leq s \leq k-1} C^\infty(\Sigma)$) by the formula

$$\Lambda[\varphi] = \hat{g} \big[\mathrm{Ext}_{\Omega, k-1}[\psi \varphi] \big], \quad \varphi \in \bigoplus_{0 \leq s \leq k-1} C^\infty(\Sigma),$$

where ψ is a smooth function on Σ supported in Ω that is equal to one in a neighborhood of $\operatorname{supp}\hat{g}\cap\Sigma$. We are required to show that $\hat{g} = \left(\mathrm{J}_\Sigma^{k-1}\right)^*[\Lambda]$, which becomes

$$\langle\hat{g},\Phi\rangle = \langle\hat{g}, \mathrm{Ext}_{\Omega,k-1}[\psi\mathrm{J}_\Sigma^{k-1}[\Phi]]\rangle$$

for every $\Phi\in\mathcal{S}(\mathbb{R}^d)$. Since $\psi = 1$ in a neighborhood of the support of $\hat{g}$, we may write

$$\langle\hat{g}, \mathrm{Ext}_{\Omega,k-1}[\psi\mathrm{J}_\Sigma^{k-1}[\Phi]]\rangle = \langle\hat{g}, \mathrm{Ext}_{\Omega,k-1}[\mathrm{J}_\Sigma^{k-1}[\Psi\Phi]]\rangle \quad\text{and}\quad \langle\hat{g},\Phi\rangle = \langle\hat{g},\Psi\Phi\rangle,$$

where Ψ is smooth function supported in Ω that equals one in a neighborhood of $\operatorname{supp}\hat{g}$. It remains to prove

$$\left\langle\hat{g}, \left(\mathrm{id} - \mathrm{Ext}_{\Omega,k-1}\circ\mathrm{J}_\Sigma^{k-1}\right)[\Psi\Phi]\right\rangle = 0.$$

Since $\operatorname{Ker}\mathrm{J}_\Sigma^{k-1}\subset\operatorname{Ker}\hat{g}$ (recall that g annihilates $_\Sigma L_p^k$), it suffices to show that

$$\mathrm{J}_\Sigma^{k-1}\circ\left(\mathrm{id} - \mathrm{Ext}_{\Omega,k-1}\circ\mathrm{J}_\Sigma^{k-1}\right) = 0,$$

which holds true since $\mathrm{J}_\Sigma^{k-1}\circ\mathrm{Ext}_{\Omega,k-1} = \mathrm{id}$ by construction of $\mathrm{Ext}_{\Omega,k-1}$. $\qquad\square$

Lemma 3.4. *The set*

$$\left\{g\in L_{p'} \,\Big|\, \exists\Lambda\in\bigoplus_{0\leq s\leq k-1} C_0^\infty(\Sigma) \quad\text{such that}\quad \hat{g} = \left(\mathrm{J}_\Sigma^{k-1}\right)^*[\Lambda]\right\}$$

is dense in $\operatorname{Ann}_{L_{p'}}(_\Sigma L_p^k)$ *if* $p > 1$. *In the case* $p = 1$, *this set is weakly dense.*

Proof. The case $p = 1$ had been considered in [7]. We repeat the argument for the general case here. Let g be a function in the said annihilator. After applying a partition of unity, we may assume that the corresponding vector-valued distribution Λ provided by Lemma 3.1 is supported in a chart neighborhood V of a point $\xi\in\Sigma$, as it will only be necessary to sum a finite number of such pieces. We may also suppose that $\xi = 0$ and

$$(37)\qquad\qquad \Sigma\cap V = \{(\zeta, h(\zeta))\,|\,\zeta\in U\},$$

here U is a neighborhood of the origin in $\mathbb{R}^{d-1}$ and $h\colon\mathbb{R}^{d-1}\to\mathbb{R}$ is a smooth function such that $h(0) = 0$, $\nabla h(0) = 0$ (see Subsection 7.1 for details). By our assumptions on the principal curvatures of Σ, the second differential $\frac{\partial^2 h}{\partial\zeta^2}$ is non-degenerate on U. Consider the operator S that makes $\Sigma\cap V$ flat:

$$(38)\qquad\qquad S[\Phi](\xi) = \Phi\left(\xi_{\bar{d}}, \xi_d + h(\xi_{\bar{d}})\right), \quad \Phi\in C_0^\infty(V), \quad \xi\in\mathbb{R}^d.$$

We use the notation $\xi = (\xi_{\bar{d}}, \xi_d)$, so ξ_d is the last coordinate of ξ and $\xi_{\bar{d}}\in\mathbb{R}^{d-1}$ is the vector consisting of first $d-1$ coordinates.

Let ψ be a compactly supported smooth function on $\mathbb{R}^{d-1}$ with unit integral, let $\psi_n(\zeta) = n^{d-1}\psi(n\zeta)$ be its dilations. The function $\xi\mapsto\psi_n(\xi_{\bar{d}})$ is also denoted by ψ_n. Consider the family of operators D_n, $n\geq n_0$, n_0 is sufficiently large, given by the rule

$$(39)\qquad\qquad D_n[\Phi] = \mathcal{F}^{-1}\left[S^{-1}\left[S[\Psi\hat{\Phi}]*\psi_n\right]\right], \quad \Phi\in\mathcal{S}(\mathbb{R}^d).$$

Here $\Psi\in C_0^\infty(V)$ is a function that equals one on the support of Λ. It is clear that the D_n are uniformly bounded as operators on $L_2(\mathbb{R}^d)$. Lemma 4.2 in [7] says that the D_n are also (uniformly in n) bounded as operators on L_1, and since the dual operators have an identical structure they are also bounded on L_∞. By interpolation, the D_n are uniformly bounded on $L_{p'}$. Also, since $\{\psi_n\}_{n\in\mathbb{N}}$ is an approximate identity,

$$\|D_n[\Phi] - \Phi\|_{L_{p'}} \to 0 \quad\text{when}\quad \Phi\in\mathcal{S}(\mathbb{R}^d).$$

Thus, for every $g \in L_{p'}$, $p' < \infty$, we have $\|D_n[g] - g\|_{L_{p'}} \to 0$. It remains to notice that D_n maps compactly supported distributions of the form

$$\mathcal{F}^{-1}\Big[\big(\mathrm{J}_\Sigma^{k-1}\big)^*[\Lambda]\Big], \quad \Lambda \in \bigoplus_{0 \le s \le k-1} \big(C^\infty(\Sigma)\big)',$$

to the ones for which $\Lambda \in \bigoplus_{0 \le s \le k-1} C_0^\infty(\Sigma)$. Thus, if $g \in \mathrm{Ann}_{L_{p'}}({}_\Sigma L_p^k)$ is a function with bounded spectrum, then $D_n[g] \in \mathrm{Ann}_{L_{p'}}({}_\Sigma L_p^k)$, $D_n[g]$ is generated by smooth Λ, and $D_n[g] \to g$ in $L_{p'}$. $\qquad\square$

Proof of Proposition 1.3. By Lemma 3.1, it suffices to show that any function $g \in \mathrm{Ann}_{L_{p'}}({}_\Sigma L_p^{k+1})$ can be approximated by functions in $\mathrm{Ann}_{L_{p'}}({}_\Sigma L_p^k)$ when $k \ge \frac{d}{p} - \frac{d+1}{2}$, and $k > \frac{d-1}{2}$ when $p = 1$. By Lemma 3.4, we may assume that

(40) $$\hat{g} = \big(\mathrm{J}_\Sigma^k\big)^*[\Lambda], \qquad \Lambda \in \bigoplus_{0 \le s \le k} C_0^\infty(\Sigma).$$

It suffices to prove that $\Lambda_k = 0$, where $\Lambda = (\Lambda_0, \Lambda_1, \ldots, \Lambda_k)$. We may suppose that Λ is supported in a neighborhood V of a point on Σ. We may also assume (37) and replace normal derivatives by derivatives with respect to ξ_d:

$$\hat{g} = \sum_{s=0}^{k} \frac{\partial^s \tilde{\Lambda}_s}{\partial \xi_d^s},$$

where $\tilde{\Lambda}_s$ are distributions generated by complex measures on $\Sigma \cap V$ whose densities with respect to the Lebesgue measure on Σ are smooth functions. Note that $\Lambda_k = 0$ whenever $\tilde{\Lambda}_k = 0$. Since each function $\tilde{\Lambda}_s$ has smooth density with respect to the Lebesgue measure on Σ, one may use the stationary phase method to compute the asymptotics of $\mathcal{F}^{-1}[\tilde{\Lambda}_s]$ at infinity (see, e.g. [22]):

$$\mathcal{F}^{-1}[\tilde{\Lambda}_s] = e(x)|x|^{-\frac{d-1}{2}} + O(|x|^{-\frac{d+1}{2}})$$

for all x such that $x \parallel \mathrm{n}_\Sigma(\xi)$ for some $\xi \in \Sigma$ with $\tilde{\Lambda}_s(\xi) \ne 0$. Here $e(x)$ is a non-zero oscillating factor with constant amplitude that depends on h and the density of $\tilde{\Lambda}_s$. This shows that

$$|\hat{g}(x)| \asymp |x|^{k-\frac{d-1}{2}}, \quad x \to \infty, \quad x \parallel \mathrm{n}_\Sigma(\xi), \tilde{\Lambda}_k(\xi) \ne 0.$$

On the other hand, $\hat{g} \in L_{p'}$, which for $p > 1$ requires $p'(k - \frac{d-1}{2}) < -d$, equivalently $k < \frac{d}{p} - \frac{d+1}{2}$, contradicting our assumptions. Therefore, $\Lambda_k = 0$ and, thus, $\mathrm{Ann}_{L_{p'}}({}_\Sigma L_p^{k+1}) = \mathrm{Ann}_{L_{p'}}({}_\Sigma L_p^k)$. If $p = 1$ the contradiction is reached provided $k - \frac{d-1}{2} > 0$.

To show that ${}_\Sigma L_p^k = {}_\Sigma L_p^\infty$, we note that the annihilator of the latter space consists of all $L_{p'}$ functions whose Fourier transform is supported on Σ, recall the Schwartz theorem that any distribution supported on Σ may be represented in the form $\big(\mathrm{J}_\Sigma^l\big)^*[\zeta]$ for some l, and use the reasoning above. $\qquad\square$

3.2. Coincidence of ${}_\Sigma L_p^k$ and the spaces defined as kernels of restriction operators. We relate the ${}_\Sigma L_p^k$ spaces with restriction operators. Consider a neighborhood V such that (37) holds true. We may redefine V in such a way that

$$V = \{\xi \in \mathbb{R}^d \mid |\xi_d - h(\xi_{\bar{d}})| < \delta, \ \xi_{\bar{d}} \in U\}.$$

This gives a natural parametrization of V by $U \times (-\delta, \delta)$. We will need the translated copies of Σ:

$$\Sigma_r = \Sigma + (0, 0, \ldots, 0, r), \quad r \in (-\delta, \delta).$$

Note that this definition depends on the choice of U.

Consider the restriction operators

(41) $$\mathcal{R}_{\Sigma,r}[f] = \hat{f}|_{V \cap \Sigma_r}, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Definition 3.5. *We say that the statement $\mathcal{R}(\Sigma, p, s)$ holds true if the $\mathcal{R}_{\Sigma, r}$ admit continuous extensions as $L_p(\mathbb{R}^d) \to H^{-s}(V \cap \Sigma_r)$ operators for any choice of U, and the norms of these extensions are uniform in r (however, we do not require any uniformity with respect to U).*

We say that $\mathcal{R}^k(\Sigma, p, s)$ is true if $\mathcal{R}^{k-1}(\Sigma, p, s)$ is true and for any choice of U the operators $\mathcal{R}_{\Sigma, r}$ extend continuously from the domain

$$\{f \in \mathcal{S}(\mathbb{R}^d) \mid \forall l < k \quad \nabla^l \hat{f} = 0 \text{ on } \Sigma\}$$

to a family of mappings ${}_\Sigma L_p^k \to H^{-s}(V \cap \Sigma_r)$ whose norms are bounded uniformly by $C|r|^k$.

Remark 3.6. In the definitions above it is important to be consistent with regard to the construction of local Sobolev norms on Σ_r. When we discuss $\mathcal{R}(\Sigma, p, s)$, we will define the Sobolev norm by the rule (28) for each particular choice of U, h, and r.

Definition 3.7. *For a fixed $s \geq 0$, define the set ${}_\Sigma \tilde{L}_p^k$ by the formula*

$${}_\Sigma \tilde{L}_p^k = \left\{ f \in L_p(\mathbb{R}^d) \,\middle|\, \forall U \quad \left\| \mathcal{R}_{\Sigma, r}[f] \right\|_{H^{-s}(V \cap \Sigma_r)} = o(|r|^{k-1}) \right\}.$$

Note that it is unclear whether ${}_\Sigma \tilde{L}_p^k$ is closed in L_p or not.

Lemma 3.8. *Suppose that Σ has non-vanishing curvature, $p \in [1, \infty)$, and $\mathcal{R}^{k-1}(\Sigma, p, s)$ holds. Then, ${}_\Sigma \tilde{L}_p^k = {}_\Sigma L_p^k$.*

Proof. It is clear that ${}_\Sigma L_p^k \subset \mathrm{clos}({}_\Sigma \tilde{L}_p^k)$, as ${}_\Sigma \tilde{L}_p^k$ contains all Schwartz functions whose Fourier transform vanishes to order $k-1$ on Σ. In fact, thanks to the uniform convergence implied by condition $\mathcal{R}^{k-1}(\Sigma, p, s)$ it is even true that ${}_\Sigma L_p^k \subset {}_\Sigma \tilde{L}_p^k$. So it suffices to show that ${}_\Sigma \tilde{L}_p^k \subset {}_\Sigma L_p^k$. Assume the contrary. By the Hahn–Banach theorem,

$$\exists f \in {}_\Sigma \tilde{L}_p^k, \ g \in \mathrm{Ann}_{L_{p'}}\left({}_\Sigma L_p^k \right), \qquad \langle f, g \rangle \neq 0.$$

By Lemma 3.4, we may assume that g is of the form

$$(42) \qquad\qquad \hat{g} = (\mathrm{J}_\Sigma^{k-1})^*[\zeta], \qquad \zeta \in \bigoplus_{0 \leq s \leq k-1} C_0^\infty(\Sigma).$$

Applying the same reasoning as in the proof of Lemma 3.4, we may also assume that ζ has compact support within a chart neighborhood V where $\Sigma \cap V$ is the graph of a smooth function $h : U \subset \mathbb{R}^{d-1} \to \mathbb{R}$, $h(0) = 0$, and $\nabla h(0) = 0$.

Recall the "flattening" operator S defined in (38). There exists another set of functions $\tilde{\zeta} \in \bigoplus_{0 \leq s \leq k-1} C_0^\infty(U)$ such that $(S^{-1})^*(\mathrm{J}_\Sigma^{k-1})^*[\zeta] = (\mathrm{J}_{\mathbb{R}^{d-1}}^{k-1})^*[\tilde{\zeta}]$. If one considers each component of ζ as an element of $C_0^\infty(U)$ via the parametrization of $\Sigma \cap V$, then the components of $\tilde{\zeta}$ are constructed from ζ, its gradients (in $\mathbb{R}^{d-1}$) up to order $k-1$, and the partial derivatives of h.

Let ψ be a compactly supported function on the unit interval whose integral equals one. Consider its dilations $\psi_n(\xi_d) = n\psi(n\xi_d)$ and the formal convolution in the d-th variable

$$(\mathrm{J}_{\mathbb{R}^{d-1}}^{k-1})^*[\tilde{\zeta}](\xi_{\bar{d}}, \cdot) * \psi_n(\xi_d) = \sum_{s=0}^{k-1} \tilde{\zeta}_s(\xi_{\bar{d}}) \frac{\partial^s \psi_n}{\partial \xi_d^s}(\xi_d),$$

which is a function in $C_0^\infty(U \times [-\frac{1}{n}, \frac{1}{n}])$. This function is bounded pointwise by the maximum size of $|\psi_n^{(k-1)}(\xi_d)|$, which is approximately n^k. All of its partial derivatives in the $\xi_{\bar{d}}$ directions are bounded by n^k as well, because those derivatives act on $\tilde{\zeta}$, not on ψ_n.

Now define $g_n \in \mathcal{S}(\mathbb{R}^d)$ by the formula

$$g_n(\xi_{\bar{d}}, \xi_d) = S^*[(\mathrm{J}_{\mathbb{R}^{d-1}}^{k-1})^*[\tilde{\zeta}](\xi_{\bar{d}}, \cdot) * \psi_n] = S^*[(S^{-1})^*(\mathrm{J}_\Sigma^{k-1})^*[\zeta](\xi_{\bar{d}}, \cdot) * \psi_n]$$

$$= S^*[(S^{-1})^*[\hat{g}](\xi_{\bar{d}}, \cdot) * \psi_n].$$

It should be clear that convolution in the ξ_d direction commutes with operators S^* and its inverse, thus the construction simplifies to

$$\hat{g}_n(\xi_{\bar{d}}, \xi_d) = \hat{g}(\xi_{\bar{d}}, \cdot) * \psi_n(\xi_d), \quad \text{or} \quad g_n(x) = g(x)\check{\psi}(nx_d).$$

It follows that $g_n \to g$ in $L_{p'}$ (in the case $p' = \infty$ we have weak-* convergence relative to L_1 instead), which means that, $\langle f, g_n \rangle \to \langle f, g \rangle$. On the other hand,

$$\|\mathcal{R}_{\Sigma,r}[g_n]\|_{H^s(U)} \lesssim \begin{cases} n^k, & |r| \lesssim n^{-1}; \\ 0, & |r| \gtrsim n^{-1}. \end{cases}$$

That gives a bound

$$|\langle f, g_n \rangle| = |\langle \hat{f}, \hat{g}_n \rangle| \lesssim \int_{-\frac{1}{n}}^{\frac{1}{n}} \int_{\Sigma_r} |\langle \hat{f}, \hat{g}_n \rangle| \, d\sigma dr \lesssim \frac{1}{n} \cdot n^k \cdot o(n^{1-k}) = o(1),$$

forcing $\langle f, g \rangle = \lim_{n \to \infty} \langle f, g_n \rangle = 0$. This contradicts the original assertion that $\langle f, g \rangle \neq 0$. $\square$

Definition 3.9. *We say that the statement $\mathcal{R}_{\mathrm{w}}^k(\Sigma, p, s)$ holds true if the mapping*

$$\mathcal{S}(\mathbb{R}^d) \ni f \mapsto \nabla^k f|_\Sigma \in C^\infty(\Sigma)$$

extends to a bounded linear operator between the spaces $_\Sigma L_p^k$ and $H^{-s}(K \cap \Sigma)$ for any compact set K.

It is explained in the Remark 7.3 below that $\mathcal{R}^k(\Sigma, p, s)$ leads to $\mathcal{R}_{\mathrm{w}}^k(\Sigma, p, s)$.

We end this subsection with an analog of Lemma 3.8 for HDR inequalities. The proof is direct, i.e. does not use duality.

