THE HELMHOLTZ EQUATION WITH L^p DATA AND BOCHNER-RIESZ MULTIPLIERS

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ABSTRACT. We prove the existence of L^2 solutions to the Helmholtz equation $(-\Delta - 1)u = f$ in $\mathbb{R}^n$ assuming the given data f belongs to $L^{(2n+2)/(n+5)}(\mathbb{R}^n)$ and satisfies the "Fredholm condition" that $\hat{f}$ vanishes on the unit sphere. This problem, and similar results for the perturbed Helmholtz equation $(-\Delta - 1)u = -Vu + f$, are connected to the Limiting Absorption Principle for Schrödinger operators.

The same techniques are then used to prove that a wide range of $L^p \mapsto L^q$ bounds for Bochner-Riesz multipliers are improved if one considers their action on the closed subspace of functions whose Fourier transform vanishes on the unit sphere.

We consider the existence of a well-defined solution map for the Helmholtz equation in Euclidean space

$$\begin{cases} (-\Delta - 1)u = f \text{ in } \mathbb{R}^n \\ \qquad\qquad u \in L^2(\mathbb{R}^n) \end{cases} \tag{1}$$

By conjugating with dilations, the same problem can be posed with an operator $(-\Delta - \lambda^2)$, $\lambda > 0$ with minimal modification. These equations are translation invariant, so it would be desirable to choose f from a function space whose norm is also translation invariant. Our goal is to establish existence of solutions and a norm bound for u in terms of the L^p-norm of the given data f, provided f is formally orthogonal to all plane waves of unit frequency.

The Fourier dual formulation of (1) is

$$\begin{cases} \hat{u}(\xi) = \dfrac{\hat{f}(\xi)}{|\xi|^2 - 1} \\ \quad \hat{u} \in L^2(\mathbb{R}^n) \end{cases} \tag{2}$$

with respect to the definition $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)\, dx$. The corresponding Plancherel identity is $\|\hat{u}\|_2 = (2\pi)^{n/2}\|u\|_2$.

It is immediately clear from (2) that solutions should be unique, as $|\xi|^2 - 1$ is nonzero almost everywhere and the Fourier Transform is (a scalar multiple of) a unitary map between $L^2(dx)$ and $L^2(d\xi)$. One can also infer that solutions exist only if $\hat{f}$ vanishes on the unit sphere in a suitable sense, and also the restrictions of $\hat{f}$ to the sphere of radius r must be controlled as r approaches 1.

It would be sufficient, for example, if the map $S(r) = \hat{f}(r\,\cdot)\big|_{\mathbb{S}^{n-1}}$ (taking $\mathbb{R}_+$ into $L^2(\mathbb{S}^{n-1})$) was Hölder continuous of order $\gamma > \frac{1}{2}$ at $r = 1$ and vanished there.

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Then the scalar restriction function

$$F(r) = \|\hat{f}\|^2_{L^2(r\mathbb{S}^{n-1})} \tag{3}$$

would be $O(|r-1|^{2\gamma})$, and the formula

$$\|\hat{u}\|_2^2 = \int_0^\infty \frac{F(r)}{(r^2-1)^2}\,dr \tag{4}$$

would be locally integrable at $r = 1$.

In fact the desired continuity can be achieved if $\hat{f}$ belongs to the Sobolev space $W^{1,2}(\mathbb{R}^n)$. While $S(r)$ is only Hölder continuous of order exactly $\frac{1}{2}$, the one-dimensional Hardy inequality suffices to establish integrability of (4). This argument plays a central role in Agmon's bootstrapping method for the decay of eigenfunctions of a Schrödinger operator [1]. For the Helmholtz equation in particular the following result is proved there.

Theorem 1 (Agmon)**.** *Suppose $(1+|x|)^\beta f \in L^2(\mathbb{R}^n)$ for some $\beta > \frac{1}{2}$, and $\hat{f}$ vanishes on the unit sphere in the L^2-trace sense. Then there exists a unique function u such that $(-\Delta - 1)u = f$ and $(1+|x|)^{\beta-1}u \in L^2(\mathbb{R}^n)$.*

It is not obvious that a similar result should hold for $f \in L^p(\mathbb{R}^n)$ without weights, regardless of the exponent, as an L^p condition typically doesn't guarantee that $S(r)$ is Hölder continuous of any positive order. Nevertheless an L^2 solution operator for the Helmholtz equation exists for data in a narrow range of L^p spaces.

Theorem 2. *Let $n \geq 3$ and $\max(1, \frac{2n}{n+4}) \leq p \leq \frac{2n+2}{n+5}$, with $(n,p) \neq (4,1)$. Suppose $f \in L^p(\mathbb{R}^n)$ and $\hat{f}$ vanishes on the unit sphere in the L^2-trace sense.*

There exists a unique $u \in L^2(\mathbb{R}^n)$ such that $(-\Delta - 1)u = f$. Furthermore, $\|u\|_2 \leq C_{n,p}\|f\|_p$.

There is no statement in dimensions 1 or 2 because $\frac{2n+2}{n+5} < 1$. When $n = 1$ it should suffice to allow $e^{\pm i|x|}f$ to belong to the Hardy space $H^{2/3}(\mathbb{R})$. It is less clear what cancellation conditions might be required for $n = 2$.