Lemma 3.10. *For any function $f \in L_p$ such that $\phi \hat{f}|_\Sigma \in H^\ell(\Sigma)$, where $\phi \in C_0^\infty(\mathbb{R}^d)$, there exists a sequence $\{f_n\}_n$ of Schwartz functions such that*

$$\|f - f_n\|_{L_p} + \big\| \phi(\hat{f} - \hat{f}_n)\big|_\Sigma \big\|_{H^\ell(\Sigma)} \to 0.$$

Proof. As usual, we may assume that ϕ and $\hat{f}$ are supported in a chart neighborhood V of a point $\xi \in \Sigma$. We may also suppose that $\xi = 0$ and (37), where U is a neighborhood of the origin in $\mathbb{R}^{d-1}$ and $h \colon \mathbb{R}^{d-1} \to \mathbb{R}$ is a smooth function such that $h = 0$, $\nabla h = 0$ (see Subsection 7.1 for details). By Proposition 1.3, in the regime $\sigma_p \leq 0$, the set of Schwartz functions whose Fourier transform vanishes on Σ, is dense in L_p, and there is nothing to prove. Let us assume $\sigma_p > 0$.

We construct the functions F_n by the rule

$$F_n(x) = f(x)\Psi\left(\frac{x}{a_n}\right),$$

where Ψ is a fixed Schwartz function with $\Psi(0) = 1$ and bounded spectrum, and a_n is a large number such that $\|f - F_n\|_{L_p} \leq 2^{-n}$. The functions F_n approximate f in L_p norm, however, their Fourier transforms may have infinite $H^\ell(\Sigma)$ norms. There is a control on a weaker quantity, namely, Theorem 1.4 (in the case $k = 0$) says that

$$\|\phi(\hat{f} - \hat{F}_n)\|_{H^{-s}(\Sigma)} \lesssim 2^{-n} \tag{43}$$

for sufficiently large s.

Let now $f_n = D_n[F_n]$, where D_n is the Domar operator (39). We need to prove two limit identities

$$\|f - f_n\|_{L_p} \to 0, \quad \text{and} \quad \|\hat{f} - \hat{f}_n\|_{H^\ell(\Sigma)} \to 0.$$

The first identity is simple since

$$\|f - f_n\|_{L_p} \leq \|f - D_n f\|_{L_p} + \|D_n[f - F_n]\|_{L_p} \lesssim o(1) + 2^{-n}$$

by the properties of the operators D_n (see the proof of Lemma 3.4). For the second identity, we write

$$\|\phi(\hat{f} - \hat{f}_n)\|_{H^\ell(\Sigma)} \leq \|\phi\mathcal{F}[f - D_nf]\|_{H^\ell(\Sigma)} + \big\|\phi\mathcal{F}\big[D_n[f - F_n]\big]\big\|_{H^\ell(\Sigma)}.$$

Note that D_n convolves the restriction to Σ of the Fourier transform of the function with ψ_n (see (39)). Thus, the first summand tends to zero by the approximation of identity properties (and since $\hat{f}|_\Sigma \in H^\ell$), and the second summand is bounded by $O(n^{\ell+s}2^{-n})$ by formula (43). $\qquad\square$

3.3. Proofs of "if" part in Theorem 1.4 and Theorem 1.6. Since $\mathcal{R}^k(\Sigma, p, s)$ leads to $\mathcal{R}_{\mathrm{w}}^k(\Sigma, p, s)$, Theorem 1.4 follows from the lemma below.

Lemma 3.11. *The statement $\mathcal{R}^k(\Sigma, p, 0)$ holds true if $p \in [1, \frac{2d+2}{d+3+4k}]$. For every $p \in [1, \frac{2d}{d+1+2k})$ there exists $s \leq \max(0, [2k - \sigma_p] + 1, 2k - \kappa_p)$ such that $\mathcal{R}^k(\Sigma, p, s)$ is true. When $p = 1$ and $\sigma_1 = \frac{d-1}{2} \in \mathbb{N}$, the value $s = \max(0, 2k - \frac{d-1}{2})$ suffices. Finally, in odd dimensions $\mathcal{R}^{\frac{d-1}{2}}(\Sigma, 1, s)$ holds for $s > \frac{d-1}{2}$.*

Proof. Consider the case $s = 0$. It suffices to prove the bound

$$\big\|\psi(\cdot)\hat{f}(\cdot, h(\cdot) + r)\big\|_{L^2(U)}^2 \lesssim_\psi |r|^{2k}\|f\|_{{}_\Sigma L_p^k}^2, \quad \psi \in C_0^\infty(U).$$

By definition of ${}_\Sigma L_p^k$, we may assume that f is a Schwartz function. Then, the function Θ given by the rule

$$(44) \qquad\qquad \Theta(r) = \big\|\psi(\cdot)\hat{f}(\cdot, h(\cdot) + r)\big\|_{L^2(U)}^2$$

is smooth. We need to prove $|\Theta(r)| \lesssim |r|^{2k}\|f\|_{{}_\Sigma L_p^k}^2$, and for that, it suffices to show an inequality and several equalities.

The inequality is

$$(45) \qquad\qquad \forall r \in (-\delta, \delta) \quad \left|\frac{\partial^{2k}\Theta}{\partial r^{2k}}(r)\right| \lesssim \|f\|_{L_p}^2,$$

which follows from Theorem 1.16 (take $\alpha = 0, \beta = 2k, \gamma = 0$, Σ_r in the role of Σ, and notice that (30) is satisfied automatically, (32) is equivalent to $p \in [1, \frac{2d+2}{d+3+4k}]$, and (31) follows from (32) in this case).

The equalities are

$$(46) \qquad\qquad \forall j \in [0..2k - 1] \quad \frac{\partial^j\Theta}{\partial r^j}(0) = 0.$$

Indeed, we use the product rule:

$$(47) \qquad \frac{\partial^j}{\partial r^j}\left[\big\|\psi(\cdot)\hat{f}(\cdot, h(\cdot) + r)\big\|_{L^2(U)}^2\right]\bigg|_{r=0} = \sum_{i \leq j} C_j^i \left\langle \psi(\cdot)\frac{\partial^i\hat{f}}{\partial\xi_d^i}(\cdot, h(\cdot)), \ \psi(\cdot)\frac{\partial^{j-i}\hat{f}}{\partial\xi_d^{j-i}}(\cdot, h(\cdot))\right\rangle_{L^2(U)}$$

and notice that in each scalar product on the right hand-side, one of the functions is identically zero since either $i < k$ or $j - i < k$.

It remains to combine (45), (46), and the Taylor integral remainder formula to complete the proof in the case $s = 0$.

When $2k > \kappa_p$, the choice of $\alpha = 0$, $\beta = 2k$, and $\gamma = 0$ is no longer available in Theorem 1.16. Suppose $p > 1$. In order to use the product-rule argument above, one must set $2\alpha + \beta = 2k$, and it is desirable to keep α as small as possible since $\gamma \geq \alpha$ is a prominent lower bound for s. We can apply Theorem 1.16 with $\alpha = [2k - \sigma_p] + 1$, $\beta = 2k - 2\alpha$ (note that $\beta \geq 0$ and $\alpha \geq 0$ here), and $s = \gamma = \max(\alpha, 2k - \kappa_p)$, then follow the above steps for

$$\Theta_1(r) = \left\|\psi(\cdot)\Big(\frac{\partial}{\partial r}\Big)^\alpha \hat{f}(\cdot, h(\cdot) + r)\right\|_{H^{-s}(U)}^2$$

to conclude that $\frac{\partial^\beta \Theta_1}{\partial r^\beta}(r) \lesssim \|f\|_{L_p}^2$ for all $r \in (-\delta, \delta)$, and every lower order derivative vanishes at $r = 0$ because $f \in {}_\Sigma L_p^k$. Thus $\|\psi(\cdot)(\frac{\partial}{\partial r})^\alpha \hat{f}(\cdot, h(\cdot) + r)\|_{H^{-s}(U)} \lesssim |r|^{\beta/2}\|f\|_{{}_\Sigma L_p^k}$. Furthermore, $(\frac{\partial}{\partial r})^j \hat{f}(\cdot, h(\cdot) + r)$ is assumed to vanish at $r = 0$ for each $0 \le j < \alpha \le k$. The Taylor remainder formula and the Minkowski inequality show that $\|\hat{f}(\cdot, h(\cdot) + r)\| \lesssim |r|^{\alpha + \beta/2}\|f\|_{{}_\Sigma L_p^k}$, and we previously set $\alpha + \frac{\beta}{2} = k$.

When $p = 1$ and $k < \sigma_1 = \frac{d-1}{2} \in \mathbb{N}$, it is also permissible to apply Theorem 1.16 with $\alpha = 2k - \frac{d-1}{2}$, $\beta = 2k - 2\alpha$, and $s = \gamma = 2k - \frac{d-1}{2}$. In the endpoint case $p = 1, k = \frac{d-1}{2} \in \mathbb{N}$, Corollary 7.8 below directly states that $|\Theta_1(r)| \lesssim \|f\|_{L_1}^2$ for $s > \frac{d-1}{2}$ and $\alpha = k = \frac{d-1}{2}$. $\qquad\square$

Proof of Theorem 1.6. Clearly, ${}_\Sigma L_p^{k+1} \subset \text{Ker}\,\mathcal{R}^k$ whenever $\mathcal{R}^k$ is suitably defined as a map from ${}_\Sigma L_p^{k+1}$ into $H_{\text{loc}}^{-s}(\Sigma)$. To prove the reverse embedding, it suffices to show that $\text{Ker}\,\mathcal{R}^k \subset {}_\Sigma \tilde{L}_p^{k+1}$, since by Lemmas 3.11 (with the value of s specified there) and 3.8, we have ${}_\Sigma L_p^{k+1} = {}_\Sigma \tilde{L}_p^{k+1}$ for these choices of p and s.

We first consider the case $s = 0$, $p \in [1, \frac{2d+2}{d+3+4k}]$. By Definition 3.7, we need to prove

$$\Theta(r) = o(|r|^{2k}), \quad r \to 0, \quad f \in \text{Ker}\,\mathcal{R}^k$$

where the function Θ is defined by (44). By Taylor integral remainder formula and (46), we simply need to show a slight refinement of (45):

$$(48) \qquad\qquad \lim_{r \to 0}\left|\frac{\partial^{2k}\Theta}{\partial r^{2k}}(r)\right| = 0, \quad f \in \text{Ker}\,\mathcal{R}^k.$$

Note that (45) holds for all $f \in L_p$. By approximating any such f by Schwartz functions, we see that $\frac{\partial^{2k}\Theta}{\partial r^{2k}}(r)$ is also continuous in r. If $f \in {}_\Sigma L_p^k$, the computation in (47) with $j = 2k$ shows that $\frac{\partial^{2k}\Theta}{\partial r^{2k}}(0) = C_{2k}^k\|\psi\frac{\partial^k \hat{f}}{\partial \xi_d^k}\|_{L^2(U)}^2$, and for $f \in \text{Ker}\,\mathcal{R}^k \subset {}_\Sigma L_p^k$, the norm on the right hand-side is zero.

The remaining case is essentially the same. This time $\frac{\partial^\beta \Theta_1}{\partial r^\beta}(r)$ is continuous for all $f \in L_p$, the lower order derivatives vanish at $r = 0$ for $f \in {}_\Sigma L_p^k$, and finally $\frac{\partial^\beta \Theta_1}{\partial r^\beta}(0) = 0$ if $f \in \text{Ker}\,\mathcal{R}^k$. Thus $\frac{\partial^\beta \Theta_1}{\partial r^\beta}(r) = o(|r|^\beta)$ and the rest of the integrations are the same as in Lemma 3.11. $\qquad\square$

3.4. **Proof of Corollary 1.14.** Let X be the vector space of functions $\{f \in L_p(\mathbb{R}^d) \mid \phi\hat{f} \in H^\ell(\Sigma)\}$ equipped with the norm $\|f\|_X = \|f\|_{L_p} + \|\phi\hat{f}\|_{H^\ell(\Sigma)}$. This space contains all functions in the Schwartz class, and convergence with respect to the Schwartz class topology implies convergence in the norm of X. Thus every bounded linear functional on X belongs to the class of distributions $\mathcal{S}'(\mathbb{R}^d)$.

Lemma 3.10 asserts that the Schwartz class is dense in X. To show completeness of X, observe that by the $k = 0$ case of Proposition 1.1 (i.e. by [6]), if $f_n \to f$ in L_p, then there exists $s \ge \max(0, 2k - \kappa_p)$ so that $\phi\hat{f}_n \to \phi\hat{f}$ in $H^{-s}(\Sigma)$. Every Cauchy sequence in X has $\phi\hat{f}_n$ convergent to a limit in the stronger topology of $H^\ell(\Sigma)$, and the limit must be $\phi\hat{f}$ as well.

We may identify $f \in X$ with the ordered pair $(f, \phi\hat{f})$. This gives an isometric embedding of X into $L_p(\mathbb{R}^d) \times H^\ell(\Sigma)$. Its image is closed, so the Hahn–Banach theorem implies that every linear functional $\rho \in X'$ extends to a functional on $L_p(\mathbb{R}^d) \times H^\ell(\Sigma)$. Using Parseval's identity there exists $F_\rho \in L_{p'}(\mathbb{R}^d)$ and $g_\rho \in H^{-\ell}(\Sigma)$ with norms bounded by that of ρ and which satisfy

$$\rho(f) = \int_{\mathbb{R}^d} F_\rho f \, dx + \int_{\mathbb{R}^d} (\phi g_\rho \, d\sigma)^{\vee} f \, dx.$$

The defining property of $\text{HDR}(\Sigma, k, s, \ell, p)$ expressed in (12) is that the linear map $f \mapsto \phi D^\alpha \hat{f}|_\Sigma$ is continuous from X to $H^{-s}(\Sigma)$. The dual map, taking $g \mapsto (ix)^\alpha(\phi g \, d\sigma)^{\vee}$ therefore is bounded from $H^s(\Sigma)$ to X', with elements of X' described as above.

Remark 3.12. Due to the use of the Hahn–Banach theorem in this argument, we do not have a construction for F_α and g_α in Corollary 1.14. In fact, it is not proved here that these two functions can be chosen to depend linearly on $g \in H^s(\Sigma)$.

4. Proof of Theorem 1.16

4.1. Pointwise estimates of the kernel.
The quadratic inequality (29) is equivalent to its bilinear version

$$(49) \qquad \left| \left(\frac{\partial}{\partial r} \right)^\beta \left\langle \frac{\partial^\alpha \hat{f}}{\partial \xi_d^\alpha} \psi, \frac{\partial^\alpha \hat{g}}{\partial \xi_d^\alpha} \psi \right\rangle_{\dot{H}^{-\gamma}(\Sigma_r)} \Big|_{r=0} \right| \lesssim \|f\|_{L_p} \|g\|_{L_p}.$$

We denote the bilinear form we estimate by $\boldsymbol{B}$ and its kernel by $\boldsymbol{K}$:

$$(50) \qquad \boldsymbol{B}(f,g) = \left(\frac{\partial}{\partial r} \right)^\beta \left\langle \frac{\partial^\alpha \hat{f}}{\partial \xi_d^\alpha} \psi, \frac{\partial^\alpha \hat{g}}{\partial \xi_d^\alpha} \psi \right\rangle_{\dot{H}^{-\gamma}(\Sigma_r)} \Big|_{r=0} = \iint\limits_{\mathbb{R}^{2d}} f(x) g(y) \boldsymbol{K}(x,y) \, dx \, dy.$$

We also recall the notation

$$x = (x_{\bar{d}}, x_d), \quad \text{where} \quad x_{\bar{d}} = (x_1, x_2, \dots, x_{d-1}),$$

for $x \in \mathbb{R}^d$.

Proposition 4.1. *The kernel $\boldsymbol{K}$ defined in* (50) *satisfies the bound*

$$(51) \qquad |\boldsymbol{K}(x,y)| \lesssim (1 + |x_d|)^{\alpha - \gamma} (1 + |y_d|)^{\alpha - \gamma} (1 + |x_d - y_d|)^{\beta + \gamma - \frac{d-1}{2}}, \quad \gamma \in \left[0, \frac{d-1}{2} \right).$$

Remark 4.2. One can track the "numerology" of conditions (30), (31), and (32) from this proposition. The boundedness of $\boldsymbol{B}$ on $L_1 \times L_1$ is equivalent to the uniform boundedness of $\boldsymbol{K}$. The right hand-side of (51) is uniformly bounded exactly when these three conditions hold for $p = 1$ (they reflect the behavior of the kernel along the directions $x_d = y_d$, $x_d = 1$, and $x_d = -y_d$ respectively).

In the case $\gamma = 0$, the inequality (51) follows from the standard Van der Corput lemma, because in this case

$$\boldsymbol{K}(x,y) = (-1)^\alpha (2\pi i)^{2\alpha + \beta} x_d^\alpha y_d^\alpha (x_d - y_d)^\beta \int_U e^{2\pi i (\langle x_{\bar{d}} - y_{\bar{d}}, \zeta \rangle + (x_d - y_d) h(\zeta))} |\psi(\zeta)|^2 \, d\zeta,$$

the angular brackets denote the standard scalar product in $\mathbb{R}^{d-1}$.

So, we assume $\gamma > 0$ in what follows. We start with explicit formulas for the kernel $\boldsymbol{K}$:

$$(52) \quad \boldsymbol{K}(x,y) = (2\pi i x_d)^\alpha (-2\pi i y_d)^\alpha \left(\frac{\partial}{\partial r} \right)^\beta \left\langle \hat{\delta}_x \psi, \hat{\delta}_y \psi \right\rangle_{\dot{H}^{-\gamma}(\Sigma_r)} \Big|_{r=0} \overset{(28)}{=}$$

$$C_{d,\gamma} (2\pi i x_d)^\alpha (-2\pi i y_d)^\alpha \left(\frac{\partial}{\partial r} \right)^\beta \left[\iint\limits_{U \times U} e^{2\pi i (\langle \zeta, x_{\bar{d}} \rangle - \langle \eta, y_{\bar{d}} \rangle + x_d (h(\zeta) + r) - y_d (h(\eta) + r))} \psi(\zeta) \overline{\psi(\eta)} |\zeta - \eta|^{2\gamma - d - 1} \, d\zeta \, d\eta \right] \Big|_{r=0} =$$

$$C_{d,\gamma} (-1)^\alpha (2\pi i)^{2\alpha + \beta} x_d^\alpha y_d^\alpha (x_d - y_d)^\beta \iint\limits_{U \times U} e^{2\pi i (\langle \zeta, x_{\bar{d}} \rangle - \langle \eta, y_{\bar{d}} \rangle + x_d h(\zeta) - y_d h(\eta))} \psi(\zeta) \overline{\psi(\eta)} |\zeta - \eta|^{2\gamma - d - 1} \, d\zeta \, d\eta.$$

We want to pass to the dyadic version of the Bessel semi-norm, namely,

$$\|f\|_{\dot{H}^{-\gamma}(\Sigma_r)}^2 = C_{d,\gamma} \sum_{k \geq 0} 2^{k(d-1-2\gamma)} \iint\limits_{U \times U} f(\zeta, h_r(\zeta)) \overline{f(\eta, h_r(\eta))} \varphi(2^k |\zeta - \eta|) \, d\zeta \, d\eta, \quad \gamma \in \left(0, \frac{d-1}{2} \right),$$

based on the formula

$$(53) \qquad |\zeta - \eta|^{2\gamma - d + 1} = \sum_{k \geq 0} 2^{k(d-1-2\gamma)} \varphi(2^k |\zeta - \eta|).$$

The function $\varphi \in C_0^\infty(\mathbb{R})$ is supported outside zero and non-negative.

We substitute formula (53) into (52) and split $\boldsymbol{K}$ into a dyadic sum:

$$\boldsymbol{K}(x,y) = C_{d,\gamma}(-1)^\alpha (2\pi i)^{2\alpha+\beta} \sum_{k \geq 0} I_k(x,y),$$

where

$$I_k(x,y) = 2^{k(d-1-2\gamma)} x_d^\alpha y_d^\alpha (x_d - y_d)^\beta \iint\limits_{U \times U} e^{2\pi i(\langle \zeta, x_{\bar{d}}\rangle - \langle \eta, y_{\bar{d}}\rangle + x_d h(\zeta) - y_d h(\eta))} \psi(\zeta)\overline{\psi(\eta)}\varphi(2^k|\zeta - \eta|)\, d\zeta\, d\eta.$$

Lemma 4.3. *For any x, y and any $k \geq 0$,*

$$|I_k(x,y)| \lesssim 2^{k(d-1-2\gamma)}(1+|x_d|)^{\alpha - \frac{d-1}{2}}(1+|y_d|)^{\alpha - \frac{d-1}{2}}|x_d - y_d|^\beta.$$

Proof. The integral in the formula for I_k may be thought of as the $(2d-2)$-dimensional Fourier integral:

$$I_k(x,y) = 2^{k(d-1-2\gamma)} x_d^\alpha y_d^\alpha (x_d - y_d)^\beta \mathcal{F}_{(\zeta,\eta)\mapsto(x_{\bar{d}},y_{\bar{d}})}\left[e^{2\pi i(x_d h(\zeta) - y_d h(\eta))}\psi(\zeta)\overline{\psi(\eta)}\varphi(2^k|\zeta-\eta|)\right](x_{\bar{d}}, -y_{\bar{d}}).$$

It suffices to prove the inequality

$$(54) \qquad \left\|\mathcal{F}_{(\zeta,\eta)\mapsto(x_{\bar{d}},y_{\bar{d}})}\left[e^{2\pi i(x_d h(\zeta) - y_d h(\eta))}\psi(\zeta)\overline{\psi(\eta)}\varphi(2^k|\zeta-\eta|)\right]\right\|_{L_\infty} \lesssim (1+|x_d|)^{-\frac{d-1}{2}}(1+|y_d|)^{-\frac{d-1}{2}}.$$

We represent the function we apply the Fourier transform to as a product of two functions

$$(\zeta,\eta) \mapsto e^{2\pi i(x_d h(\zeta) - y_d h(\eta))}\psi(\zeta)\overline{\psi(\eta)} \quad \text{and} \quad (\zeta,\eta) \mapsto \varphi(2^k|\zeta-\eta|).$$

By the Van der Corput lemma, the Fourier transform of the first function is uniformly (in $(x_{\bar{d}}, y_{\bar{d}})$) bounded by the right hand-side of (54). It remains to notice that the Fourier transform of the second function is a complex measure whose total variation is bounded uniformly in k. This is easiest to see by making a linear change of variables from (ζ, η) to $(\zeta, \zeta - \eta)$. $\qquad\square$

Define the number $k_0 \geq 0$ by the rule

$$(55) \qquad 2^{2k_0} = \frac{(1+|x_d|)(1+|y_d|)}{1+|x_d - y_d|}.$$

Lemma 4.4. *For any x, y and any $k \geq k_0$,*

$$|I_k(x,y)| \lesssim 2^{-2k\gamma}(1+|x_d|)^\alpha (1+|y_d|)^\alpha (1+|x_d - y_d|)^{\beta - \frac{d-1}{2}}.$$

Proof. Let $|y_d| \geq |x_d|$. It suffices to prove the estimate

$$\left|\iint\limits_{U \times U} e^{2\pi i(\langle \zeta, x_{\bar{d}}\rangle - \langle \eta, y_{\bar{d}}\rangle + x_d h(\zeta) - y_d h(\eta))}\psi(\zeta)\overline{\psi(\eta)}\varphi(2^k|\zeta-\eta|)\, d\zeta\, d\eta\right| \lesssim 2^{-k(d-1)}(1+|x_d - y_d|)^{-\frac{d-1}{2}}.$$

We introduce new variables $(\theta, \eta) = (2^k(\zeta - \eta), \eta)$ and disregard oscillations in the θ variable:

$$\left|\iint\limits_{U \times U} e^{2\pi i(\langle \zeta, x_{\bar{d}}\rangle - \langle \eta, y_{\bar{d}}\rangle + x_d h(\zeta) - y_d h(\eta))}\psi(\zeta)\overline{\psi(\eta)}\varphi(2^k|\zeta-\eta|)\, d\zeta\, d\eta\right| =$$

$$2^{-k(d-1)}\left|\iint\limits_{\mathbb{R}^{2d-2}} e^{2\pi i(\langle \eta + 2^{-k}\theta, x_{\bar{d}}\rangle - \langle \eta, y_{\bar{d}}\rangle + x_d h(\eta + 2^{-k}\theta) - y_d h(\eta))}\psi(\eta + 2^{-k}\theta)\overline{\psi(\eta)}\varphi(|\theta|)\, d\theta\, d\eta\right| \lesssim$$

$$2^{-k(d-1)} \sup_{|\theta|\lesssim 1}\left|\int\limits_{\mathbb{R}^{d-1}} e^{2\pi i(\langle \eta, x_{\bar{d}} - y_{\bar{d}}\rangle + x_d h(\eta + 2^{-k}\theta) - y_d h(\eta))}\psi(\eta + 2^{-k}\theta)\overline{\psi(\eta)}\, d\eta\right|.$$

It remains to prove

$$\left| \int\limits_{\mathbb{R}^{d-1}} e^{2\pi i(\langle \eta, x_{\bar{d}} - y_{\bar{d}} \rangle + x_d h(\eta + 2^{-k}\theta) - y_d h(\eta))} \psi(\eta + 2^{-k}\theta) \overline{\psi(\eta)} \, d\eta \right| \lesssim (1 + |x_d - y_d|)^{-\frac{d-1}{2}}, \quad |\theta| \lesssim 1.$$

The function $\psi(\cdot + 2^{-k}\theta)$ is uniformly (with respect to k and θ) bounded in any Schwartz norm, so its Fourier transform is an L_1-function whose norm is bounded independently of k and θ. Thus, it suffices to prove

$$(56) \qquad \left| \int\limits_{\mathbb{R}^{d-1}} e^{2\pi i(\langle \eta, x_{\bar{d}} - y_{\bar{d}} \rangle + x_d h(\eta + 2^{-k}\theta) - y_d h(\eta))} \overline{\psi(\eta)} \, d\eta \right| \lesssim (1 + |x_d - y_d|)^{-\frac{d-1}{2}}$$

uniformly in $x_{\bar{d}}, y_{\bar{d}}$. This inequality is trivial if $|x_d - y_d| \lesssim 1$, so we assume the quantity $|x_d - y_d|$ is sufficiently large. We represent the non-linear part of the phase function as

$$x_d h(\eta + 2^{-k}\theta) - y_d h(\eta) = (x_d - y_d)\Phi_{\theta, x_d, y_d}(\eta),$$

where

$$\Phi_{\theta, x_d, y_d}(\eta) = h(\eta) + \frac{x_d}{x_d - y_d}\big(h(\eta + 2^{-k}\theta) - h(\eta)\big) = h(\eta) + O\Big(\frac{2^{-k}|x_d|}{|x_d - y_d|}\Big)$$

since $|\theta| \lesssim 1$. Note that

$$\frac{2^{-k}|x_d|}{|x_d - y_d|} \leq \frac{2^{-k_0}|x_d|}{|x_d - y_d|} \overset{(55)}{=} \frac{|x_d|\sqrt{1 + |x_d - y_d|}}{|x_d - y_d|\sqrt{1 + |x_d|}\sqrt{1 + |y_d|}} \lesssim \frac{1}{\sqrt{1 + |x_d - y_d|}} \ll 1,$$

when $|y_d| \geq |x_d|$, $|x_d - y_d|$ is sufficiently large, and $k \geq k_0$. In particular, the Hessians of the functions in the family $\{\Phi_{\theta, x_d, y_d}\}_{\theta, x_d, y_d}$ take the form $\frac{\partial^2 h}{\partial \eta^2}(\eta) + O(\frac{2^{-k}|x_d|}{|x_d - y_d|})$ and are uniformly invertible. By similar reasons, the functions in the family $\{\Phi_{\theta, x_d, y_d}\}_{\theta, x_d, y_d}$ are uniformly bounded in any Schwartz norm. The version of Littman's lemma from [7] leads to (56). $\qquad\square$

Proof of Proposition 4.1. We use Lemmas 4.3 and 4.4 (the case $\gamma = 0$ has already been considered, so we assume $\gamma > 0$ here):

$$|\mathbf{K}(x,y)| \lesssim C_{d,\gamma} \sum_{k \geq 0} |I_k(x,y)| \lesssim$$

$$\sum_{k \leq k_0} 2^{(d-1-2\gamma)k}(1+|x_d|)^{\alpha - \frac{d-1}{2}}(1+|y_d|)^{\alpha - \frac{d-1}{2}}|x_d - y_d|^{\beta} + \sum_{k \geq k_0} 2^{-2k\gamma}(1+|x_d|)^{\alpha}(1+|y_d|)^{\alpha}(1+|x_d - y_d|)^{\beta - \frac{d-1}{2}} \lesssim$$

$$2^{(d-1-2\gamma)k_0}(1+|x_d|)^{\alpha - \frac{d-1}{2}}(1+|y_d|)^{\alpha - \frac{d-1}{2}}|x_d - y_d|^{\beta} + 2^{-2k_0\gamma}(1+|x_d|)^{\alpha}(1+|y_d|)^{\alpha}(1+|x_d - y_d|)^{\beta - \frac{d-1}{2}} \lesssim$$

$$(1+|x_d|)^{\alpha - \gamma}(1+|y_d|)^{\alpha - \gamma}(1+|x_d - y_d|)^{\beta + \gamma - \frac{d-1}{2}}.$$

$$\square$$

4.2. Interpolation.
To prove the "if" part of Theorem 1.16 for the case $p > 1$, we will have to work with "slices" of the kernel $\mathbf{K}$. For any x_d and y_d, define the kernel $\mathbf{K}_{x_d, y_d} \colon \mathbb{R}^{2d-2} \to \mathbb{C}$ by the formula

$$\mathbf{K}_{x_d, y_d}(x_{\bar{d}}, y_{\bar{d}}) = \mathbf{K}(x,y), \quad x = (x_{\bar{d}}, x_d), \ y = (y_{\bar{d}}, y_d).$$

Define the bilinear forms $\mathbf{B}_{x_d, y_d}$ accordingly

$$\mathbf{B}_{x_d, y_d}[f, g] = \iint\limits_{\mathbb{R}^{2d-2}} f(x_{\bar{d}})g(y_{\bar{d}})\mathbf{K}_{x_d, y_d}(x_{\bar{d}}, y_{\bar{d}}) \, dx_{\bar{d}} \, dy_{\bar{d}},$$

here f and g are functions on $\mathbb{R}^{d-1}$. Proposition 4.1 now may be restated as

$$(57) \qquad \left\|\boldsymbol{B}_{x_d,y_d}\right\|_{L_1 \times L_1} \lesssim (1+|x_d|)^{\alpha-\gamma}(1+|y_d|)^{\alpha-\gamma}(1+|x_d-y_d|)^{\beta+\gamma-\frac{d-1}{2}}, \quad \gamma \in \left[0, \frac{d-1}{2}\right).$$

Lemma 4.5. *For any* $\gamma \in [0, \frac{d-1}{2})$,

$$\left\|\boldsymbol{B}_{x_d,y_d}\right\|_{L_{\frac{2d-2}{d-1+2\gamma}} \times L_{\frac{2d-2}{d-1+2\gamma}}} \lesssim (1+|x_d|)^{\alpha-\gamma}(1+|y_d|)^{\alpha-\gamma}(1+|x_d-y_d|)^{\beta}.$$

Proof. Let λ_{x_d} and λ_{y_d} be the Lebesgue measures on the hyperplanes

$$\{w \in \mathbb{R}^d \mid w_d = x_d\} \quad \text{and} \quad \{w \in \mathbb{R}^d \mid w_d = y_d\}.$$

Then,

$$\boldsymbol{B}_{x_d,y_d}[f,g] = \boldsymbol{B}[f\,d\lambda_{x_d}, g\,d\lambda_{y_d}]$$

if we interpret f and g as functions of d variables that do not depend on x_d. With this formula in hand, we may re-express $\boldsymbol{B}_{x_d,y_d}$:

$$(58) \quad \boldsymbol{B}_{x_d,y_d}[f,g] = \left(\frac{\partial}{\partial r}\right)^\beta \left\langle \frac{\partial^\alpha[\hat{f}(\xi)e^{2\pi i x_d \xi_d}]}{\partial \xi_d^\alpha}\psi(\xi), \frac{\partial^\alpha[\hat{g}(\xi)e^{2\pi i y_d \xi_d}]}{\partial \xi_d^\alpha}\psi(\xi)\right\rangle_{\dot{H}^{-\gamma}(\Sigma_r)}\Bigg|_{r=0} =$$

$$(2\pi i x_d)^\alpha(-2\pi i y_d)^\alpha(2\pi i(x_d-y_d))^\beta \langle \hat{f}e^{2\pi i x_d h(\cdot)}\psi, \hat{g}e^{2\pi i y_d h(\cdot)}\psi\rangle_{\dot{H}^{-\gamma}(\Sigma_0)} =$$

$$(-1)^\alpha(2\pi i)^{2\alpha+\beta}x_d^\alpha y_d^\alpha (x_d-y_d)^\beta \int\limits_{\mathbb{R}^{d-1}} \left[f*\mathcal{F}_{\zeta\mapsto z}\left[e^{2\pi i x_d h(\zeta)}\psi(\zeta)\right]\right](z)\cdot\overline{\left[g*\mathcal{F}_{\zeta\mapsto z}\left[e^{2\pi i y_d h(\zeta)}\psi(\zeta)\right]\right](z)}\cdot|z|^{-2\gamma}\,dz.$$

Therefore, it suffices to prove the bound

$$(59) \quad \left|\int\limits_{\mathbb{R}^{d-1}} \left[f*\mathcal{F}_{\zeta\mapsto z}\left[e^{2\pi i x_d h(\zeta)}\psi(\zeta)\right]\right](z)\cdot\overline{\left[g*\mathcal{F}_{\zeta\mapsto z}\left[e^{2\pi i y_d h(\zeta)}\psi(\zeta)\right]\right](z)}\cdot|z|^{-2\gamma}\,dz\right| \lesssim$$

$$(1+|x_d|)^{-\gamma}(1+|y_d|)^{-\gamma}\|f\|_{L_{\frac{2d-2}{d-1+2\gamma}}}\|g\|_{L_{\frac{2d-2}{d-1+2\gamma}}}.$$

We postulate the inequality

$$(60) \qquad \left\|f*\mathcal{F}_{\zeta\mapsto z}\left[e^{2\pi i x_d h(\zeta)}\psi(\zeta)\right]\right\|_{L_{\frac{2d-2}{d-1-2\gamma},2}} \lesssim (1+|x_d|)^{-\gamma}\|f\|_{L_{\frac{2d-2}{d-1+2\gamma}}}.$$

The space on the left is the Lorentz space, see [3] for definitions. Inequality (60) immediately leads to (59):

$$\left|\int\limits_{\mathbb{R}^{d-1}} \left[f*\mathcal{F}_{\zeta\mapsto z}\left[e^{2\pi i x_d h(\zeta)}\psi(\zeta)\right]\right](z)\cdot\overline{\left[g*\mathcal{F}_{\zeta\mapsto z}\left[e^{2\pi i y_d h(\zeta)}\psi(\zeta)\right]\right](z)}\cdot|z|^{-2\gamma}\,dz\right| \lesssim$$

$$\left\|f*\mathcal{F}_{\zeta\mapsto z}\left[e^{2\pi i x_d h(\zeta)}\psi(\zeta)\right]\right\|_{L_{\frac{2d-2}{d-1-2\gamma},2}}\cdot\left\|g*\mathcal{F}_{\zeta\mapsto z}\left[e^{2\pi i y_d h(\zeta)}\psi(\zeta)\right]\right\|_{L_{\frac{2d-2}{d-1-2\gamma},2}}\cdot\||z|^{-2\gamma}\|_{L_{\frac{d-1}{2\gamma},\infty}} \overset{(60)}{\lesssim}$$

$$(1+|x_d|)^{-\gamma}(1+|y_d|)^{-\gamma}\|f\|_{L_{\frac{2d-2}{d-1+2\gamma}}}\|g\|_{L_{\frac{2d-2}{d-1+2\gamma}}}.$$

We are required to prove (60). Let us denote the operator we want to estimate by $\boldsymbol{C}_{x_d}$:

$$\boldsymbol{C}_{x_d}[f] = f*\mathcal{F}_{\zeta\mapsto z}\left[e^{2\pi i x_d h(\zeta)}\psi(\zeta)\right].$$

By the Plancherel theorem,

$$\|\boldsymbol{C}_{x_d}\|_{L_2\to L_2} \lesssim 1.$$

By the Van der Corput lemma,

$$\|\boldsymbol{C}_{x_d}\|_{L_1\to L_\infty} \lesssim (1+|x_d|)^{-\frac{d-1}{2}}.$$

The real interpolation formulas (see [3], §5.3)

$$[L_1, L_2]_{\frac{2\gamma}{d-1}, \frac{2d-2}{d-1+2\gamma}} = L_{\frac{2d-2}{d-1+2\gamma}}; \quad [L_\infty, L_2]_{\frac{2\gamma}{d-1}, \frac{2d-2}{d-1+2\gamma}} = L_{\frac{2d-2}{d-1-2\gamma}, \frac{2d-2}{d-1+2\gamma}}$$

lead to the inequality

$$\|\boldsymbol{C}_{x_d}\|_{L_{\frac{2d-2}{d-1+2\gamma}} \to L_{\frac{2d-2}{d-1-2\gamma}, 2}} \lesssim \|\boldsymbol{C}_{x_d}\|_{L_{\frac{2d-2}{d-1+2\gamma}} \to L_{\frac{2d-2}{d-1-2\gamma}, \frac{2d-2}{d-1+2\gamma}}} \lesssim (1 + |x_d|)^{-\gamma},$$

which is exactly (60). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Interpolation between (57) and Lemma 4.5 leads to the inequality

$$(61) \qquad \|\boldsymbol{B}_{x_d, y_d}\|_{L_p \times L_p} \lesssim (1 + |x_d|)^{\alpha - \gamma} (1 + |y_d|)^{\alpha - \gamma} (1 + |x_d - y_d|)^{\beta + \gamma - \frac{d-1}{p} + \frac{d-1}{2}}$$

for $p \in [1, \frac{2d-2}{d-1+2\gamma}]$. Let us restrict our attention to this case for a while. To finish the proof of Theorem 1.16, we invoke a version of the Stein–Weiss inequality (Theorem 7.4 in Subsection 7.2 below):

$$\left|\boldsymbol{B}[f, g]\right| = \left|\iint_{\mathbb{R} \times \mathbb{R}} \boldsymbol{B}_{x_d, y_d}\big[f(\cdot, x_d), g(\cdot, y_d)\big] \, dx_d \, dy_d\right| \overset{(61)}{\lesssim}$$

$$\iint_{\mathbb{R} \times \mathbb{R}} (1 + |x_d|)^{\alpha - \gamma} (1 + |y_d|)^{\alpha - \gamma} (1 + |x_d - y_d|)^{\beta + \gamma - \frac{d-1}{p} + \frac{d-1}{2}} \|f(\cdot, x_d)\|_{L_p(\mathbb{R}^{d-1})} \|g(\cdot, y_d)\|_{L_p(\mathbb{R}^{d-1})} \, dx_d \, dy_d \overset{\text{Th.7.4}}{\lesssim}$$

$$\|f\|_{L_p} \|g\|_{L_p}$$

provided $a = \gamma - \alpha$ and $b = -\beta - \gamma - \frac{d-1}{2} + \frac{d-1}{p}$ satisfy the requirements of Theorem 7.4. The inequality (30) leads to $a \geq 0$, the requirement (31) leads to $a + b \geq 1 - \frac{1}{p}$ (with the same exclusion of the endpoint case if $p > 1$), and (32) gives $2a + b \geq 2 - \frac{2}{p}$. The case $b = 1$ and $p = 2$ is impossible ($\beta + \gamma$ is negative in this case). The "if" part of Theorem 1.16 is proved in the case $p \in [1, \frac{2d-2}{d-1+2\gamma}]$.