The lower exponent bound of $\frac{2n}{n+4}$ comes from Sobolev embedding. It can be disregarded if one applies any sort of cutoff to remove high frequencies. As a special case, the sharp cutoff at $|\xi| = 1$ leaves a Bochner-Riesz multiplier of order -1. For further discussion of these operators we adopt the definition

$$(S^\alpha f)\hat{\ }(\xi) = (1-|\xi|^2)_+^\alpha \hat{f}(\xi). \tag{5}$$

For $\alpha \leq -1$ we define S^α by (formal) positivity of the operator rather than by analytic continuation. This preserves the multiplicative structure $S^\alpha S^\beta = S^{\alpha+\beta}$, however it comes at the cost that S^α will not have a bounded action on general Schwartz functions once $\alpha \leq -1$.

Never the less, S^α may behave well when applied to functions whose Fourier transform vanishes on the unit sphere, as stated below.

Theorem 3. *Let $n \geq 2$ and $\frac{1}{2} \leq \alpha < \frac{3}{2}$. Suppose $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \frac{2n+2}{n+1+4\alpha}$ with $(\alpha, p) \neq (\frac{1}{2}, \frac{2n+2}{n+3})$, and suppose $\hat{f}$ vanishes on the unit sphere.*

Then $\|S^{-\alpha}f\|_2 \lesssim \|f\|_p$.

Both Theorems 2 and 3 are easily derived from the following statement, which is our main technical result.

Proposition 4. *Let $n \geq 2$ and $\frac{1}{2} < \alpha < \frac{3}{2}$. Suppose $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \frac{2n+2}{n+1+4\alpha}$. There is a constant C_α such that*

$$\left| \int_{\frac{1}{2}<|\xi|<\frac{3}{2}} \frac{|\hat{f}(\xi)|^2}{((1-|\xi|^2)^2+\varepsilon^2)^\alpha}\, d\xi - \frac{C_\alpha}{\varepsilon^{2\alpha-1}} \|\hat{f}\|^2_{L^2(\mathbb{S}^{n-1})} \right| \lesssim \|f\|_p^2 \tag{6}$$

with a constant that remains bounded in the limit $\varepsilon \to 0$.

In both theorems, it is given that $\hat{f}$ vanishes on the unit sphere, eliminating the $\varepsilon^{1-2\alpha}\|\hat{f}\|^2_{L^2(\mathbb{S}^{n-1})}$ term from the left side of (6). Assuming Proposition 4 holds, the same inequality is then true with $\varepsilon = 0$ by monotone convergence. The Hausdorff-Young inequality is more than sufficient to bound the left-side integral over the center region $\{|\xi| < \frac{1}{2}\}$ for any $f \in L^p$, $1 \leq p \leq 2$.

For Theorem 2, let $\chi \in C_c^\infty(\mathbb{R}^n)$ be any smooth cutoff that is identically 1 in the ball $\{|\xi| \leq \frac{5}{4}\}$ and has support in the ball of radius $\frac{3}{2}$. Theorem 2 then reduces to the $\alpha = 1$ case of Proposition 4 combined with a Sobolev embedding estimate for the high frequency tail $\frac{1-\chi}{|\xi|^2-1}\hat{f}$. In a similar manner, all cases of Theorem 3 with $\alpha > \frac{1}{2}$ follow from the Proposition by applying the multiplier of the unit ball, which is bounded on $L^2(\mathbb{R}^n)$.

Finally, if $\alpha = \frac{1}{2}$ and $p \in [1, \frac{2n+2}{n+3})$, we have already established Theorem 3 for the pair (β, p) with $\beta = \min\big((n+1)(\frac{1}{2p} - \frac{1}{4}), 1\big)$. Since $\beta > \frac{1}{2}$, it follows that $\|S^{-1/2}f\|_2 \leq \|S^{-\beta}f\|_2$ by Plancherel's formula.

Sharpness of the upper exponent $\frac{2n+2}{n+1+4\alpha}$ is verified using a Knapp-type example. Let $\hat{f}$ be a smooth compactly supported function, suitably scaled to have support in the slab $\{|\xi'| \leq \delta,\ 1-2\delta^2 \leq \xi_n \leq 1-\delta^2\}$, where $\xi' = (\xi_1, \xi_2, \ldots, \xi_{n-1}) \in \mathbb{R}^{n-1}$ and unit height. Then $|f(x)| \sim \delta^{n+1}$ over the dual region $\{|x'| \leq \delta^{-1},\ |x_n| \leq \delta^{-2}\}$ and has rapid decay elsewhere. It follows that $\|f\|_p \sim \delta^{(n+1)(1-p^{-1})}$ and $\|S^{-\alpha}f\|_2 \sim \delta^{\frac{n+1}{2}-2\alpha}$. If $p > \frac{2n+2}{n+1+4\alpha}$ then $(n+1)(1-p^{-1}) > \frac{n+1}{2} - 2\alpha$ and Theorem 3 fails by taking δ to zero.

Remark 1. When $0 < \alpha < \frac{1}{2}$, no vanishing condition on the unit sphere is needed in the statement of Theorem 3. The range of viable exponents is once again $p \in [1, \frac{2n+2}{n+1+4\alpha}]$ including the endpoints [3]. The full range of $L^p \to L^q$ mappings in this regime is established in [2].