To deal with the remaining case, we start from the estimate

$$\left\|\boldsymbol{B}_{x_d, y_d}\right\|_{L_2 \times L_2} \lesssim (1 + |x_d|)^\alpha (1 + |y_d|)^\alpha (1 + |x_d - y_d|)^\beta,$$

which follows from the representation (58); we use the trivial inequality

$$\left\|f * \mathcal{F}_{\zeta \mapsto z}\big[e^{2\pi i x_d h(\zeta)} a(\zeta)\big]\right\|_{L_\infty} \lesssim \|f\|_{L_2} \|e^{2\pi i x_d h(\zeta)} \psi(\zeta)\|_{L_2} \lesssim \|f\|_{L_2}.$$

We interpolate this bound with Lemma 4.5:

$$\left\|\boldsymbol{B}_{x_d, y_d}\right\|_{L_p \times L_p} \lesssim \Big(1 + |x_d|\Big)^{\alpha - (d-1)(\frac{1}{p} - \frac{1}{2})} \Big(1 + |y_d|\Big)^{\alpha - (d-1)(\frac{1}{p} - \frac{1}{2})} (1 + |x_d - y_d|)^\beta, \quad p \in \left[\frac{2d - 2}{d - 1 + 2\gamma}, 2\right].$$

We invoke Theorem 7.4 with $a = (d-1)(\frac{1}{p} - \frac{1}{2}) - \alpha$ and $b = -\beta$. Since $b \leq 0$, the condition $a + b > 1 - \frac{1}{p}$ is stronger than $2a + b \geq 2 - \frac{2}{p}$. The condition $a + b > 1 - \frac{1}{p}$ is exactly (31). The condition $a > 0$ also follows from it:

$$\alpha \leq \alpha + \beta < \frac{d}{p} - \frac{d+1}{2} = \frac{d-1}{p} - \frac{d-1}{2} + \frac{1}{p} - 1 \leq \frac{d-1}{p} - \frac{d-1}{2}.$$

The "if" part of Theorem 1.16 is now proved.

Remark 4.6. Note that we did not use that α or β are integer provided we define our bilinear form by (52).

4.3. **Strichartz estimates.** With the same method as in the previous section, we can get a collection of sharp (up to the endpoint) Strichartz estimates. For that we need the mixed norm spaces $L_r(L_p)$:

$$(62) \qquad \|g\|_{L_r(L_p)} = \big\|\|g\|_{L_{p(x)}}\big\|_{L_{r(t)}} = \left(\int\limits_{\mathbb{R}} \Big(\int\limits_{\mathbb{R}^{d-1}} |g(x,t)|^p \, dx \Big)^{\frac{r}{p}} dt \right)^{\frac{1}{r}}, \quad g \colon \mathbb{R}^d \to \mathbb{R}.$$

We also use Theorem 7.6 here in order to work with the cases $r > 2$ as well. This provides some new information even in the case $\alpha = \beta = 0$ considered in [6]. The cases $r > 2$ were excluded in that paper and it is not clear whether the methods of [6] work in this situation.

Theorem 4.7. *The inequality*

$$\big| \boldsymbol{B}(f,g) \big| \lesssim \|f\|_{L_r(L_p)} \|g\|_{L_r(L_p)}$$

holds true if

(1) $r \in [1, 2]$ *and*
   - $p \le 2$;
   - $\gamma \ge \alpha$;
   - $\alpha + \beta < \frac{d-1}{p} + \frac{1}{r} - \frac{d+1}{2}$, *with equality permitted if $r = 1$;*
   - $2\alpha + \beta - \gamma < \frac{d-1}{p} + \frac{2}{r} - \frac{d+3}{2}$, *with equality also permitted if $r < 2$ or $\gamma > \alpha$;*

(2) $r \in (2, \infty]$ *and*
   - $p \le 2$;
   - $\gamma - \alpha > \frac{1}{2} - \frac{1}{r}$;
   - $\alpha + \beta < \frac{d-1}{p} + \frac{1}{r} - \frac{d+1}{2}$;
   - $2\alpha + \beta - \gamma < \frac{d-1}{p} + \frac{2}{r} - \frac{d+3}{2}$.

The proof is a direct application of Theorems 7.4 and 7.6. Consider the case $p \in [1, \frac{2d-2}{d-1+2\gamma}]$. Set $a = \gamma - \alpha$, $b = -\beta - \gamma - \frac{d-1}{2} + \frac{d-1}{p}$, and $p = r$ (that is, the value of p in those theorems is replaced by r). We note that the conditions of Theorem 7.4 can be summarized as $a \ge 0$, $a + b > 1 - \frac{1}{p}$ and $2a + b \ge 2 - \frac{2}{p}$. When $p = 1$, combinations with $a + b \ge 0$ are also accepted, and when $p = 2$ the case $a = 0$, $b = 1$ is excluded.

The three conditions stated in the case $r \in [1, 2]$ are equivalent to $a \ge 0$, $a + b > 1 - \frac{1}{r}$, and $2a + b > 2 - \frac{2}{r}$, respectively. The three conditions stated in the case $r \in (2, \infty]$ are equivalent to the conditions in Theorem 7.6, namely $a > \frac{1}{2} - \frac{1}{r}$, $a + b > 1 - \frac{1}{r}$, and $2a + b > 2 - \frac{2}{r}$.

Consider the case $p \in [\frac{2d-2}{d-1+2\gamma}, 2]$ and set $a = (d-1)(\frac{1}{p} - \frac{1}{2}) - \alpha$, $b = -\beta$, and $p = r$ in the same sense as above. Since $b \le 0$, the condition $a + b > \frac{1}{p}$ is stronger than $2a + b \ge 2 - \frac{2}{p}$. The condition $a + b > \frac{1}{p}$ is equivalent to $\alpha + \beta < \frac{d-1}{p} + \frac{1}{r} - \frac{d+1}{2}$ with equality permitted if $r = 1$. In the case $r \le 2$, the requirement $a \ge 0$ is rewritten as $\alpha < \frac{d-1}{p} - \frac{d-1}{2}$. It also follows from $\alpha + \beta \le \frac{d-1}{p} + \frac{1}{r} - \frac{d+1}{2}$. The condition $a > \frac{1}{2} - \frac{1}{r}$ arising in the case $r \ge 2$ follows from the same inequality.

5. Robust estimates

5.1. **Introduction to "numerology".**

Remark 5.1. We are mostly interested in the case $k > s$ in (12). We claim that in the "subcritical" case $k \le s$, the second term on the right hand-side of this inequality is unnecessary. If $k \le s$ and $\mathrm{HDR}(\Sigma, k, s, \ell, p)$ is true, then a simpler inequality

$$(63) \qquad \|\phi \nabla^k \hat{f}\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L_p(\mathbb{R}^d)}$$

also holds true. Indeed, if $\mathrm{HDR}(\Sigma, k, s, \ell, p)$ is true, then (15) and (17) are valid. However, in this case, these conditions are also sufficient for (63) to be true (see Theorem 1.16 and Figure 1 below).
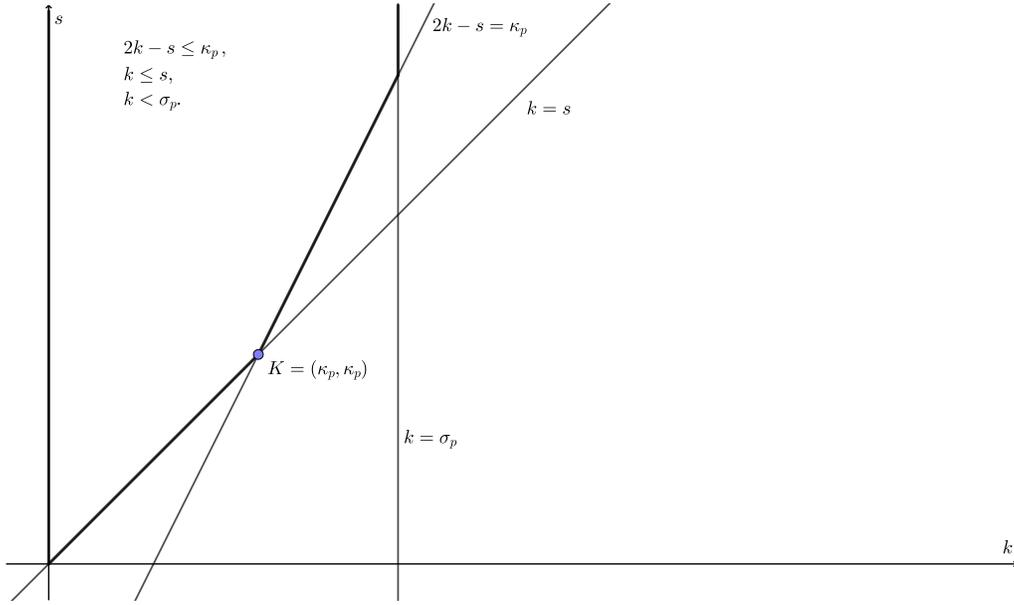
FIGURE 1. Diagram for the case $k \leq s$.

Remark 5.2. In the "supercritical" case $k > s$, the condition (15) follows from (17) since $\kappa_p \leq \sigma_p$ in this case (see Figure 1 as well). Note also that (16) is equivalent to

$$(64) \qquad s \geq \left(1 + \frac{\ell}{\kappa_p}\right)k - \ell.$$

This inequality, in its turn, leads to (13) provided $\kappa_p \geq 0$ (which is true by (17)). Thus, in the case $k > s$, the conditions in Proposition 1.11 are reduced to (14), (64), and (17).

In our proof, the parameters k and s will be varied, however, ℓ and p will be steady. It appears convenient to draw diagrams of admissible pairs (k, s). We have already drawn such a diagram for the case $k \leq s$ (Figure 1). For our first attempt to the "numerology", we neglect the integer nature of k and imagine this parameter is real positive. We have three inequalities in the subcritical case: $k \leq s$, (15), and (17). The cases of equality correspond to lines on the diagram, and all three inequalities are satisfied inside the domain bounded by the bold broken line. We also note that the lines $2k - s = \kappa_p$ and $k = s$ intersect at the point (κ_p, κ_p), which we denote by K.

Now we pass to the "supercritical" case $k > s$. We need to draw two additional lines $k = s + 1$ and

$$(65) \qquad s = \left(1 + \frac{\ell}{\kappa_p}\right)k - \ell,$$

which correspond to (14) and (16) respectively. The structure of the domain of admissible parameters will depend on the mutual disposition of these two lines and the line $s = 2k - \kappa_p$. Before we classify the cases of disposition, we note that the line (65) passes through K. There is one more nice point lying on it: the point $L = (0, -\ell)$. We will consider the cases $\kappa_p < 2$ and $\kappa_p \geq 2$ separately.

Case $\kappa_p < 2$. In this case, the condition (14) is unnecessary, it follows from (17) and $s \geq 0$. This case, in its turn, is naturally split into subcases $\ell \leq \kappa_p$ (see Figure 2, note that the broken line has a non-trivial angle at K) and $\ell > \kappa_p$ (see Figure 3), note that (16) follows from (17) when $\ell \geq \kappa_p$.
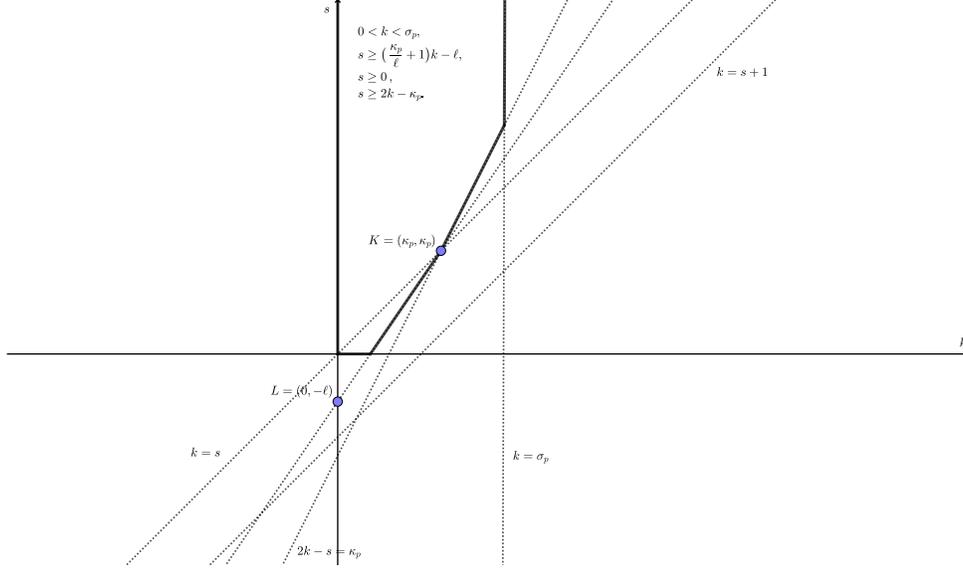

FIGURE 2. Diagram for the case $\ell \leq \kappa_p < 2$.

Case $\kappa_p \geq 2$. In this case, there will be three subcases: $\ell \leq \frac{\kappa_p}{\kappa_p - 1}$ (if this inequality turns into equality, then KL passes through the point $(1, 0)$), $\frac{\kappa_p}{\kappa_p - 1} < \ell \leq \kappa_p$, and $\ell > \kappa_p$. In the first case, the condition (14) is unnecessary (see Figure (4)). In the second case, all the conditions are required (see Figure 5). In the third case, the condition (16) is unnecessary (see Figure 6).

5.2. **Convexity properties of the function N.** It is useful to consider the expression

$$N(k, s) = \left\| \psi(\cdot) \frac{\partial^k \hat{f}}{\partial \xi_d^k}(\cdot, h(\cdot)) \right\|_{\dot{H}^{-s}}^2$$

as a function of the parameters k and s. We always assume k is a non-negative integer and s is a non-negative real. Since we will be working with points in the (k, s)-plane, we will give names to some regions there.

Definition 5.3. *Let $d, p, \ell,$ and h be fixed. The domain*

$$\left\{ (k, s) \in \mathbb{R}^2 \;\middle|\; k \in (0, \sigma_p), s \geq 0, s \geq k - 1, 2k - s \leq \kappa_p \right\}$$

is called the friendly region. The domain where $s \geq k$ is called the subcritical region. The set of all points (k, s) such that $\mathrm{HDR}_{\mathrm{loc}}(h, k, s, \ell, p)$ holds true is called the HDR-domain.
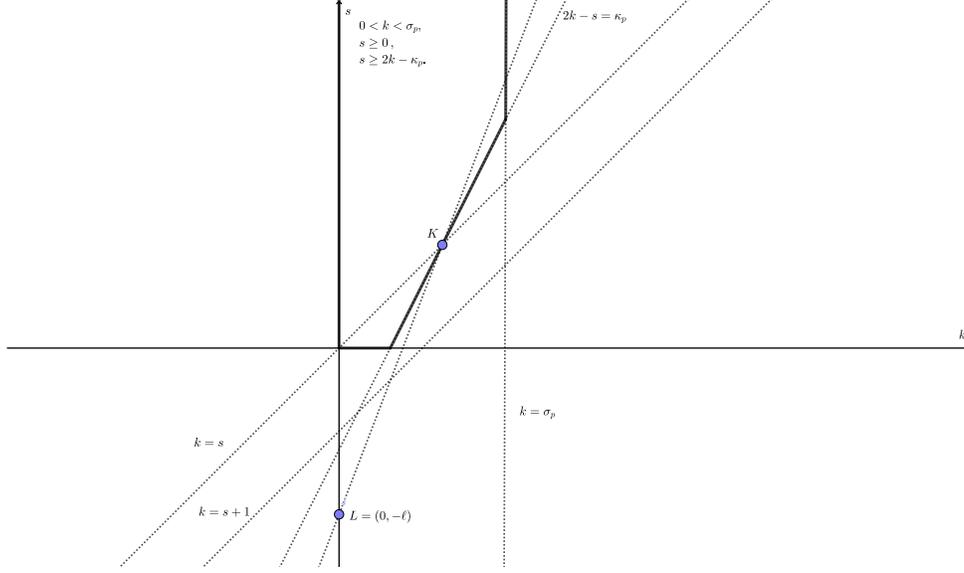
FIGURE 3. Diagram for the case $\kappa_p < 2$ and $\ell > \kappa_p$.

If X is an arbitrary point in the (k, s) plane, k_X will usually denote its k-coordinate, and s_X will denote its s-coordinate.

Remark 5.4. The HDR-domain lies inside the friendly domain (by Proposition 1.11).

Lemma 5.5. *For any k and s, there exists a constant C such that the inequality*

$$N(k, s) \leq C\|f\|_{L_p}^2 + \sqrt{N(k_1, s_1)N(k_2, s_2)}, \quad 2k = k_1 + k_2, 2s = s_1 + s_2,$$
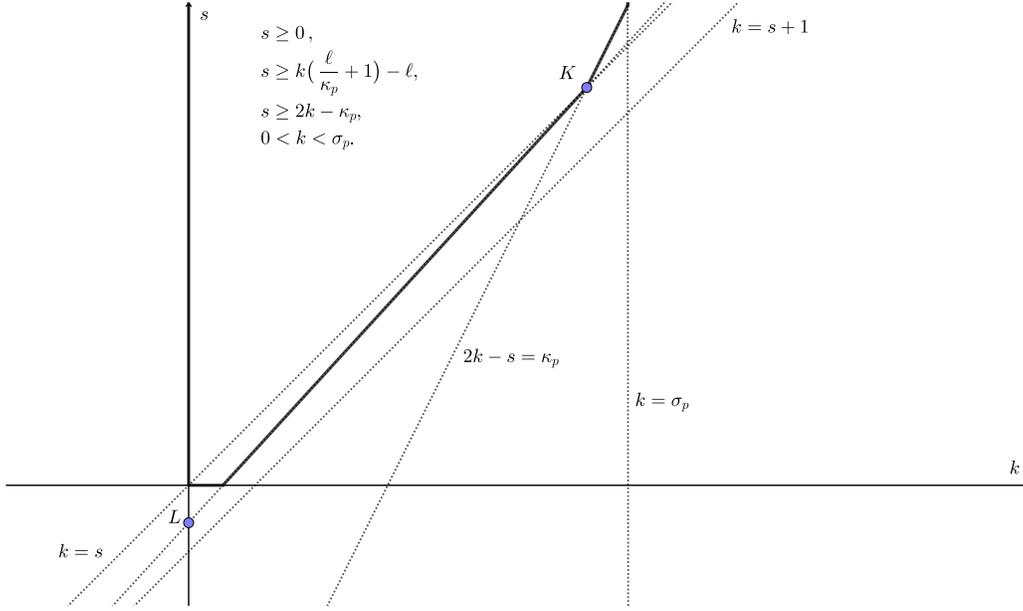
is true provided (k, s) lies in the friendly region and $0 \leq k_j < \sigma_p$ for $j = 1, 2$.

We will need an "algebraic" lemma that will link the quantities $N(k, s)$, $N(k_1, s_1)$, and $N(k_2, s_2)$ together.