Remark 2. The statement of Proposition 4 is not true for $\alpha > 1$ if integration is limited to the inner annulus $\{\frac{1}{2} < |\xi| < 1\}$. An additional remainder term of order $\varepsilon^{2-2\alpha}$ is present in that case. The same remainder term appears with the opposite sign if one integrates over the outer annulus $\{1 < |\xi| < \frac{3}{2}\}$. This illustrates a difference in behavior between "one-sided" and "two-sided" Bochner-Riesz multipliers of order below -1, with the former being modestly more singular than the latter.

Remark 3. The endpoint case $\alpha = \frac{1}{2}$, $p = \frac{2n+2}{n+3}$ is quite delicate. The conclusion is certainly false if one does not assume that $\hat{f}$ vanishes on the unit sphere. In one dimension it remains false even with the vanishing condition. Since $S^{-\frac{1}{2}}$ in one dimension is closely related to the fractional integral operator $I_{1/2}$, a stronger condition that $e^{\pm ix}f$ belongs to the Hardy space $H^1(\mathbb{R})$ is needed to guarantee that $S^{-1/2}f \in L^2(\mathbb{R})$.

The one-dimensional counterexamples do not generalize well to $n \geq 2$. We believe it is an open problem whether Theorem 3 is true in these endpoint cases.

Theorem 1 plays an important role in the spectral theory of Schrödinger operators $H = -\Delta + V(x)$ with a short-range potential. Namely, it is used in a bootstrapping argument to show that any singular part of the essential spectrum of H must contain embedded eigenvalues. Thus the spectral measure on compact subsets of $[0, \infty) \setminus \sigma_{pp}(H)$ is absolutely continuous and satisfies an assortment of uniform mapping properties. In Section 2 we present a similar bootstrapping application using Theorem 2 as the primary device. These results are contained within the more general Limiting Absorption Principle of Ionescu and Schlag [4], and serve as an instructive special case.

The discussion of perturbed Schrödinger operators naturally raises the question of whether there is a similar existence theorem for the Helmholtz equation $(-\Delta + V - 1)u = f$. In Section 3 we use resolvent identities to obtain an affirmative answer.

Theorem 5. *Let $n \geq 3$, $p_0 = \frac{2n+2}{n+5}$, and suppose $V \in L^{\frac{n+1}{2}}(\mathbb{R}^n)$. There is a subspace $X \subset L^{p_0}(\mathbb{R}^n)$, isomorphic to the subspace $X_0 \subset L^{p_0}(\mathbb{R}^n)$ of functions whose Fourier transform vanishes on the unit sphere, with the following property: For each $f \in X$ there exists a unique $u \in L^2(\mathbb{R}^n)$ such that $(-\Delta + V - 1)u = f$. Furthermore, $\|u\|_2 \leq C_n \|f\|_{p_0}$.*

The spaces X and X_0 are formally linked via the action of the wave operators for $-\Delta + V$. For potentials where the wave operators are known to be bounded on $L^{p_0}(\mathbb{R}^n)$ one can prove Theorem 5 directly via the intertwining relations. However the given integrability condition $V \in L^{\frac{n+1}{2}}(\mathbb{R}^n)$ is a sharp threshold for the absolute continuity of spectral measure in both [4] and [6], and it seems unlikely that wave operators are well behaved outside of L^2 in this generality.

Remark 4. Versions of Theorem 5 can be stated for $1 \leq p < \frac{2n+2}{2+5}$ as well. The conditions on V are dictated by the range of resolvent estimates available on $L^p(\mathbb{R}^n)$, and will be more restrictive than when $p = \frac{2n+2}{n+5}$.

Finally, it is possible to extend Theorem 3 further by interpolation with other known estimates for Bochner-Riesz operators, subject to a few technical limitations. In this paper we do not assemble a full catalog of such estimates but instead consider a family of bounds that are sharp with respect to Knapp counterexamples. The following result is proved in Section 4.

Theorem 6. *Let $n \geq 2$ and $\beta \in (\frac{1}{2}, \frac{3}{2})$ with $\beta \leq \frac{n+1}{4}$. Suppose $f \in L^{\frac{2n+2}{n+1+4\beta}}(\mathbb{R}^n)$ with $\hat{f}$ vanishing on the unit sphere, and $\alpha \in [\beta, 2\beta]$ with $\alpha < 2$. Then*

$$\|S^{-\alpha} f\|_{\frac{2n+2}{n+1-4(\alpha-\beta)}} \lesssim \|f\|_{\frac{2n+2}{n+1+4\beta}}. \tag{7}$$

Remark 5. The restriction $\alpha < 2$ may be removed if one instead considers the analytic family of operators $\tilde{S}^{-\alpha} = \Gamma(1-\alpha)^{-1} S^{-\alpha}$. This will be evident in the proof.

1. Proof of Proposition 4

The proof of Proposition 4 mirrors that of the sharp Stein-Tomas restriction theorem. We follow the exposition in [7] most closely.