Lemma 5.6. *For any $k, k_1, k_2 \in \mathbb{Z}_+$ such that $2k = k_1 + k_2$, there exist coefficients $c_1, c_2, \ldots, c_{|k_1-k|}$ such that*

$$(66) \quad (-1)^{|k-k_1|}\left\|\psi(\cdot)\frac{\partial^k \hat{f}}{\partial \xi_d^k}(\cdot, h(\cdot))\right\|_{\dot{H}^{-s}}^2 - \Re\left\langle\psi(\cdot)\frac{\partial^{k_1} \hat{f}}{\partial \xi_d^{k_1}}(\cdot, h(\cdot)), \psi(\cdot)\frac{\partial^{k_2} \hat{f}}{\partial \xi_d^{k_2}}(\cdot, h(\cdot))\right\rangle_{\dot{H}^{-s}} =$$

$$\sum_{j=1}^{|k-k_1|} c_j \frac{\partial^{2j}}{\partial r^{2j}}\left\|\psi(\cdot)\frac{\partial^{k-j} \hat{f}}{\partial \xi_d^{k-j}}(\cdot, h(\cdot) + r)\right\|_{\dot{H}^{-s}}^2$$

for any function f and any s.

FIGURE 4. Diagram for the case $\kappa_p \geq 2$ and $\ell \leq \frac{\kappa_p}{\kappa_p - 1}$.

Proof. First, by the Newton–Leibniz formula,

$$\frac{\partial^{2j}}{\partial r^{2j}}\left\|\psi(\cdot)\frac{\partial^{k-j}\hat{f}}{\partial \xi_d^{k-j}}(\cdot, h(\cdot)+r)\right\|_{\dot{H}^{-s}}^2 = C_{2j}^j\left\langle\psi(\cdot)\frac{\partial^k\hat{f}}{\partial\xi_d^k}(\cdot, h(\cdot)), \psi(\cdot)\frac{\partial^k\hat{f}}{\partial\xi_d^k}(\cdot, h(\cdot))\right\rangle_{\dot{H}^{-s}} +$$

$$2\sum_{i=0}^{j-1} C_{2j}^i\left\langle\psi(\cdot)\frac{\partial^{k+j-i}\hat{f}}{\partial\xi_d^{k+j-i}}(\cdot, h(\cdot)), \psi(\cdot)\frac{\partial^{k+i-j}\hat{f}}{\partial\xi_d^{k+i-j}}(\cdot, h(\cdot))\right\rangle_{\dot{H}^{-s}}.$$

Thus, it is clear that

$$\Re\left\langle\psi(\cdot)\frac{\partial^{k_1}\hat{f}}{\partial\xi_d^{k_1}}(\cdot, h(\cdot)), \psi(\cdot)\frac{\partial^{k_2}\hat{f}}{\partial\xi_d^{k_2}}(\cdot, h(\cdot))\right\rangle_{\dot{H}^{-s}}$$

is a linear combination of all the other terms in the identity (66). The only non-trivial question is why does the term

$$\left\|\psi(\cdot)\frac{\partial^k\hat{f}}{\partial\xi_d}(\cdot, h(\cdot))\right\|_{\dot{H}^{-s}}^2$$

have coefficient $(-1)^{|k-k_1|}$. For this we observe that the binomial coefficients that appear in (66) once the Newton–Leibniz formula is applied are the same ones that arise in the trigonometric identity

$$(-1)^{|k-k_1|} - \cos(2|k-k_1|\theta) = \sum_{j=1}^{|k-k_1|} c_j[2\cos\theta]^{2j}.$$

The result follows by evaluating the trigonometric sum at $\theta = \frac{\pi}{2}$. $\qquad\qquad\square$
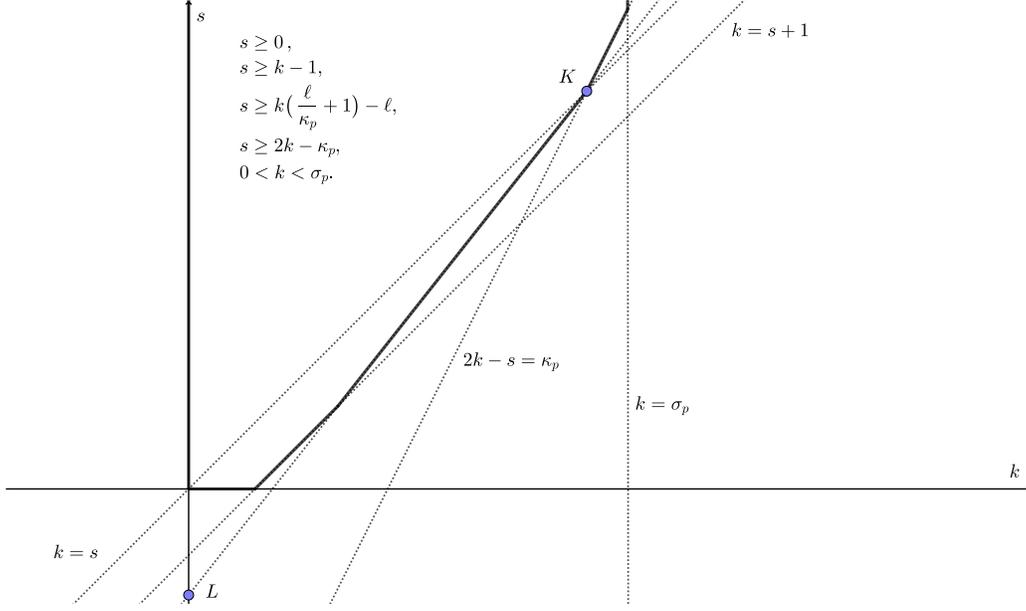
FIGURE 5. Diagram for the case $\kappa_p \geq 2$ and $\frac{\kappa_p}{\kappa_p - 1} < \ell \leq \kappa_p$.

Proof of Lemma 5.5. Lemma 5.6 says that

$$\left\| \psi(\cdot) \frac{\partial^k \hat{f}}{\partial \xi_d^k}(\cdot, h(\cdot)) \right\|_{\dot{H}^{-s}}^2 \leq$$

$$\left| \left\langle \psi(\cdot) \frac{\partial^{k_1} \hat{f}}{\partial \xi_d^{k_1}}(\cdot, h(\cdot)), \psi(\cdot) \frac{\partial^{k_2} \hat{f}}{\partial \xi_d^{k_2}}(\cdot, h(\cdot)) \right\rangle_{\dot{H}^{-s}} \right| + C \sum_{j=1}^{|k-k_1|} \left| \frac{\partial^{2j}}{\partial r^{2j}} \left\| \psi(\cdot) \frac{\partial^{k-j} \hat{f}}{\partial \xi_d^{k-j}}(\cdot, h(\cdot) + r) \right\|_{\dot{H}^{-s}}^2 \right|.$$

By the Cauchy–Schwarz inequality, the first summand on the right can be estimated by $\sqrt{N(k_1, s_1) N(k_2, s_2)}$. All the remaining terms are bounded by $C\|f\|_{L_p}^2$, provided (29) holds true with $(\alpha, \beta, \gamma, p) = (k-j, 2j, s, p)$ for any $j = 1, 2, \ldots, |k - k_1|$. By Theorem 1.16, this holds exactly when

$$\begin{aligned} k - j &\leq s; \\ k + j &< \sigma_p; \qquad\qquad j = 1, 2, \ldots, |k - k_1|. \\ 2k - s &\leq \kappa_p, \end{aligned}$$

The first list of conditions turns into $k - 1 \leq s$. So, the first and the third conditions are fulfilled inside the friendly region. The second list is reduced to $k_1, k_2 < \sigma_p$. □

Corollary 5.7. *For any k and s, there exists a constant C such that the inequality*

$$N(k, s) \leq C\|f\|_{L_p}^2 + \frac{N(k_1, s_1) + N(k_2, s_2)}{2}, \quad 2k = k_1 + k_2, \ 2s = s_1 + s_2,$$

is true provided (k, s) lies in the friendly region and $0 \leq k_j < \sigma_p$ for $j = 1, 2$.
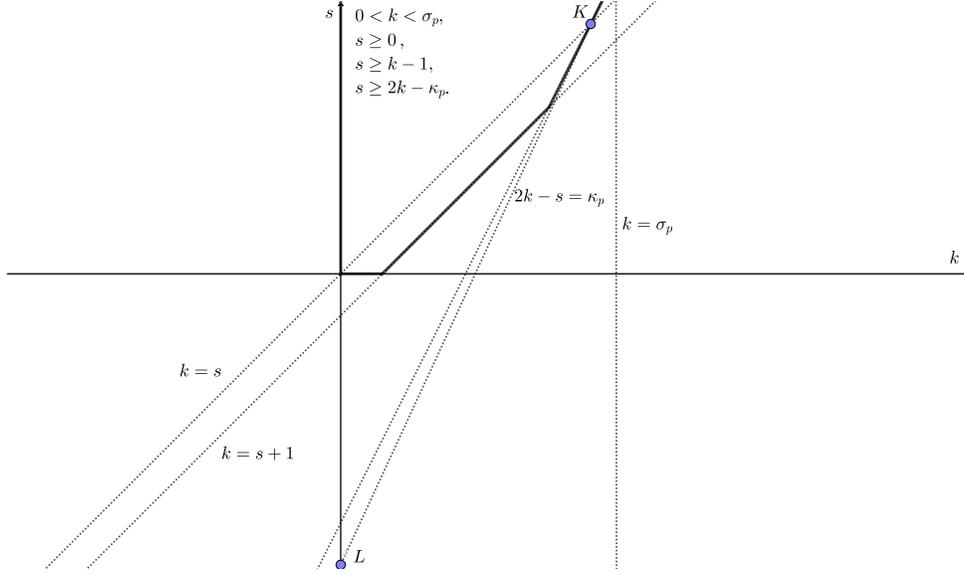
FIGURE 6. Diagram for the case $2 \le \kappa_p \le \ell$.

Lemma 5.8. *Let $a\colon \{0\} \cup [M..N] \to \mathbb{R}$ be a finite sequence and let $2M \le N$. Assume that*

$$\forall k \in [M+1..N-1] \qquad a_k \le 1 + \frac{a_{k+1} + a_{k-1}}{2},$$

$a_M \le 1 + \frac{a_0 + a_{2M}}{2}$, and $a_0 \le 1$, $a_N \le 1$. Then, $a_M \lesssim 1$.

Proof. Consider the sequence $\{b_k\}_k$, $b_k = a_k + k^2$. Its terms satisfy the inequalities

$$\forall k \in [M+1..N-1] \qquad b_k \le \frac{b_{k+1} + b_{k-1}}{2}$$

and $b_M \le \frac{b_0 + b_{2M}}{2}$. In particular, $\{b_k\}_k$ is convex on $[M..N]$. We also subtract the linear function $k\frac{b_N - b_0}{N} + b_0$ from it:

$$c_k = b_k - \left(k\frac{b_N - b_0}{N} + b_0 \right), \quad k \in \{0\} \cup [M..N].$$

The sequence $\{c_k\}_k$ is convex on $[M..N]$, equals zero at the endpoints 0 and N, and also satisfies the inequality $2c_M \le c_{2M}$. Thus, $c_M \le 0$ (otherwise, $c_{2M} \ge 2c_M \ge c_M$, which contradicts the convexity of c on the interval $[M..N]$). Therefore, $b_M \le \frac{k}{N}(b_N - b_0)$, and finally, $a_M \le \frac{M(N^2+1)}{N}$. $\qquad \square$

Remark 5.9. In fact, we have proved that $a_k \lesssim 1$ for any $k \in [M..N]$.

Remark 5.10. Using the homogeneity, one can replace the assumptions of Lemma 5.8 by

$$\forall k \in [M+1..N-1] \qquad a_k \le C + \frac{a_{k+1} + a_{k-1}}{2},$$

$a_M \le C + \frac{a_0 + a_{2M}}{2}$, and $a_0 \le C$, $a_N \le C$ for some positive constant C. Then, $a_M \lesssim C$.

Corollary 5.11. *The* HDR-*domain is convex in the sense that if* (k, s) *is a convex combination of* (k_1, s_1) *and* (k_2, s_2) *(we assume* $k, k_1, k_2 \in \mathbb{Z}_+$*), and the latter two points belong to the* HDR-*domain, then the former point lies in it as well.*

Proof. Consider the line passing through our three points. Let $X_0, X_2, \ldots, X_N$ be all the points with integer first coordinates lying on the segment connecting (k_1, s_1) and (k_2, s_2) (we enumerate the points in such a way that the k-coordinate increases with the index). Consider also the sequence

$$a_j = N(k_{X_j}, s_{X_j}), \quad j = 0, 1, \ldots, N.$$

By Corollary 5.7, this sequence satisfies the inequality

(67) $$a_j \leq C\|f\|_{L_p}^2 + \frac{a_{j-1} + a_{j+1}}{2}, \quad j = 1, 2, \ldots, N-1.$$

By the assumption, $a_0, a_N \lesssim \|f\|_{L_p}^2 + \|\psi(\cdot)\hat{f}(\cdot, h(\cdot))\|_{H^\ell}^2$. Thus, by Lemma 5.8 with $M = 0$ (in the light of Remark 5.10), a_j is bounded by $C(\|f\|_{L_p}^2 + \|\psi\hat{f}\|_{H^\ell(\Sigma)}^2)$. In particular, (k, s) belongs to the HDR-domain. $\square$

Corollary 5.12. *Let X be a point with natural k-coordinate lying in the intersection of friendly and subcritical domains. Suppose that the point Y lies on the segment LX, has natural first coordinate k_Y, and lies in the friendly domain. If $2k_Y \leq k_X$, then Y lies belongs to the* HDR-*domain.*

Proof. The proof of this corollary is very much similar to the proof of the previous one. We consider all the points on the segment LX that have integer first coordinates and lie inside the friendly domain. Suppose the leftmost of them has first coordinate M, let us call our points $Y_M, Y_{M+1}, \ldots, Y_N$ (so, $Y_N = X$). We also add the point $Y_0 = L$ to our sequence and consider the numbers

$$a_j = N(k_{Y_j}, s_{Y_j}), \quad j = 0, M, M+1, M+2, \ldots, N.$$

These numbers satisfy the inequality (67) for $j \in [M+1..N-1]$. Moreover, Corollary 5.7 provides the inequality

$$a_M \leq C\|f\|_{L_p}^2 + \frac{a_0 + a_{2M}}{2}.$$

Note that $2M \leq N$ since $N = k_X$ and $k_Y \in [M..N]$ since Y lies in the friendly domain and $k_X \geq 2k_Y$. At the endpoint M, we have the inequality

$$N(k_X, s_X) \lesssim \|f\|_{L_p}^2$$

since X lies in the subcritical part of the friendly domain (this inequality is the case $\beta = 0$ in Theorem 1.16). Thus, $a_N \lesssim \|f\|_{L_p}^2$. Clearly, $a_0 \leq \|\psi(\cdot)\hat{f}(\cdot, h(\cdot))\|_{H^\ell}^2$. So, Lemma 5.8 says all the points $Y_M, Y_{M+1}, \ldots, Y_N$ belong to the HDR-domain. In particular, Y does. $\square$

Our general strategy will be to apply Corollary 5.12 to the points X close to the point $K = (\kappa_p, \kappa_p)$. This will enable us to obtain "almost extremal points" of the HDR region, after that, we will apply Corollary 5.11 to pass to convex hulls. Before we pass to the cases, we explain the obstructions that prevent us from proving the sufficiency of the conditions in Proposition 1.11. They are of two types. First, we are able to work with points whose first coordinates are integers only. However, in the general case, the extremal points of the domain of admissible parameters need not necessarily have integer first coordinates. So, we cannot prove (and even formulate) HDR for them. This makes the convex hull we obtain smaller (we are able to reach only some "integer" points close to the extremal points) than it should be. The second obstruction is more severe. The problem comes from the inequality $2k_Y \leq k_X$ in Corollary 5.12. That restricts our "extremal points" from having too large k-coordinate, roughly speaking, their k-coordinates should satisfy $2k < \sigma_p$, if we want to apply Corollary 5.12. This will result in a considerable gap between our results and the conditions listed in Proposition 1.11 in the case when $\ell \geq 2$.
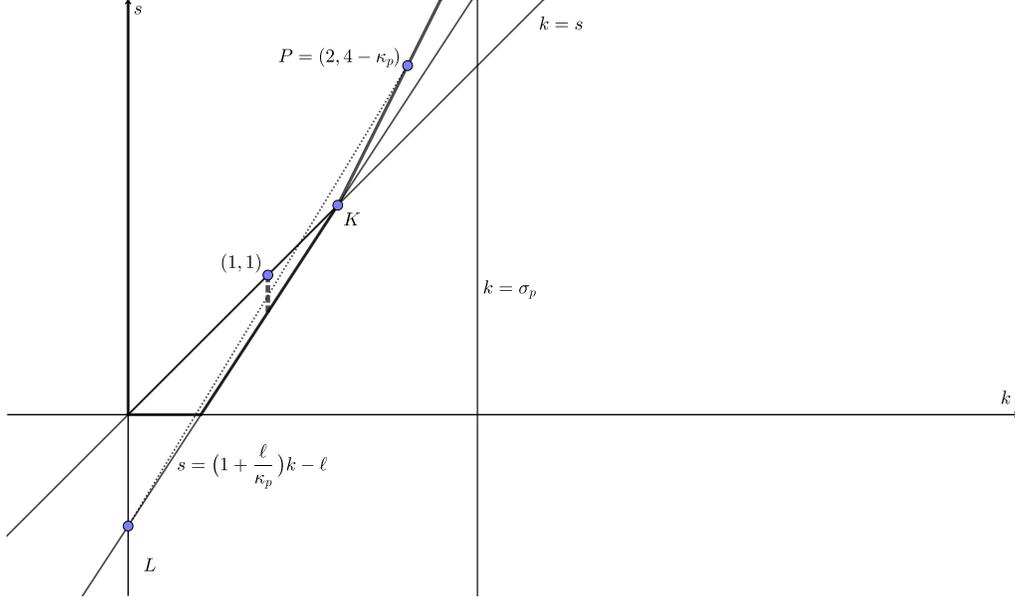
Now we pass to the cases.

FIGURE 7. What we can reach in the case $\ell \leq \kappa_p < 2$.

5.3. Statement of results by cases.

Case $\ell \leq \kappa_p < 2$**.** Our reasonings are illustrated by Figure 7. Clearly, here we are interested in the case $k = 1$ only (because if $k \geq 2$ and (k, s) lies in the HDR domain, then $k \leq s$ automatically). We consider the point $P = (2, 4 - \kappa_p)$ and assume P lies in the friendly region, that is, $2 < \sigma_p$. We draw a segment that connects P with L (it is the slant punctured segment on Figure 7). It crosses the line $k = 1$ at the point $(1, 2 - \frac{\kappa_p + \ell}{2})$. We apply Corollary 5.12 to the points P as X and $(1, 2 - \frac{\kappa_p + \ell}{2})$ as Y and obtain the theorem below.

Theorem 5.13. *Let* $\ell \leq \kappa_p < 2$*, let* $2 < \sigma_p$*. Then,* $\mathrm{HDR}_{\mathrm{loc}}(h, 1, s, \ell, p)$ *holds true provided*

$$s \geq \min\left(1, 2 - \frac{\kappa_p + \ell}{2}\right).$$

Case $\kappa_p < 2$, $\ell > \kappa_p$**.** Our reasonings are illustrated by Figure 8. This case is simpler than the previous one. We only need $2 < \sigma_p$ here. In this case, if $(1, s)$ lies on the vertical punctured segment, then it is an average of L and a point inside the intersection of the friendly domain with the subcritical domain. Thus, Corollary 5.12 leads to the theorem below.