Let σ_r denote the surface measure on $r\mathbb{S}^{n-1}$ inherited from its embedding in $\mathbb{R}^n$. The main estimate will be a bound on $\langle K_1^\varepsilon * f, f\rangle$, where

$$K_1^\varepsilon = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\check{\sigma}_r - \check{\sigma}_1}{((1-r^2)^2+\varepsilon^2)^\alpha}\,dr = \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\check{\sigma}_{1+s} - \check{\sigma}_1}{(s^2(2+s)^2+\varepsilon^2)^\alpha}\,ds. \tag{8}$$

This is almost equal to the lefthand expression in (6), with the only discrepancy arising in the coefficient of $\langle \check{\sigma}_1 * f, f\rangle$. More precisely,

$$\begin{aligned}[\text{Left side of (6)}] - \langle K_1^\varepsilon * f, f\rangle &= \Big(\int_{\frac{1}{2}}^{\frac{3}{2}} ((1-r^2)^2+\varepsilon^2)^{-\alpha}\,dr - \frac{C_\alpha}{\varepsilon^{2\alpha-1}}\Big)\langle \check{\sigma}_1 * f, f\rangle \\ &= O(1)\langle \check{\sigma}_1 * f, f\rangle.\end{aligned}$$

Since $f \in L^p(\mathbb{R}^n)$ with $p \le \frac{2n+2}{n+3}$, the $O(1)$ term can be absorbed into the right side of (6) by the Stein-Tomas theorem.

The integrand in (8) may become highly singular at $s = 0$ as ε decreases. However the denominator is approximately an even function of s, while the leading order behavior of the numerator is an odd function. To be precise, let

$$\begin{aligned} A_{even}(s) &= \frac{1}{2}\left(\frac{1}{(s^2(2+s)^2+\varepsilon^2)^\alpha} + \frac{1}{(s^2(2-s)^2+\varepsilon^2)^\alpha}\right) \\ A_{odd}(s) &= \frac{1}{2}\left(\frac{1}{(s^2(2+s)^2+\varepsilon^2)^\alpha} - \frac{1}{(s^2(2-s)^2+\varepsilon^2)^\alpha}\right).\end{aligned}$$

Then

$$K_1^\varepsilon = \frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}} \Big(A_{even}(s)(\check{\sigma}_{1+s} - 2\check{\sigma}_1 + \check{\sigma}_{1-s}) + A_{odd}(s)(\check{\sigma}_{1+s} - \check{\sigma}_{1-s})\Big)\,ds \tag{9}$$

The main size bounds for A_{even} and A_{odd} are:

$$|A_{even}(s)| \lesssim s^{-2\alpha}, \quad |A_{odd}(s)| \lesssim s^{1-2\alpha} \quad \text{uniformly in } \varepsilon > 0. \tag{10}$$

It is a common practice to estimate the inverse Fourier transform of a surface measure by decomposing the surface into smaller regions where stationary phase methods can be applied. Consider a conical decomposition $\sum_{j=1}^n \eta_j(\frac{\xi}{|\xi|}) = 1$ where each smooth cutoff η_j is supported in the region where $|\xi_j| \sim |\xi|$. One may symmetrize so that each η_j is invariant under reflections across any one of the coordinate planes. Then (8) splits into a directional sum

$$K_1^\varepsilon = \sum_{j=1}^n \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(\eta_j\sigma_{1+s})\check{} - (\eta_j\sigma_1)\check{}}{(s^2(2+s)^2+\varepsilon^2)^\alpha}\,ds. \tag{11}$$

Let K_2^ε denote the $j = n$ term of this sum and write coordinates in $\mathbb{R}^n$ as (x', x_n) or (ξ', ξ_n). We will make further estimates on K_2^ε as a representative element.

Inside the support of $\eta_n\sigma_r$, the relationship $\xi_n = \pm(r^2 - |\xi'|^2)^{1/2}$ expresses ξ_n as a smooth function of ξ' on each hemisphere. Then the inverse Fourier transform of $\eta_n\sigma_r$ takes the form

$$(\eta_n\sigma_r)\check{}(x', x_n) = (2\pi)^{-n}\sum_{\pm}\int_{\mathbb{R}^{n-1}} \frac{r\,\eta_n\big(\frac{\xi'}{r}, \pm\frac{\sqrt{r^2-|\xi'|^2}}{r}\big)}{\sqrt{r^2-|\xi'|^2}} e^{i(x'\cdot\xi' \pm x_n\sqrt{r^2-|\xi'|^2})}\,d\xi'.$$

For $\frac{1}{2} < r < \frac{3}{2}$, the Hessian of the phase function is bounded below by x_n times the $(n-1)$-identity matrix and the initial fraction is a uniformly smooth function. This leads to the pointwise bound

$$|(\eta_n \sigma_r)\check{}(x', x_n)| \lesssim (1 + |x_n|)^{\frac{1-n}{2}}.$$

for r in this range. Furthermore one can differentiate with respect to r under the integral sign to obtain bounds

$$|\partial_r^k (\eta_n \sigma_r)\check{}(x', x_n)| \lesssim (1 + |x_n|)^{\frac{1-n}{2}+k}. \tag{12}$$