Theorem 5.14. *Let* $\kappa_p < 2$*,* $\kappa_p < \ell$*, let* $2 < \sigma_p$*. Then,* $\mathrm{HDR}_{\mathrm{loc}}(h, 1, s, \ell, p)$ *holds true provided*
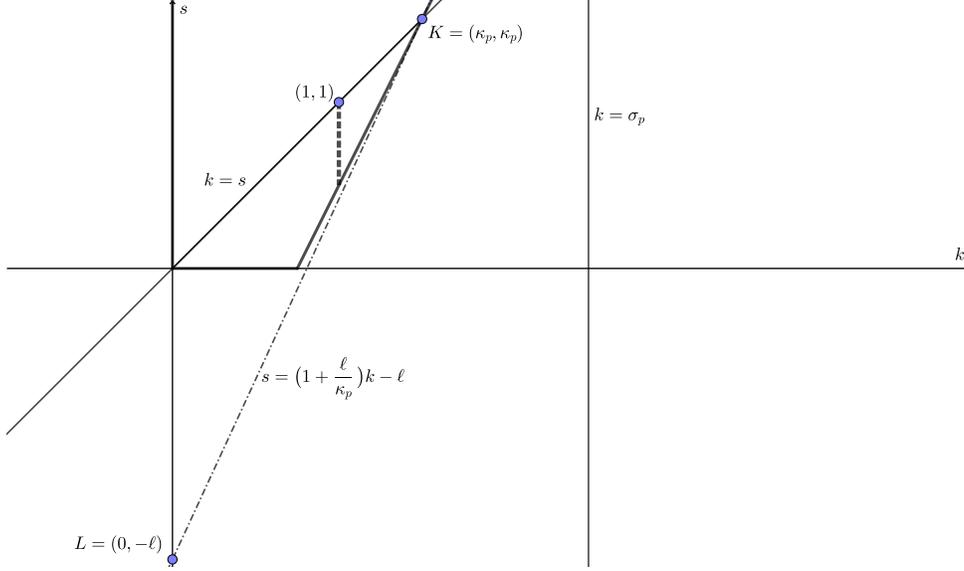
$$s \geq \min(1, 2 - \kappa_p).$$

FIGURE 8. What we can reach in the case $\kappa_p < 2$ and $\ell > \kappa_p$.

Case $2 \leq \kappa_p$ **and** $\ell \leq \frac{\kappa_p}{\kappa_p - 1}$**.** Our reasonings are illustrated by Figure 9. We introduce two auxiliary points P and Q:

$$P = (\lceil \kappa_p \rceil, 2\lceil \kappa_p \rceil - \kappa_p); \quad Q = ([\kappa_p], [\kappa_p]).$$

We have used two types of the notion "integer part of a number", see formula (20).

   We connect the point L to P and Q. Since the point Q lies in the intersection of friendly and subcritical regions, Corollary 5.12 applied to Q in the role of X says that $\mathrm{HDR}_{\mathrm{loc}}(h, k, s_k, \ell, p)$ is true for all pairs (k, s_k) such that $(k, s_k) \in LQ$, in other words

$$s_k = -\ell \frac{[\kappa_p] - k}{[\kappa_p]} + k.$$

Clearly, the same assertion is true for larger s when k is fixed. The situation with the point P is slightly more complicated: it may lie outside the friendly region if its k-coordinate is too large. If it is not so (i.e. $\lceil \kappa_p \rceil < \sigma_p$), then we may apply Corollary 5.12 to the point P in the role of X and achieve $\mathrm{HDR}_{\mathrm{loc}}(h, k, s_k, \ell, p)$ is true for all pairs (k, s_k) such that $(k, s_k) \in LP$, in other words

$$s_k = -\ell \frac{\lceil \kappa_p \rceil - k}{\lceil \kappa_p \rceil} + 2k - \frac{\kappa_p k}{\lceil \kappa_p \rceil}.$$

We summarize our results.

Theorem 5.15. *Let $2 \leq \kappa_p$ and let $\ell \leq \frac{\kappa_p}{\kappa_p - 1}$. If $\lceil \kappa_p \rceil \geq \sigma_p$, then $\mathrm{HDR}_{\mathrm{loc}}(h, k, s, \ell, p)$ holds true if*

$$s \geq -\ell \frac{[\kappa_p] - k}{[\kappa_p]} + k, \quad k \leq [\kappa_p].$$
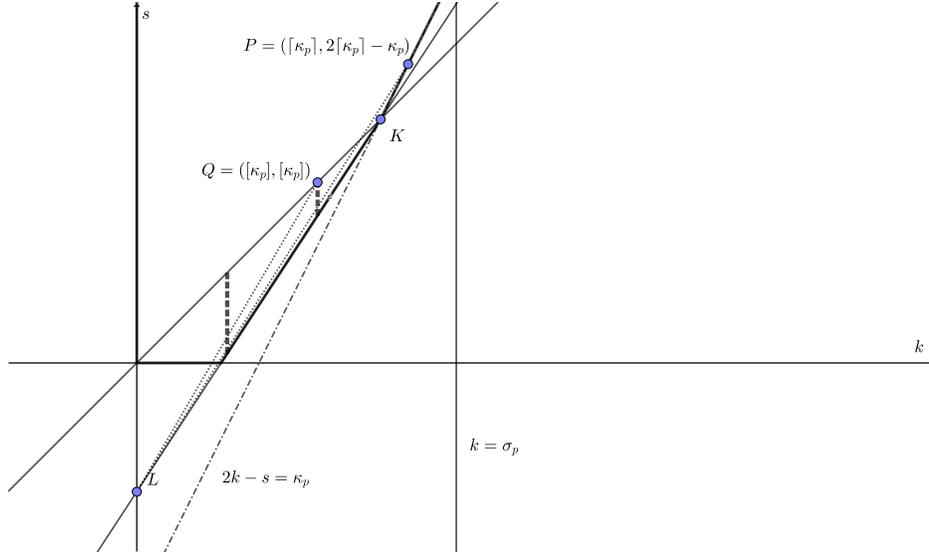
FIGURE 9. What we can reach in the case $2 \leq \kappa_p$ and $\ell \leq \frac{\kappa_p}{\kappa_p - 1}$.

If $\lceil \kappa_p \rceil < \sigma_p$, then $\mathrm{HDR}_{\mathrm{loc}}(h, k, s, \ell, p)$ holds true if

$$(68) \qquad s \geq \min \left(-\ell \frac{\lceil \kappa_p \rceil - k}{\lfloor \kappa_p \rfloor} + k, -\ell \frac{\lceil \kappa_p \rceil - k}{\lceil \kappa_p \rceil} + 2k - \frac{\kappa_p k}{\lceil \kappa_p \rceil} \right), \quad k \leq \lfloor \kappa_p \rfloor.$$

Remark 5.16. If $k = \lceil \kappa_p \rceil$ and $\mathrm{HDR}_{\mathrm{loc}}(h, k, s, \ell, p)$ holds true, then $s \geq k$.

Case $2 \leq \kappa_p$, $\frac{\kappa_p}{\kappa_p - 1} \leq \ell$. This case will be split into many subcases. We will need to construct two sequences of points generated by P and Q.

The points Q_j, $j = 1, 2, \ldots, \lfloor \kappa_p \rfloor$, are generated by Q. Namely,

$$Q_j = \begin{cases} LQ \cap \{(j, s) \mid s \in \mathbb{R}\}, & \text{if this point lies above the line } s = k - 1; \\ (j, j - 1), & \text{in the other case.} \end{cases}$$

The point Q_j may be described as the lowest possible point on the line $\{(j, s) \mid s \in \mathbb{R}\}$ that lies above the segment LQ and belongs to the friendly domain. See Figure 10.

Lemma 5.17. *For any $j < \lfloor \frac{\ell - 1}{\ell} \kappa_p \rfloor$, we have $Q_j = (j, j - 1)$. For $j \geq \lceil \frac{\ell - 1}{\ell} \kappa_p \rceil$, all points Q_j lie on the line LQ.*

Proof. The equation of the line LQ is

$$s = -\ell \frac{\lfloor \kappa_p \rfloor - k}{\lfloor \kappa_p \rfloor} + k.$$

FIGURE 10. Construction of the points Q_j.

To prove the first half of the lemma, it suffices to verify the inequality

$$j - 1 \geq -\ell\frac{[\kappa_p] - j}{[\kappa_p]} + j$$

when $j \leq [\frac{\ell-1}{\ell}\kappa_p] - 1$. This may be rewritten as

$$\ell j \leq (\ell - 1)[\kappa_p].$$

Clearly, it suffices to prove this inequality for the largest possible $j = [\frac{\ell-1}{\ell}\kappa_p] - 1$. In this case, we arrive at

$$\ell\Big[\frac{\ell - 1}{\ell}\kappa_p\Big] \leq (\ell - 1)[\kappa_p] + \ell.$$

We estimate the left hand-side with $(\ell - 1)\kappa_p$, which, in its turn, does not exceed $(\ell - 1)[\kappa_p] + (\ell - 1)$. The first assertion of the lemma is proved.

Similar to the previous reasoning, it suffices to verify the inequality

$$j - 1 \leq -\ell\frac{[\kappa_p] - j}{[\kappa_p]} + j$$

when $j \geq \lceil\frac{\ell-1}{\ell}\kappa_p\rceil$, to prove the second assertion of the lemma. This may be rewritten as

$$j \geq \frac{\ell - 1}{\ell}[\kappa_p],$$

which follows from $j \geq \frac{\ell-1}{\ell}\kappa_p$, which is weaker than our assumption $j \geq \lceil\frac{\ell-1}{\ell}\kappa_p\rceil$. So, we have proved the second half of the lemma. $\qquad\square$

The lemma says that, among all the points Q_j, only those with the indices $[\frac{\ell-1}{\ell}\kappa_p] - 1$, $[\frac{\ell-1}{\ell}\kappa_p]$, and $\lceil\frac{\ell-1}{\ell}\kappa_p\rceil$, may be the extremal points of the accessible domain.

The points P_j, $j = 1, 2, \ldots, [\kappa_p] - 1$, are generated by P in a similar manner:

$$P_j = \begin{cases} PL \cap \{(j, s) \mid s \in \mathbb{R}\}, & \text{if this point lies above the line } s = k - 1; \\ (j, j-1), & \text{in the other case.} \end{cases}$$

We also consider the point $P_{[\kappa_p]}$ separately:

$$P_{[\kappa_p]} = \begin{cases} PL \cap \{(j, s) \mid s \in \mathbb{R}\}, & \text{if this point lies above the line } s = 2k - \kappa_p; \\ ([\kappa_p], 2[\kappa_p] - \kappa_p), & \text{in the other case.} \end{cases}$$

Remark 5.18. The point $P_{[\kappa_p]}$ lies on LP if and only if $\kappa_p \geq \ell$.

Unfortunately, there is no analog of the first assertion of Lemma 5.17 for the points P_j. Here we can only say that for small j the points P_j lie on the line $s = k - 1$ and then at some moment they jump to the line LP. However, this "moment" can happen much earlier than $\frac{\ell-1}{\ell}\kappa_p$. We can only bound it from above.

Lemma 5.19. *For $j \geq \lceil\frac{\ell-1}{\ell}\kappa_p\rceil$, all points P_j lie on the line LP.*

Proof. Consider the case $\ell \leq \kappa_p$ first. The equation of the line LP is

$$s = -\ell\frac{\lceil\kappa_p\rceil - k}{\lceil\kappa_p\rceil} + \frac{k}{\lceil\kappa_p\rceil}(2\lceil\kappa_p\rceil - \kappa_p).$$

So, we need to verify the inequality

$$j - 1 \leq -\ell\frac{\lceil\kappa_p\rceil - j}{\lceil\kappa_p\rceil} + \frac{j}{\lceil\kappa_p\rceil}(2\lceil\kappa_p\rceil - \kappa_p).$$

This may be rewritten as

$$\frac{(\ell-1)\lceil\kappa_p\rceil}{\lceil\kappa_p\rceil - \kappa_p + \ell} \leq j.$$

So, it suffices to prove

$$\frac{(\ell-1)\lceil\kappa_p\rceil}{\lceil\kappa_p\rceil - \kappa_p + \ell} \leq \left\lceil\frac{\ell-1}{\ell}\kappa_p\right\rceil.$$

We will prove a stronger inequality

$$\frac{(\ell-1)\lceil\kappa_p\rceil}{\lceil\kappa_p\rceil - \kappa_p + \ell} \leq \frac{\ell-1}{\ell}\kappa_p,$$

which is equivalent to

$$\ell\lceil\kappa_p\rceil \leq \kappa_p\Big(\lceil\kappa_p\rceil - \kappa_p + \ell\Big).$$

This may be restated as

$$\ell(\lceil\kappa_p\rceil - \kappa_p) \leq \kappa_p(\lceil\kappa_p\rceil - \kappa_p),$$

which is true under our assumption $\ell \leq \kappa_p$.

In the other case $\ell > \kappa_p$, we have

$$\left\lceil\frac{\ell-1}{\ell}\kappa_p\right\rceil \geq \lceil(\kappa_p - 1)+\rceil > [\kappa_p],$$

so the statement of the lemma is empty in this case (we consider the points P_j with $j \leq [\kappa_p]$ only). $\square$

Lemma 5.20. *If j is a number between 1 and $\lceil\frac{\ell-1}{\ell}\kappa_p\rceil$ and $2j \leq [\kappa_p]$, then Q_j belongs to the HDR-domain. If $2j \leq \lceil\kappa_p\rceil$ and P belongs to the friendly region, then P_j belongs to the HDR domain.*

Note that P belongs to the friendly region if and only if $\lceil \kappa_p \rceil < \sigma_p$.

Proof. We prove the second assertion, the proof of the first one is completely similar. We consider two cases: P_j lies on LP and above LP. In the first case, we may apply Corollary 5.12 with P in the role of X and P_j in the role of Y. In the second case, we may apply the same corollary with P_j in the role of Y and the point of intersection of the lines LP_j and $\{(\lceil \kappa_p \rceil, s) \mid s \in \mathbb{R}\}$ in the role of X (the latter point lies above P since P_j lies above the segment LP, and thus belongs to the friendly domain). $\qquad \square$

We finally summarize our results.

Theorem 5.21. *Assume* $2 \leq \kappa_p$, $\frac{\kappa_p}{\kappa_p - 1} \leq \ell$. *The* HDR *domain contains the convex hull of points specified below. We always include the points* $(0,0)$, $(1,0)$, *and* Q *in our list. The other points are specified in the following table*:

	$\lceil \kappa_p \rceil < \sigma_p$	$\lceil \kappa_p \rceil \geq \sigma_p$
$2\lceil \frac{\ell-1}{\ell}\kappa_p \rceil \leq [\kappa_p]$	$Q_{\lceil \frac{\ell-1}{\ell}\kappa_p \rceil}, Q_{[\frac{\ell-1}{\ell}\kappa_p]}, Q_{[\frac{\ell-1}{\ell}\kappa_p]-1}, P_{\lceil \frac{\ell-1}{\ell}\kappa_p \rceil}, P_{[\frac{\ell-1}{\ell}\kappa_p]}$	$Q_{\lceil \frac{\ell-1}{\ell}\kappa_p \rceil}, Q_{[\frac{\ell-1}{\ell}\kappa_p]}, Q_{[\frac{\ell-1}{\ell}\kappa_p]-1}$
$2\lceil \frac{\ell-1}{\ell}\kappa_p \rceil \leq [\kappa_p] < 2\lceil \frac{\ell-1}{\ell}\kappa_p \rceil \leq \lceil \kappa_p \rceil$	$Q_{[\frac{\ell-1}{\ell}\kappa_p]}, Q_{[\frac{\ell-1}{\ell}\kappa_p]-1}, P_{\lceil \frac{\ell-1}{\ell}\kappa_p \rceil}, P_{[\frac{\ell-1}{\ell}\kappa_p]}$	$Q_{[\frac{\ell-1}{\ell}\kappa_p]}, Q_{[\frac{\ell-1}{\ell}\kappa_p]-1}$
$2\lceil \frac{\ell-1}{\ell}\kappa_p \rceil \leq [\kappa_p] < \lceil \kappa_p \rceil < 2\lceil \frac{\ell-1}{\ell}\kappa_p \rceil$	$Q_{[\frac{\ell-1}{\ell}\kappa_p]}, Q_{[\frac{\ell-1}{\ell}\kappa_p]-1}, P_{\lceil \frac{\ell-1}{\ell}\kappa_p \rceil}$	$Q_{[\frac{\ell-1}{\ell}\kappa_p]}, Q_{[\frac{\ell-1}{\ell}\kappa_p]-1}$
$[\kappa_p] < 2\lceil \frac{\ell-1}{\ell}\kappa_p \rceil \leq \lceil \kappa_p \rceil < 2\lceil \frac{\ell-1}{\ell}\kappa_p \rceil$	$Q_{[\frac{\ell-1}{\ell}\kappa_p]-1}, P_{\lceil \frac{\ell-1}{\ell}\kappa_p \rceil}$	$Q_{[\frac{\ell-1}{\ell}\kappa_p]-1}$
$\lceil \kappa_p \rceil < 2\lceil \frac{\ell-1}{\ell}\kappa_p \rceil$	$Q_{[\frac{[\kappa_p]}{2}]}, P_{[\frac{[\kappa_p]}{2}]}$	$Q_{[\frac{[\kappa_p]}{2}]}$

This theorem is a straightforward consequence of Lemma 5.20.

We note that the cases $\ell < \kappa_p$ and $\ell \geq \kappa_p$ are the same for our result (our answer in these cases are given by the last row in the table above, at least when $\kappa_p \geq 3$). However, the forms of the HDR-domain suggested by Proposition 1.11 differ in these cases (see Figures 5 and 6).

Proof of Theorem 1.12. Since κ_p is assumed to be a nonnegative integer, we have $K = P = Q$. Therefore, all the points Q_j and P_j lie on the lines LK and $s = k - 1$. Since $p > 1$, we have $\kappa_p = [\sigma_p] < \sigma_p$ as well.

When $0 \leq \ell \leq \frac{\kappa_p}{\kappa_p - 1}$, Theorem 5.15 immediately implies that for points (k, s) whose first coordinate is a positive integer, $\text{HDR}(\Sigma, k, s, \ell, p)$ holds whenever conditions (15) and (16) are satisfied; (68) turns into (16).

When $\ell > \frac{\kappa_p}{\kappa_p - 1}$ it remains to apply Theorem 5.21 and decode its results.

For the case $2\lceil \frac{\ell-1}{\ell}\kappa_p \rceil \leq \kappa_p$, the points $Q_{[\frac{\ell-1}{\ell}\kappa_p]-1}$, $Q_{[\frac{\ell-1}{\ell}\kappa_p]}$, and $Q_{\lceil \frac{\ell-1}{\ell}\kappa_p \rceil}$ straddle the intersection of the lines $s = k - 1$ and LK. The convex hull of these three Q_j together with points K and $(1,0)$ contains every point along $s = \max(k - 1, k - \ell\frac{(\kappa_p - k)}{\kappa_p})$ with $k = 1, 2, \ldots, \kappa_p$. Thus $\text{HDR}(\Sigma, k, s, \ell, p)$ holds provided (14), (15) and (16) are all satisfied.