Taylor remainder estimates then imply that

$$|(\eta_n \sigma_{1+s})\check{}(x) - (\eta_n \sigma_{1-s})\check{}(x)| \lesssim \min(|s|(1 + |x_n|), 1)(1 + |x_n|)^{\frac{1-n}{2}},$$
$$|(\eta_n \sigma_{1+s})\check{}(x) - 2(\eta_n \sigma_1)\check{}(x) + (\eta_n \sigma_{1-s})\check{}(x)| \lesssim \min(s^2(1 + |x_n|)^2, 1)(1 + |x_n|)^{\frac{1-n}{2}}$$

while $|s| < \frac{1}{2}$. Plugging these and (10) into the appropriately modified version of (9), one concludes that

$$|K_2^\varepsilon(x)| \lesssim (1 + |x_n|)^{\frac{4\alpha-1-n}{2}}. \tag{13}$$

In other words, for a fixed choice of x_n, the restricted convolution operator

$$Tg(x') = \int_{\mathbb{R}^{n-1}} K_2^\varepsilon(x' - y', x_n) g(y')\, dy'$$

maps $L^1(\mathbb{R}^{n-1})$ to $L^\infty(\mathbb{R}^{n-1})$ with operator norm controlled by $(1 + |x_n|)^{\frac{4\alpha-1-n}{2}}$.

One can also determine the size of T as an operator on $L^2(\mathbb{R}^{n-1})$. This bound is given by the essential supremum of the x'-Fourier transform of the convolution kernel K_2^ε. Since K_2^ε is a superposition of the inverse Fourier transforms of $(\eta_n \sigma_s)$ as in (11), the x'-Fourier transform reverses the procedure in all except the x_n variable. More precisely,

$$\int_{\mathbb{R}^{n-1}} e^{-i\xi' \cdot x'} K_2^\varepsilon(x', x_n)\, dx' = \frac{1}{2\pi} \int_{\{\xi'\} \times \mathbb{R}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{ix_n \xi_n}(\sigma_{1+s} - \sigma_1)\eta_n}{(s^2(2+s)^2 + \varepsilon^2)^\alpha}\, ds\, d\xi_n.$$

If the integral over s is split into even and odd contributions as in (9), the result is

$$\begin{aligned}
&\int_{\mathbb{R}^{n-1}} e^{-i\xi' \cdot x'} K_2^\varepsilon(x', x_n)\, dx' \\
&\qquad = \frac{1}{4\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} \bigg(A_{even}(s) \int_{\{\xi'\} \times \mathbb{R}} e^{ix_n \xi_n}(\sigma_{1+s} - 2\sigma_1 + \sigma_{1-s})\eta_n\, d\xi' \\
&\qquad\qquad + A_{odd}(s) \int_{\{\xi'\} \times \mathbb{R}} e^{ix_n \xi_n}(\sigma_{1+s} - \sigma_{1-s})\eta_n\, d\xi' \bigg)\, ds
\end{aligned}$$

For a fixed choice of $\xi' \in \mathbb{R}^{n-1}$ and radius $r > 0$, the line $\{\xi'\} \times \mathbb{R}$ intersects the support of σ_r only when $\xi_n = \pm\sqrt{r^2 - |\xi'|^2}$. Thus for $|\xi'| < r$

$$\int_{\{\xi'\} \times \mathbb{R}} e^{ix_n \xi_n} \sigma_r \eta_n\, d\xi' = \frac{2\cos\left(x_n \sqrt{r^2 - |\xi'|^2}\right) \eta_n\left(\frac{\xi'}{r}, \sqrt{1 - (|\xi'|/r)^2}\right)}{\sqrt{1 - (|\xi'|/r)^2}}$$

and is zero otherwise. The denominator accounts for the angle of intersection between the line and surface. It is bounded away from zero within the support of η_n, so the integral expression is a smooth bounded function of ξ' and r. Within the range $\frac{1}{2} < r < \frac{3}{2}$, its first two derivatives with respect to r are bounded by $(1+|x_n|)$ and $(1+|x_n|)^2$ respectively. It follows that

$$\begin{aligned}\Big|\int_{\mathbb{R}^{n-1}} e^{-i\xi'\cdot x'} K_2^\varepsilon(x', x_n)\,dx'\Big| &\lesssim \int_{-\frac{1}{2}}^{\frac{1}{2}} A_{even}(s)\max(s^2(1+|x_n|)^2, 1) \\ &\qquad\qquad + A_{odd}(s)\max(|s|(1+|x_n|), 1)\,ds \\ &\lesssim (1+|x_n|)^{2\alpha-1}\end{aligned}$$

and therefore T is a bounded operator on $L^2(\mathbb{R}^{n-1})$ with norm comparable to $|x_n|^{2\alpha-1}$. Interpolating with the previous $L^1 \to L^\infty$ bound shows that

$$\|Tg\|_{L^{p'}(\mathbb{R}^{n-1})} \lesssim (1+|x_n|)^{2\alpha+\frac{n-3}{2}+\frac{1-n}{p}}\|g\|_{L^p(\mathbb{R}^{n-1})}, \quad 1 \le p \le 2.$$

Returning to the action of K_2^ε on functions in $\mathbb{R}^n$, these estimates imply that

$$\begin{aligned}\big\|K_2^\varepsilon * f\big\|_{p'} &\lesssim \Big\|\int_{-\infty}^{\infty}(1+|x_n-y_n|)^{2\alpha+\frac{n-3}{2}+\frac{1-n}{p}}\|f(\,\cdot\,, y_n)\|_{L^p(\mathbb{R}^{n-1})}\,dy_n\Big\|_{L^{p'}(\mathbb{R})} \\ &\lesssim \|f\|_p \end{aligned} \tag{14}$$

provided $2\alpha + \frac{n-3}{2} + \frac{1-n}{p} \le \frac{2}{p} - 2$, or more simply $1 \le p \le \frac{2n+2}{4\alpha+n+1}$. The last step is a restatement of the Hardy-Littlewood-Sobolev inequality in one dimension.