In the case $2\lceil \frac{\ell-1}{\ell}\kappa_p \rceil > \kappa_p$ we have $2\lceil \frac{\ell-1}{\ell}\kappa_p \rceil > \lceil \kappa_p \rceil$. Thus, this case is described in the intersection of the last row and first column in the table above. We observe that the points $P_{[\frac{\kappa_p}{2}]}$ and $Q_{[\frac{\kappa_p}{2}]}$ coincide at the location $([\frac{\kappa_p}{2}], [\frac{\kappa_p}{2}] - 1)$. Then convex combinations of $P_{[\frac{\kappa_p}{2}]}$ and $(0,1)$ form a segment of the line $k = s + 1$, and convex combinations of $P_{[\frac{\kappa_p}{2}]}$ and K form a segment of the line $s \geq k - \frac{\kappa_p - k}{\kappa_p - [\frac{\kappa_p}{2}]}$. It follows that $\text{HDR}(\Sigma, k, s, \ell, p)$ holds provided (14), (15), and (18) are satisfied. $\qquad \square$

6. Sharpness

In this section, we consider the case where Σ is the paraboloid $\{\xi_d = |\xi_{\bar{d}}|^2\}$ as a representative example. We also assume that $\varphi \in C_0^\infty(\mathbb{R}^{d-1})$ is non-negative.

6.1. **Surface measure conditions.** Let χ be a smooth function of one variable supported in $[1,2]$ such that $\hat{\chi}^{(k)}(0) = 1$. Consider the functions f_n defined as

$$\hat{f}_n(\xi) = \varphi(\xi_{\bar{d}})\hat{\chi}\big(2^n(\xi_d - |\xi_{\bar{d}}|^2)\big).$$

The function f_n can be written explicitly:

$$f_n(x) = 2^{-n}\mathcal{F}_{\xi \to x}\Big[\varphi d\mathbb{P}\Big]\chi(2^{-n}x_d),$$

here $d\mathbb{P}$ is the Lebesgue measure on the paraboloid Σ. It is easy to observe two formulas:

$$\frac{\partial^k \hat{f}_n}{\partial \xi_d^k}(\xi) = 2^{nk}\varphi(\xi_{\bar{d}}), \quad \xi \in \Sigma; \qquad \|f_n\|_{L_p} \asymp 2^{n\sigma_p}.$$

We will also need the functions

$$f_{\leq N} = \sum_{n=0}^{N}(1+n)^{-1}2^{-nk}f_n.$$

Sharpness of (7). Assume $\phi = 1$ on the support of φ. We plug $f_{\leq N}$ into (3). The left hand-side is bounded away from zero by

$$\Big\|\frac{\partial^k \hat{f}_{\leq N}}{\partial \xi_d^k}\Big\|_{H^{-s}(\Sigma)} = \|\varphi\|_{H^{-s}(\Sigma)}\sum_{0 \leq n \leq N}(1+n)^{-1} \asymp \log(N).$$

As for the L_p-norm, we note that the functions f_n have disjoint supports, so,

$$\|f_{\leq N}\|_{L_p} = \Big(\sum_{n \leq N}\|f_n\|_{L_p}^p\Big)^{\frac{1}{p}} \asymp \Big(\sum_{n \leq N}(1+n)^{-p}\,2^{np(\sigma_p - k)}\Big)^{\frac{1}{p}}.$$

Since the left hand-side of (3) tends to infinity as $N \to \infty$, the right hand-side cannot be uniformly bounded. This means (7) holds true if $p > 1$. In the case $p = 1$, we get $k \leq \frac{d-1}{2}$ instead.

Necessity of $p < \frac{2d}{d+1+2k}$ **in Theorem 1.4.** Add the requirements $\hat{\chi}^{(j)}(0) = 0$ for all $j < k$. Then, f_n and $f_{\leq N}$ belong to $_{\Sigma}L_p^k$ and the same reasoning gives the necessity of $k < \sigma_p$ in (9) and (10), which is exactly $p < \frac{2d}{d+1+2k}$. As usual, the cases $k = \frac{d-1}{2}$ are permitted if $p = 1$.

Necessity of (15). We plug exactly the same functions $f_{\leq N}$ into (12). The H^{-s}-norm on the left hand-side and the L_p-norm on the right hand-side behave in the same manner as previously. Since we have assumed $\hat{\chi}(0) = 0$, there is no summand $\|\hat{f}\|_{H^\ell(\Sigma)}$ on the right hand-side.

Necessity of (31). As it was mentioned earlier, the quadratic inequality (29) is equivalent to its bilinear version (49). We work with the latter expression here. The functions f and g will be constructed from the functions f_n in a slightly different manner from before. To define g, we take χ that satisfies $\hat{\chi}^{(\alpha)}(0) = 1$ and set $g = f_0$. For the function f, we require $\hat{\chi}^{(j)}(0) = 0$ for all $j < k = \alpha + \beta$ and $\hat{\chi}^{(\alpha+\beta)}(0) = 1$, and set $f = f_{\leq N}$ (with $k = \alpha + \beta$). We plug these functions f and g into (49) and use the Newton–Leibniz formula (we assume $\psi = 1$ on the support of φ)

$$\Big|\Big(\frac{\partial}{\partial r}\Big)^\beta\Big\langle\frac{\partial^\alpha \hat{f}}{\partial \xi_d^\alpha}\psi, \frac{\partial^\alpha \hat{g}}{\partial \xi_d^\alpha}\psi\Big\rangle_{\dot{H}^{-\gamma}(\Sigma_r)}\Big|_{r=0}\Big| = \Big|\Big\langle\frac{\partial^{\alpha+\beta}\hat{f}}{\partial \xi_d^{\alpha+\beta}}\psi, \frac{\partial^\alpha \hat{g}}{\partial \xi_d^\alpha}\psi\Big\rangle_{\dot{H}^{-\gamma}(\Sigma)}\Big| = \|\varphi\|_{\dot{H}^{-\gamma}(\Sigma)}^2\sum_{n=0}^{N}\frac{1}{1+n} \asymp \log N.$$

On the right hand-side, we have

$$\|g\|_{L_p} \asymp 1, \quad \|f_{\leq N}\|_{L_p} \asymp \Big(\sum_{n \leq N}(1+n)^{-p}\,2^{np(\sigma_p - \alpha - \beta)}\Big)^{\frac{1}{p}}.$$

So, the necessity of (31) is proved.

Necessity of $\alpha + \beta < \frac{d-1}{p} + \frac{1}{r} - \frac{d+1}{2}$ **in Theorem 4.7.** This is proved in the same manner as in the previous paragraph. One should only replace the formula for the L_p-norm of f_n with

$$\|f_n\|_{L_r(L_p)} \asymp 2^{n(\frac{d-1}{p} + \frac{1}{r} - \frac{d+1}{2})}.$$

6.2. **Knapp examples.** We start with a Schwartz function f with compactly supported Fourier transform and define the functions f_n by the formula

$$(69) \qquad\qquad f_n(x) = n^{-\frac{d-1}{2}} f\Big(\frac{x_1}{n}, \frac{x_2}{n}, \dots, \frac{x_{d-1}}{n}, \frac{x_d}{n^2}\Big).$$

By homogeneity,

$$\Big\|\frac{\partial^k \hat{f}_n}{\partial \xi_d^k}\Big\|_{\dot{H}^{-s}(\Sigma)} = n^{2k-s}\Big\|\frac{\partial^k \hat{f}}{\partial \xi_d^k}\Big\|_{\dot{H}^{-s}(\Sigma)};$$
$$\|f_n\|_{L_p} = n^{\kappa_p}\|f\|_{L_p}.$$

with the caveat that the homogeneous Sobolev norm may already be infinite if $s \geq \frac{d-1}{2}$.

Necessity of (8). This can be obtained by simply plugging f_n into (3) and assuming $\phi = 1$ in a neighborhood of the origin.

Necessity of the condition $2k - s \leq \kappa_p$ **in Theorem 1.4.** We take $f \in {}_\Sigma L_p^k$ and note that $f_n \in {}_\Sigma L_p^k$ as well (recall that Σ is the paraboloid). It remains to plug f_n into (10) with the same assumption about ϕ.

Necessity of (17). Here we plug f_n generated by $f \in {}_\Sigma L_p$ into (12) and note that $\|\hat{f}_n\|_{H^\ell(\Sigma)} = 0$.

Necessity of (32). This follows from the formula

$$\frac{\partial^\beta}{\partial r^\beta}\Big\|\frac{\partial^\alpha \hat{f}_n}{\partial \xi_d^\alpha}\Big\|_{\dot{H}^{-\gamma}(\Sigma_r)}^2\Big|_{r=0} = n^{4\alpha+2\beta-2\gamma}\frac{\partial^\beta}{\partial r^\beta}\Big\|\frac{\partial^\alpha \hat{f}}{\partial \xi_d^\alpha}\Big\|_{\dot{H}^{-\gamma}(\Sigma_r)}^2\Big|_{r=0}.$$

Necessity of $2\alpha + \beta - \gamma \leq \frac{d-1}{p} + \frac{2}{r} - \frac{d+3}{2}$ **in Theorem 4.7.** One can prove this in the same manner as in the previous paragraph. One should only replace the formula for the L_p-norm of f_n with

$$\|f_n\|_{L_r(L_p)} \asymp 2^{n(\frac{d-1}{p} + \frac{2}{r} - \frac{d+3}{2})}.$$

6.3. **Pure shifts.** We start with a Schwartz function f and consider its shifts in the x_d direction:

$$(70) \qquad\qquad f_n(x) = f(x_1, x_2, \dots, x_{d-1}, x_d - n).$$

We also assume

$$\frac{\partial^j \hat{f}}{\partial \xi_d^j} = 0 \quad \text{on } \Sigma \text{ for } j = 1, 2, \dots, k$$

and $\hat{f} = 1$ on the support of ϕ. Then $\hat{f}_n(\xi) = f(\xi)e^{2\pi i n \xi_d}$ and

$$\Big\|\phi\frac{\partial^k \hat{f}_n}{\partial \xi_d^k}\Big\|_{H^{-s}(\Sigma)} = (2\pi n)^k\big\|\phi e^{2\pi i n|\cdot|^2}\big\|_{H^{-s}} \gtrsim n^k\Big(n^{-(d-1)}\int\limits_{\mathbb{R}^{d-1}}\Big|\check{\phi} * e^{-2\pi i\frac{|z|^2}{4n}}\Big|^2(z) \cdot (1+|z|)^{-2s}\, dz\Big)^{\frac{1}{2}}$$

$$\gtrsim n^k\Big(n^{-(d-1)}\int\limits_{|z|\lesssim n} (1+|z|)^{-2s}\, dz\Big)^{\frac{1}{2}} \gtrsim n^{k-s}.$$

Necessity of (6). This follows from the fact that $\|f_n\|_{L_p}$ does not depend on n.

Necessity of (13). Note that $\|\hat{f}_n\|_{H^\ell(\Sigma)}$ does not exceed cn^ℓ (this estimate reduces to the product rule in the case $\ell \in \mathbb{Z}_+$; the general case follows from the case $\ell \in \mathbb{Z}_+$ by the Cauchy–Schwarz inequality). Comparing the left and right parts of (12), we get (13).

Necessity of (14)**.** We consider the functions f_n generated by the rule (70) from a function $f \in {}_\Sigma L_p$ such that $\frac{\partial \hat{f}}{\partial \xi_d} = \varphi$ and $\varphi = 1$ on the support of ϕ, and all higher order (up to order k) derivatives of $\hat{f}$ vanish on Σ. Then, $f_n \in {}_\Sigma L_p$ as well, so, there is no term $\|\hat{f}_n\|_{H^\ell}$ on the right hand-side of (12). However, on left hand-side, we cannot have n^{k-s}, but only have growth n^{k-s-1} since

$$\frac{\partial^k \hat{f}_n}{\partial \xi_d^k}\bigg|_\Sigma (\xi_{\bar{d}}) = (2\pi i n)^{k-1} \varphi(\xi_{\bar{d}}) e^{2\pi i n |\xi_{\bar{d}}|^2}.$$

Thus, n^{k-s-1} should be bounded if (12) holds, which is exactly (14).

Necessity of (30)**.** Consider a Schwartz function f of d variables such that for any $j \in [0..\alpha + \beta]$ we have

$$(71) \qquad \frac{\partial^j \hat{f}}{\partial \xi_d^j} = 1 \quad \text{on } \Sigma \cap V.$$

Let f_n be generated by (70) from f. We plug f_n into (29). We first compute the "interior" derivative:

$$\frac{\partial^\alpha \hat{f}_n}{\partial \xi_d^\alpha}(\xi)\psi(\xi) = \frac{\partial^\alpha \left[\hat{f} e^{2\pi i n \xi_d}\right]}{\partial \xi_d^\alpha}(\xi)\psi(\xi) = e^{2\pi i n \xi_d}\psi(\xi)\sum_{j=0}^{\alpha} C_\alpha^j \frac{\partial^j \hat{f}}{\partial \xi_d^j}(\xi)(2\pi i n)^{\alpha - j}.$$

Therefore,

(72)

$$\left(\frac{\partial}{\partial r}\right)^\beta \left\|\frac{\partial^\alpha \hat{f}_n}{\partial \xi_d^\alpha}\psi\right\|^2_{\dot{H}^{-\gamma}(\Sigma_r)}\bigg|_{r=0} = \left(\frac{\partial}{\partial r}\right)^\beta \left\|e^{2\pi i n(|\xi_{\bar{d}}|^2 + r)}\psi(\xi_{\bar{d}})\sum_{j=0}^{\alpha} C_\alpha^j \frac{\partial^j \hat{f}}{\partial \xi_d^j}(\xi_{\bar{d}}, |\xi_{\bar{d}}|^2 + r)(2\pi i n)^{\alpha - j}\right\|^2_{\dot{H}^{-\gamma}}\bigg|_{r=0} =$$

$$\left(\frac{\partial}{\partial r}\right)^\beta \left\|e^{2\pi i n |\xi_{\bar{d}}|^2}\psi(\xi_{\bar{d}})\sum_{j=0}^{\alpha} C_\alpha^j \frac{\partial^j \hat{f}}{\partial \xi_d^j}(\xi_{\bar{d}}, |\xi_{\bar{d}}|^2 + r)(2\pi i n)^{\alpha - j}\right\|^2_{\dot{H}^{-\gamma}}\bigg|_{r=0} =$$

$$\sum_{k=0}^{\beta} C_\beta^k \left\langle e^{2\pi i n |\xi_{\bar{d}}|^2}\psi(\xi_{\bar{d}})\sum_{j=0}^{\alpha} C_\alpha^j \frac{\partial^{j+k}\hat{f}}{\partial \xi_d^{j+k}}(\xi_{\bar{d}}, |\xi_{\bar{d}}|^2)(2\pi i n)^{\alpha - j},\right.$$

$$\left. e^{2\pi i n |\xi_{\bar{d}}|^2}\psi(\xi_{\bar{d}})\sum_{j=0}^{\alpha} C_\alpha^j \frac{\partial^{j+\beta-k}\hat{f}}{\partial \xi_d^{j+\beta-k}}(\xi_{\bar{d}}, |\xi_{\bar{d}}|^2)(2\pi i n)^{\alpha - j}\right\rangle_{\dot{H}^{-\gamma}} \overset{(71)}{=}$$

$$\sum_{k=0}^{\beta} C_\beta^k \left\langle e^{2\pi i n |\xi_{\bar{d}}|^2}\psi(\xi_{\bar{d}})\sum_{j=0}^{\alpha} C_\alpha^j (2\pi i n)^{\alpha - j}, e^{2\pi i n |\xi_{\bar{d}}|^2}\psi(\xi_{\bar{d}})\sum_{j=0}^{\alpha} C_\alpha^j (2\pi i n)^{\alpha - j}\right\rangle_{\dot{H}^{-\gamma}} =$$

$$2^\beta (1 + 2\pi i n)^{2\alpha}\left\|e^{2\pi i n |\cdot|^2}\psi(\cdot)\right\|^2_{\dot{H}^{-\gamma}} = 2^\beta (1 + 2\pi i n)^{2\alpha}\int_{\mathbb{R}^{d-1}}\left|\left[\hat{\psi} * \left(n^{-\frac{d-1}{2}} e^{2\pi i \frac{|\zeta|^2}{4n}}\right)\right](z)\right|^2 |z|^{-2\gamma}\, dz$$

$$\gtrsim n^{2\alpha - d + 1}\int_{|z|\lesssim n} |z|^{-2\gamma}\, dz \gtrsim n^{2\alpha - 2\gamma}.$$

Thus, the left hand-side of (29) grows at least as fast as $n^{2\alpha - 2\gamma}$, whereas the right hand-side does not change. This proves the necessity of the condition (30).

Necessity of condition $\gamma \geq \alpha$ in Theorem 4.7. This is obtained by completely the same method in the case $r \leq 2$. For the case $r \geq 2$, we can only prove the necessity of the non-strict inequality $\gamma - \alpha \geq \frac{1}{2} - \frac{1}{r}$.

For that we slightly modify the construction above. We consider the function

$$F_n = \sum_{j=n}^{2n} \varepsilon_j f_{Aj},$$

where the functions f_j are generated by (70), A is a sufficiently large number, and ε_j are randomly chosen signs. Then,

$$\mathbb{E}\left(\frac{\partial}{\partial r}\right)^\beta \left\|\frac{\partial^\alpha \hat{F}_n}{\partial \xi_d^\alpha}\psi\right\|_{\dot{H}^{-\gamma}(\Sigma_r)}^2\bigg|_{r=0} = \sum_{j=n}^{2n}\left(\frac{\partial}{\partial r}\right)^\beta\left\|\frac{\partial^\alpha \hat{f}_{Aj}}{\partial\xi_d^\alpha}\psi\right\|_{\dot{H}^{-\gamma}(\Sigma_r)}^2\bigg|_{r=0} \overset{(72)}{\gtrsim} n^{2\alpha-2\gamma+1}.$$

On the other hand, disregarding the choice of the signs ε_j,

$$\|F_n\|_{L_r(L_p)} \asymp n^{\frac{1}{r}}$$

provided A is sufficiently large (this number is needed to diminish the influence of Schwartz tails on this almost orthogonality). It remains to choose ε_j with the largest possible quantity on the right hand-side and compare the two sides.

Necessity of condition $p \leq 2$ in Theorem 4.7. This can be obtained by a construction similar to the one described in the previous paragraph, except with functions f_n shifted in the x_1 direction instead of the x_d direction.

6.4. **Shifted Knapp example.** We need to modify the classical Knapp construction to get the necessity of (16). We take some sequence $\{D_n\}_n$ and modify the functions f_n generated by (69). Now we also shift them:

$$f_n = n^{-d-1}f\left(\frac{x_1}{n}, \frac{x_2}{n}, \dots, \frac{x_{d-1}}{n}, \frac{x_d - D_n}{n^2}\right).$$

We require $D_n \gg n^2$ and do not require the vanishing $f \in {}_\Sigma L_p$. The L_p-norms are influenced by scaling but do not depend on the size of the shifts:

$$\|f_n\|_{L_p} \asymp n^{(d+1)(\frac{1}{p}-1)}.$$

Let $\hat{f}(\zeta, |\zeta|^2)$ be $g(\zeta)$, here g is a smooth function, let us assume it is compactly supported and has non-zero integral. Then,

$$\|\hat{f}_n\|_{\dot{H}^\ell(\Sigma)} = \left\|g(n\cdot)e^{2\pi i D_n|\cdot|^2}\right\|_{\dot{H}^\ell} = n^{\ell-\frac{d-1}{2}}\left\|g(\cdot)e^{2\pi i\frac{D_n}{n^2}|\cdot|^2}\right\|_{\dot{H}^\ell} \lesssim n^{\ell-\frac{d-1}{2}}\left(\frac{D_n}{n^2}\right)^\ell.$$

The latter estimate can be proved via the product rule for the case $\ell \in \mathbb{Z}_+$ and reduced to this case with the help of the Cauchy–Schwarz inequality. Similarly,

$$\left\|\frac{\partial^k \hat{f}_n}{\partial\xi_d^k}\right\|_{\dot{H}^{-s}(\Sigma)} \asymp D_n^k\left\|g(n\cdot)e^{2\pi i D_n|\cdot|^2}\right\|_{\dot{H}^{-s}} = D_n^k n^{-s-\frac{d-1}{2}}\left\|g(\cdot)e^{2\pi i\frac{D_n}{n^2}|\cdot|^2}\right\|_{\dot{H}^{-s}} \gtrsim$$

$$D_n^k n^{-s-\frac{d-1}{2}}\left(\frac{D_n}{n}\right)^{-\frac{d-1}{2}}\left(\int_{|z|\lesssim\frac{D_n}{n^2}}|z|^{-2s}\right)^{\frac{1}{2}} \gtrsim D_n^k n^{-s-\frac{d-1}{2}}\left(\frac{D_n}{n^2}\right)^{-s}.$$

So, if (12) is true, then

$$(73) \qquad\qquad D_n^{k-s}n^{s-\frac{d-1}{2}} \lesssim D_n^\ell n^{-\ell-\frac{d-1}{2}} + n^{(d+1)(\frac{1}{p}-1)}$$

whenever $D_n \gg n^2$. We recall $k - s \leq \ell$ by (13) (the necessity of which is already proved), so, the first term on the right dominates the left hand-side when D_n is sufficiently large. We want to make D_n as small as possible in such a way that the left hand-side is still greater than the second summand on the right. Let

$$D_n = n^{\frac{\kappa_p - s}{k-s}}\log n.$$

Note that such a choice of D_n guarantees $D_n \gg n^2$ by (17) and the assumption $k > s$. Plugging it back to (73), we get

$$n^{\kappa_p - s} n^{s - \frac{d-1}{2}} (\log n)^{k-s} \lesssim n^{\frac{(\kappa_p - s)\ell}{k-s} - \ell - \frac{d-1}{2}} (\log n)^\ell,$$

which, after a tiny portion of algebra and (13), leads to

$$\frac{k\ell}{s + \ell - k} \leq \kappa_p,$$

which is (16).

7. ADDITIONAL LEMMAS AND SUPPLEMENTARY MATERIAL

7.1. **Localization argument.** We need to localize the HDR inequalities and also replace the gradient with a single directional derivative. Namely, we want to reduce $\mathrm{HDR}(\Sigma, k, s, \ell, p)$ to a collection of statements $\mathrm{HDR}_{\mathrm{loc}}(h, k, s, \ell, p)$ defined below. A similar principle works for inequalities of the type (3), (9), (10) and the proof is completely identical.

Definition 7.1. *Let the numbers k, s, ℓ, p be of the same nature as in Definition 1.9. Let U be a neighborhood of the origin in $\mathbb{R}^{d-1}$, let $h \colon U \to \mathbb{R}$ be a smooth function such that $h(0) = 0$, $\nabla h(0) = 0$, and the determinant of the Hessian of h at the origin does not vanish. Further, we assume (26). We say that the statement $\mathrm{HDR}_{\mathrm{loc}}(h, k, s, \ell, p)$ holds true if the inequality*

$$\left\| \psi(\cdot) \frac{\partial^k \hat{f}}{\partial \xi_d^k} (\cdot, h(\cdot)) \right\|_{H^{-s}(\mathbb{R}^{d-1})} \lesssim_\psi \|f\|_{L_p} + \left\| \psi(\cdot) \hat{f}(\cdot, h(\cdot)) \right\|_{H^\ell(\mathbb{R}^{d-1})}$$

holds true for any smooth function ψ supported in U.

Lemma 7.2. *The statement $\mathrm{HDR}(\Sigma, k, s, \ell, p)$ is true provided the statement $\mathrm{HDR}_{\mathrm{loc}}(h, k, s, \ell, p)$ is true for any h satisfying the conditions of Definition 7.1.*

Proof. We need to prove (12) with a fixed compactly supported smooth function ϕ. We find a smooth partition of unity $\{\Phi_n\}_n$ on Σ, each function Φ_n supported in a small ball V_n and each V_n lies in a chart neighborhood of a certain point $\xi_n \in \Sigma$. For each n fixed, we identify ξ_n with the origin of $\mathbb{R}^d$, the tangent plane $T_{\xi_n}\Sigma$ with $\mathbb{R}^{d-1}$, and get a graph representation for $\Sigma \cap V_n$:

$$\Sigma \cap V_n = \{(\zeta, h_n(\zeta)) \mid \zeta \in U_n\},$$

where U_n is a neighborhood of the origin in $\mathbb{R}^{d-1}$. If the partition of unity is sufficiently fine, then the function h_n satisfies (26). We estimate the left hand-side of (12) by the triangle inequality

$$\|\phi \nabla^k \hat{f}\|_{H^{-s}(\Sigma)} \leq \sum_n \|\phi \Phi_n \nabla^k \hat{f}\|_{H^{-s}(\Sigma)}.$$

Note that the sum on the right is, in fact, finite. We fix n. We are going to use the following algebraic fact: there exists a finite collection of vectors v_n in $\mathbb{R}^d$ such that any homogeneous polynomial of degree k is a linear combination of the monomials $\langle \cdot, v_n \rangle^k$; moreover, such vectors v_n may be chosen arbitrarily close to any fixed vector. Since the determinant of the Hessian of h_n is non-zero, the normals $\mathbf{n}_\zeta$ to Σ at the points $(\zeta, h_n(\zeta))$ cover a neighborhood of the vector $(0, 0, \ldots, 1)$ in S^{d-1} (the unit sphere in $\mathbb{R}^d$). Thus, we may choose finitely many points ζ_j in a sufficiently small neighborhood of the origin such that

$$(74) \qquad \forall \alpha \in \mathbb{Z}_+^d \text{ such that } |\alpha| = k \qquad \frac{\partial^\alpha}{\partial \xi^\alpha} \text{ is a linear combination of } \left\{ \frac{\partial^k}{\partial \mathrm{n}_{\zeta_j}^k} \right\}_j;$$

$$(75) \qquad \mathrm{n}_{\zeta_j} \nparallel T_\zeta \Sigma \text{ for any } j \text{ and any } \zeta \in U_n.$$

This allows us to write the estimate

$$(76) \qquad \|\phi\Phi_n\nabla^k\hat{f}\|_{H^{-s}(\Sigma)} \lesssim \sum_j \Big\|\phi\Phi_n\frac{\partial^k\hat{f}}{\partial \mathrm{n}_{\zeta_j}^k}\Big\|_{H^{-s}(\Sigma)}.$$

Now we restrict our attention to each point ζ_j individually. We adjust our coordinates to this point: now ζ_j is the origin, we also identify $T_{\zeta_j}\Sigma$ with $\mathbb{R}^{d-1}$. The summand corresponding to j on the right hand-side of the previous inequality transforms into

$$\Big\|\Psi\frac{\partial^k\hat{f}}{\partial\xi_d^k}\Big\|_{H^{-s}(\Sigma)},$$

where Ψ is a certain smooth function supported in V_n. By the assumption (75),

$$\Sigma\cap V_n = \Big\{\big(\zeta, h_{n,j}(\zeta)\big) \,\Big|\, \zeta \in U_{n,j}\Big\},$$

where $U_{n,j}$ is a neighborhood of the origin in $\mathbb{R}^{d-1}$, and $h_{n,j}$ satisfies (26) (with the constant $\frac{1}{4}$ instead of $\frac{1}{10}$ possibly). Take a smooth non-negative function ψ that is supported in $U_{n,j}$ and is bounded away from zero on the projection of the support of Ψ to $\mathbb{R}^{d-1}$. Then, clearly,

$$\Big\|\Psi\frac{\partial^k\hat{f}}{\partial\xi_d^k}\Big\|_{H^{-s}(\Sigma)} \lesssim \Big\|\psi(\cdot)\frac{\partial^k\hat{f}}{\partial\xi_d^k}(\cdot, h(\cdot))\Big\|_{H^{-s}(\mathbb{R}^{d-1})}.$$

We also note that the norms

$$\|g\|_{H^{-s}(\Sigma)} \quad \text{and} \quad \|g\big(\cdot, h_{n,j}(\cdot)\big)\|_{H^{-s}(\mathbb{R}^{d-1})}$$

are comparable for functions g supported on $\Sigma\cap V_n$. Thus, by $\mathrm{HDR}_{\mathrm{loc}}(h_{n,j}, k, s, \ell, p)$, we may bound each summand in (76) by

$$\|f\|_{L_p} + \|\psi(\cdot)\hat{f}\psi(\cdot, h(\cdot))\|_{H^\ell(\mathbb{R}^{d-1})} \lesssim \|f\|_{L_p} + \|\Psi\hat{f}\|_{H^\ell(\Sigma)}.$$

It remains to note that we have a finite number of summands both over j and n. $\qquad\square$

Remark 7.3. Consider Banach spaces $X_1, X_2, \ldots, X_m$ of functions on Σ such that multiplication operators

$$\varphi \mapsto \psi\varphi, \quad \varphi \in X_m,$$

are bounded on X_m whenever $\psi \in C_0^\infty(\Sigma)$. The inequality

$$\big\|\phi(\nabla^k\hat{f})|_\Sigma\big\|_{H^{-s}(\Sigma)} \lesssim_\phi \Big(\|f\|_{L_p(\mathbb{R}^d)} + \sum_{j=1}^m \|\Phi\hat{f}\|_{X_j}\Big)$$

may be reduced to local form

$$\Big\|\psi(\cdot)\frac{\partial^k\hat{f}}{\partial\xi_d^k}(\cdot, h(\cdot))\Big\|_{H^{-s}(\mathbb{R}^{d-1})} \lesssim_\psi \|f\|_{L_p} + \sum_{j=1}^m \|\hat{f}\|_{X_j}, \quad \mathrm{supp}\,\psi \subset U,$$

and U satisfies the usual assumptions, with the same argument as in the proof of Lemma 7.2. In particular, the case $X_j = \{0\}$ allows to reduce $\mathcal{R}_{\mathrm{w}}^k(\Sigma, p, s)$ to $\mathcal{R}^k(\Sigma, p, s)$ (see Definitions 3.5 and 3.9).

7.2. **A version of the Stein–Weiss inequality.**

7.2.1. *Case $p \in [1,2]$.* Let $L_p(w)$ be the weighted Lebesgue space:
$$f \in L_p(w) \quad \Leftrightarrow \quad fw \in L_p.$$
Let also $\mathcal{C}_b$ be the operator of convolution with the function $(1+|x|)^{-b}$. In this section, we work with functions on $\mathbb{R}$.

Theorem 7.4. *Let $a \geq 0$, let $p \in [1,2]$. The operator $\mathcal{C}_b$ maps the space $L_p((1+|x|)^a)$ to its dual space $L_{p'}((1+|x|)^{-a})$ if*

    (1) *$b \leq 0$ and*
- *$p = 1$ and $a + b \geq 0$;*
- *$p > 1$ and $a + b > 1 - \frac{1}{p}$;*

    (2) *$b \in (0,1)$ and $2a + b \geq 2 - \frac{2}{p}$;*

    (3) *$b = 1$ and*
- *$p < 2$;*
- *$p = 2$ and $a > 0$;*

    (4) *$b > 1$.*

Theorem 7.4 is a variation on the classical Stein–Weiss inequality from [23]. In the classical setting, the convolutional kernel and weights are homogeneous.

Remark 7.5. The conditions listed in Theorem 7.4 are also necessary.

7.2.2. *Case $p > 2$.*

Theorem 7.6. *Let $p \in (2,\infty]$, and let*

    (1) *$a + b > 1 - \frac{1}{p}$;*

    (2) *$2a + b > 2 - \frac{2}{p}$;*

    (3) *$a > \frac{1}{2} - \frac{1}{p}$.*

Then the operator $\mathcal{C}_b$ maps $L_p((1+|x|)^a)$ to $L_{p'}((1+|x|)^{-a})$.

Theorems 7.4 and 7.6 are proved by directly examining the $L_p \to L_{p'}$ boundedness of integral operators with kernel
$$K_{a,b}(x,y) = (1+|x|)^{-a}(1+|y|)^{-a}(1+|x-y|)^{-b}$$
in the cases $p = 1$, $p = 2$, and $p = \infty$, then performing complex interpolation of operators. These are elementary bounds based on $\|K_{a,b}\|_{L_\infty}$ when $p = 1$, the Schur test when $p = 2$, and on $\|K_{a,b}\|_{L_1}$ when $p = \infty$.

7.3. **Some endpoint estimates.** To formulate the endpoint version of inequality (3), we need some Besov spaces (see [3]). Given a function f, we define the Besov $B_2^{-\frac{d-1}{2},\infty}$ norm by the formula
$$\|f\|_{B_2^{-\frac{d-1}{2},\infty}} = \sup_{k \geq 0} \ 2^{-\frac{d-1}{2}k}\|P_k f\|_{L_2},$$
where the P_k, $k \geq 1$, are the Littlewood–Paley projectors on the annuli $B_{2^k}(0) \setminus B_{2^{k-1}}(0)$ and P_0 is the spectral projector on the unit ball $B_1(0)$ (the symbol $B_r(x)$ denotes the $(d-1)$-dimensional Euclidean ball of radius r centered at x). Using the standard properties of Besov spaces, one may then define Besov spaces on smooth submanifolds of $\mathbb{R}^d$ as well as on their reasonable subdomains.

Proposition 7.7. *The inequality*
$$\left\|\hat{g}(\cdot, h(\cdot))\psi(\cdot)\right\|_{B_2^{-\frac{d-1}{2},\infty}(\mathbb{R}^{d-1})} \lesssim_\psi \|g\|_{L_1((1+|x_d|)^{-\frac{d-1}{2}})}$$
is true for any h and ψ satisfying the standard requirements.

The norm in the weighted space on the right hand-side is given by the formula

$$\|g\|_{L_1((1+|x_d|)^{-\frac{d-1}{2}})} = \int_{\mathbb{R}^d} |g(x)|(1+|x_d|)^{-\frac{d-1}{2}}\, dx.$$

Similarly, $f \in L_p((1+|x_d|)^{-\alpha})$ whenever $f(x)(1+|x_d|)^{-\alpha} \in L_p$.

Proof. Since the delta measures are the extremal points of the unit ball in the space of measures, it suffices to prove the proposition for the case where g is a delta measure:

$$\left\| e^{2\pi i(\langle x_{\bar{d}}, \cdot \rangle + x_d h(\cdot))}\psi(\cdot) \right\|_{B_2^{-\frac{d-1}{2},\infty}} \lesssim (1+|x_d|)^{-\frac{d-1}{2}}.$$

By the Van der Corput Lemma (for $h(\cdot) = |\cdot|^2$, this is also the Schrödinger dispersive bound),

$$\left\| \mathcal{F}_\zeta\left[e^{2\pi i(\langle x_{\bar{d}}, \zeta \rangle + x_d h(\zeta))}\psi(\zeta) \right] \right\|_{L_\infty} \lesssim (1+|x_d|)^{-\frac{d-1}{2}}.$$

Thus, we need to prove the inequality

$$\sup_{k \geq 0} \left(2^{-k(d-1)} \int_{B_{2^k}(0)\setminus B_{2^{k-1}}(0)} (1+|x_d|)^{-(d-1)}\, dy \right)^{\frac{1}{2}} \lesssim (1+|x_d|)^{-\frac{d-1}{2}},$$

which is obvious. $\qquad\square$

Corollary 7.8. *Let d be odd. If we apply Proposition 7.7 with the function $g(x) = (-2\pi i x_d)^{\frac{d-1}{2}}f(x)$, we get the local form of the endpoint case in (3):*

$$\left\| \frac{\partial^{\frac{d-1}{2}}\hat{f}}{\partial \xi_d^{\frac{d-1}{2}}}(\cdot, h(\cdot))\psi(\cdot) \right\|_{B_2^{-\frac{d-1}{2},\infty}(\mathbb{R}^{d-1})} \lesssim_\psi \|f\|_{L_1}.$$

Using Remark 7.3, we may pass to the global form:

$$\left\| (\phi \nabla^{\frac{d-1}{2}}\hat{f})|_\Sigma \right\|_{B_2^{-\frac{d-1}{2},\infty}(\Sigma)} \lesssim_\phi \|f\|_{L_1(\mathbb{R}^d)}.$$

Since $B_2^{-\frac{d-1}{2},\infty} \hookrightarrow H^{-s}$ for $s > \frac{d-1}{2}$, we also have

$$\left\| (\phi \nabla^{\frac{d-1}{2}}\hat{f})|_\Sigma \right\|_{H^{-s}(\Sigma)} \lesssim_\phi \|f\|_{L_1(\mathbb{R}^d)}$$

for $s > \frac{d-1}{2}$.

Now we will show how to derive Proposition 1.1 from the case $k = 0$ considered in [6] and Proposition 7.7. We consider the inequality

$$\left\| \hat{g}(\cdot, h(\cdot))\psi(\cdot) \right\|_{H^{-s}(\mathbb{R}^{d-1})} \lesssim_\psi \|g\|_{L_p((1+|x_d|)^{-k})},$$

which, as we have seen, is stronger than (3). Note that in such a formulation, k might be real. We know the inequality holds true in the case $k = 0$ (from [6]) and is almost true when $k = s = \frac{d-1}{2}$, $p = 1$ (from Proposition 7.7). We claim that any triple $(k, s, \frac{1}{p})$ that satisfies the necessary conditions of Proposition 1.1 might be represented as a convex combination of the said cases:

$$\left(k, s, \frac{1}{p} \right) = \theta_+ \left(0, s_+, \frac{1}{p_+} \right) + \theta_- \left(\frac{d-1}{2}, \frac{d-1}{2}, 1 \right).$$

Solving several elementary equations, we see

$$\theta_- = \frac{2k}{d-1}, \quad \theta_+ = \frac{d-1-2k}{d-1}, \quad p_+ = p\frac{d-1-2k}{d-1-2kp}, \quad s_+ = \frac{(s-k)(d-1)}{d-1-2k}.$$

We leave to the reader the verification of the conditions

$$0 < \sigma_{p+}, \quad \text{and} \quad -s_+ \le \kappa_{p+}$$

(the easiest way to do this is to sketch the $3D$-domain of admissible $(k, s, \frac{1}{p})$) and explain how we interpolate the inequality. First, we note that linear operator that maps g to $\psi \hat{g}|_\Sigma$ does not depend on the varying parameters, so we may use the classical interpolation theory, specifically, the real interpolation method (see [3]). For the image of our operator, we use the formula

$$(H^{-s_+}, B_2^{-\frac{d-1}{2}, \infty})_{\theta_-, 2} = H^{-s},$$

see [3]. For the domain, we need to show that

$$(L_{p_+}, L_1((1 + |x_d|)^{-\frac{d-1}{2}}))_{\theta_-, 2} \supset L_p((1 + |x_d|)^{-k}).$$

In fact,

$$(L_{p_+}, L_1((1 + |x_d|)^{-\frac{d-1}{2}}))_{\theta_-, 2} = L_{p,2}((1 + |x_d|)^{-k}),$$

where the latter space is the space of all functions f such that $f(1 + |x_d|)^{-k} \in L_{p,2}$ (see [10]). It is clear that $L_p = L_{p,p} \hookrightarrow L_{p,2}$ since $p \le 2$.

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