Summing over the n pieces of the conical decomposition concludes the proof.

2. Application to embedded resonances of $-\Delta + V$

Statements like Theorem 2 are useful for constraining the spectral measure of Schrödinger operators $-\Delta+V$ with a scalar perturbation $V \in L^r(\mathbb{R}^n)$. We present a version of the Limiting Absorption Principle here; one can find a more extensive set of tools and results in [4].

Proposition 7. *Suppose $n \ge 3$ and $V \in L^{\frac{n+1}{2}}(\mathbb{R}^n)$ is real valued. The Schrödinger operator $H = -\Delta + V$ has absolutely continuous spectrum on the interval $(0,\infty)$.*

Proof. It is known that the free resolvent $R_0^+(\lambda) = \lim_{\varepsilon\to 0^+}(-\Delta - (\lambda + i\varepsilon))^{-1}$ maps $L^{\frac{2n+2}{n+3}}(\mathbb{R}^n)$ to $L^{\frac{2n+2}{n-1}}(\mathbb{R}^n)$ for each $\lambda > 0$, with operator norm proportional to $\lambda^{-1/n+1}$ [5]. One may present the resolvent of H using identities such as

$$R_V^+(\lambda) := \lim_{\varepsilon\to 0^+}(H - (\lambda+i\varepsilon))^{-1} = (I + R_0^+(\lambda)V)^{-1}R_0^+(\lambda).$$

The spectral measure of H is given by the imaginary part of $R_V^+(\lambda)$, and this measure is absolutely continuous on intervals where the perturbed resolvent maps between the dual L^p spaces named above with uniform bound. Note that $\lambda^{-1/n+1}$ is bounded on each compact set $E \subset (0,\infty)$.

Under the given condition $V \in L^{\frac{n+1}{2}}(\mathbb{R}^n)$, the factor $(I + R_0^+(\lambda)V)$ is a compact perturbation of the identity on $L^{\frac{2n+2}{n-1}}(\mathbb{R}^n)$ except when $\lambda = 0$. Furthermore, it varies continuously (in operator norm) with respect to λ. It follows that the norm of $(I+R_0^+(\lambda)V)^{-1}$ is bounded over $\lambda \in E$ unless there exists $\lambda_0 \in E$ and a function $g \in L^{\frac{2n+2}{n-1}}$ such that $g = -R_0^+(\lambda_0)Vg$.

Such a function also has the property $(R_0^+(\lambda_0)Vg, Vg) = -(g, Vg)$, where $(\,\cdot\,,\,\cdot\,)$ is the sesquilinear pairing between $L^{\frac{2n+2}{n-1}}$ and its dual. The imaginary part of the left-hand pairing is equal to $c\lambda_0^{-1/2}\|(Vg)^\wedge\|^2_{L^2(d\sigma_{\sqrt{\lambda_0}})}$, whereas the right-hand pairing is real valued, hence the Fourier transform of Vg vanishes on the sphere radius $\sqrt{\lambda_0}$. Furthermore, g is a solution of the Helmholtz equation $(-\Delta - \lambda_0)g = -Vg$.

The statement of Theorem 2 can be modified to accommodate any operator $-\Delta - \lambda_0$, $\lambda_0 > 0$, by conjugating with dilations of order $\sqrt{\lambda_0}$. Write $V = V_1 + V_2$, where V_1 is bounded and compactly supported, and $\|V_2\|_{\frac{n+1}{2}} < \delta$ for a quantity $\delta > 0$ to be chosen in a moment. We have

$$\|g\|_2 \le C_{n,\lambda_0}\|Vg\|_{\frac{2n+2}{n+5}} \le C_{n,\lambda_0}\big(\|V_1\|_{\frac{n+1}{3}}\|g\|_{\frac{2n+2}{n-1}} + \delta\|g\|_2\big). \tag{15}$$

If $C_{n,\lambda_0}\delta < \frac{1}{2}$, the last term can be moved to the left side of the inequality so that $\|g\|_2 \lesssim \|g\|_{\frac{2n+2}{n-1}}$. In other words g is an eigenfunction of H in $L^2(\mathbb{R}^n)$ with eigenvalue $\lambda_0 > 0$.

The conclusion is that H has absolutely continuous spectrum on each compact subset of $(0, \infty)$ that contains no eigenvalues. However it is also known that embedded eigenvalues do not exist if the potential is real and belongs to $L^{\frac{n+1}{2}}(\mathbb{R}^n)$ [6], so in fact the spectrum of H is absolutely continuous on the entire halfline. $\square$

3. Perturbed Helmholtz equation

Theorem 2 admits a relatively easy extension to the equation $(-\Delta+V-1)u = f$. Factorize the perturbed Helmholtz operator as

$$-\Delta + V - 1 = (I + VR_0^+(1))(-\Delta - 1)$$

where $R_0^+(1) = \lim_{\varepsilon\to 0^+}(-\Delta - (1 + i\varepsilon))^{-1}$. In this case the choice of resolvent continuations is unimportant, as both $R_0^+(1)$ and $R_0^-(1)$ act the same when applied to functions in the range of $-\Delta - 1$. Then there should exist L^2 solutions of $(-\Delta + V - 1)u = f$ whenever $f = g + VR_0^+(1)g$ and the unperturbed equation $(-\Delta - 1)u = g$ has solutions in $L^2(\mathbb{R}^n)$.

Let X_0 be the subspace of functions in $L^p(\mathbb{R}^n)$ whose Fourier transform vanishes on the unit sphere, as defined in the statement of Theorem 5. For $p = \frac{2n+2}{n+5}$, Theorem 2 indicates that the latter problem admits solutions precisely when $g \in L^p(\mathbb{R}^n)$ also belongs to $X_0 \subset L^p(\mathbb{R}^n)$. The substance of Theorem 5 is that the correspondence between g and f is an isomorphism of subspaces of $L^p(\mathbb{R}^n)$. This statement is proved below.

Proposition 8. *Assume the conditions of Theorem 5, namely that $p = \frac{2n+2}{n+5}$ and $V \in L^{\frac{n+1}{2}}(\mathbb{R}^n)$. Let $J : X_0 \to L^p(\mathbb{R}^n)$ be the inclusion map. The linear operator $J + VR_0^+(1) : X_0 \to L^p(\mathbb{R}^n)$ is an isomorphism onto its range.*

The space $X \subset L^p(\mathbb{R}^n)$ in Theorem 5 is precisely the range of $J + VR_0^+(1)$.

Proof. The fact that it is a bounded operator is a direct consequence of Theorem 2, which effectively states that $R_0^+(1)$ is a bounded map from X_0 to $L^2(\mathbb{R}^n)$. It is injective by the result in [6], as $R_0^+(1)g$ would be an L^2 eigenfunction of $-\Delta+V-1$ for any g in the nullspace of $J + VR_0^+(1)$.

In fact $VR_0^+(1)$ is a compact operator from X_0 into $L^p(\mathbb{R}^n)$. For smooth compactly supported V it acts compactly on the larger domain $L^p(\mathbb{R}^n)$. Approximating $V \in L^{\frac{n+1}{2}}(\mathbb{R}^n)$ preserves compactness of $VR_0^+(1)$ over the restricted domain X_0.

The argument that $(J + VR_0^+(1))X_0 \subset L^p(\mathbb{R}^n)$ is closed is nearly identical to the analogous statement in the Fredholm Alternative. Let $f_n = (J + VR_0^+(1))g_n$ be a sequence converging to $f \in L^p$. If g_n has a bounded subsequence, then by compactness $VR_0^+(1)g_n$ has a convergent subsequence and so does $g_n = f_n - VR_0^+(1)g_n$. The limit point $g \in X_0$ satisfies $(J + VR_0^+(1))g = f$.

If $\lim_{n\to\infty} \|g_n\|_p = +\infty$, consider the normalized functions $\tilde{g}_n = g_n/\|g_n\|_p$. This sequence satisfies $(J + VR_0^+(1))\tilde{g}_n \to 0$, and by compactness there is a convergent subsequence of $VR_0^+(1)\tilde{g}_n$ with limit $-g$. Then the same subsequence of $\tilde{g}_n$ converges to g, which has unit norm and belongs to the nullspace of $J+VR_0^+(1)$. That would violate the injectivity property of the map.

Having ruled out unbounded (subsequences of) g_n, it follows that $f \in (J + VR_0^+(1))X_0$ as in the first case, making the range closed. By the closed graph theorem, $J + VR_0^+(1)$ is then an isomorphism onto its range. □

4. Extensions via Interpolation

The subspace of L^p consisting of functions whose Fourier transform vanishes on the unit sphere is not particularly well suited to interpolation. The Fourier-vanishing condition not preserved by lattice operations or by the complex-analytic families used in the Riesz-Thorin theorem. As a further impediment, it is not obvious that one can approximate each element by a sequence of simple functions (or compactly supported functions, or Schwartz functions) whose Fourier transforms also vanish on the sphere.

Fortunately, Theorem 6 can be proved by complex interpolation of operators and of the target L^q space, where none of these problems arise. The argument proceeds as follows. Suppose f and g are simple functions with compact support. Let $\tilde{S}^z$ be the "analytic" Bochner-Riesz operators defined by

$$\tilde{S}^z = \frac{1}{\Gamma(z+1)} S^z$$

for real-valued $z > -1$, and by analytic continuation to $z \in \mathbb{C}$.

The key observation is that for $\operatorname{Re} z > -2$, and for functions whose Fourier transform vanishes on the unit sphere, $\Gamma(z+1)\tilde{S}^z f = S^z f$ (The singularity at $z = -1$ is removable in this case). Proposition 4 establishes the same observation about "two-sided" Bochner-Riesz operators over the larger range $\operatorname{Re} z > -3$.

It is true by construction that the function $\langle \tilde{S}^z f, g\rangle$ is holomorphic in z for any pair of simple functions f and g. This remains true by uniform convergence in the halfplane $\operatorname{Re} z > -\frac{1}{2} - \frac{2\beta n}{n+1}$ if we take limits to a generic element $f \in L^{\frac{2n+2}{n+1+4\beta}}$. Then

$$G(z) := \Gamma(z+1)\langle \tilde{S}^z f, g\rangle$$

is meromorphic over the same domain, with residues at the negative integers determined by $\langle \tilde{S}^{-k} f, g\rangle$. Since $\tilde{S}^{-1}$ agrees (up to a scalar multiple) with convolution against $\check{\sigma}_1$, if we further assume that $\hat{f}$ vanishes on the unit sphere then in fact the singularity of $G(z)$ at $z = -1$ is removable. The slightly modified function

$$\tilde{G}(z) := (z+2)\Gamma(z+1)\langle \tilde{S}^z f, g\rangle = (z+2)G(z)$$

is meromorphic with poles at the negative integers $k \le -3$.

Assuming once again that $\hat{f}$ vanishes on the unit sphere, Theorem 3 provides a bound on the line $z = -\beta + i\mu$,

$$\left\|\Gamma(1-\beta+i\mu)\tilde{S}^{-\beta+i\mu} f\right\|_2 \lesssim \|f\|_{\frac{2n+2}{n+1+4\beta}}. \tag{16}$$

The constant does not depend on μ because any one of the Fourier multipliers $(1-|\xi|^2)^{i\beta}$ is an isometry on L^2. Therefore

$$|\tilde{G}(-\beta+i\mu)| \lesssim (1+|\mu|)\|f\|_{\frac{2n+2}{n+1+4\beta}}\|g\|_2.$$

On the line $z=-2\beta+i\mu$, we need the following estimates.

Proposition 9. *Let $\frac{1}{2}<\beta<\frac{3}{2}$ and $\beta \le \frac{n+1}{4}$. The inequality*

$$\big\|(2-2\beta+i\mu)\Gamma(1-2\beta+i\mu)\tilde{S}^{-2\beta+i\mu}f\big\|_{\frac{2n+2}{n+1-4\beta}} \lesssim (1+|\mu|)\|f\|_{\frac{2n+2}{n+1+4\beta}} \tag{17}$$

holds uniformly for all $\mu\in\mathbb{R}$ and all $f\in L^{\frac{2n+2}{n+1+4\beta}}(\mathbb{R}^n)$.

Sketch of Proof. Proposition 9 follows from the same argument as the endpoint Stein-Tomas theorem, using the fact that the convolution kernel of $\tilde{S}(z)$ has an asymptotic description

$$\tilde{S}^z(|x|) \sim \frac{C_n}{|x|^{\frac{n+1}{2}+z}}\cos\Big(|x|-\frac{(n-3)\pi}{4}+\frac{\pi}{2}z\Big)$$

for large $|x|$. Note that for complex z, oscillations of the cosine function in this formula have amplitude approximately $e^{(\pi/2)\operatorname{Im} z}$.

For $z=-2\beta+i\mu$, the prefactor $(z+2)\Gamma(z+1)$ is dominated by $(1+|z|)e^{-(\pi/2)\operatorname{Im} z}$, using Stirling's approximation when μ is large, and the absence of poles for $\operatorname{Re} z > -3$ when μ is small. Hence the product $(z+2)\Gamma(1+z)\tilde{S}^z$ enjoys mapping estimates that are uniform in μ along this line. □

Consequently $|\tilde{G}(-2\beta+i\mu)| \le (1+|\mu|)\|f\|_{\frac{2n+2}{n+1+4\beta}}\|g\|_{\frac{2n+2}{n+1+4\beta}}$. If one constructs g_z to be a holomorphic family of simple functions (as in Riesz-Thorin interpolation) that belong isometrically to $L^2(\mathbb{R}^n)$ along the line $\operatorname{Re} z=-\beta$, and to $L^{\frac{2n+2}{n+1+4\beta}}(\mathbb{R}^n)$ along the line $\operatorname{Re} z=-2\beta$. it follows from the Three-Lines Theorem that

$$|(z+2)\Gamma(z+1)\langle\tilde{S}^z f, g_z\rangle| \lesssim (1+|\operatorname{Im} z|)\|f\|_{\frac{2n+2}{n+1+4\beta}}\|g_z\|_{\frac{2n+2}{n+1-4(\operatorname{Re} z+\beta)}}$$

For a fixed real value $\beta\le\alpha\le 2\beta$, one can arrange for $g_{-\alpha}$ to be any simple function. It follows by duality and density of simple functions that

$$\|\Gamma(z+1)\tilde{S}^{-\alpha}f\|_{\frac{2n+2}{n+1-4(\alpha-\beta)}} \lesssim \frac{1}{|2-\alpha|}\|f\|_{\frac{2n+2}{n+1+4\beta}}.$$

For $\alpha<2$ and with the assumption that $\hat{f}$ vanishes on the unit sphere, the left hand function is exactly $S^{-\alpha}f$. This is the norm bound claimed in Theorem 6.

As a final note, we observe that estimates can also be made for $\alpha<\beta$ by interpolating between Theorem 2 and other known bounds for Bochner-Riesz operators. To give a simple example, $S^{-\alpha}$ maps $L^p(\mathbb{R}^2)$ to itself for each $\alpha<-\frac{1}{2}$, and if $\hat{f}$ vanishes on the unit circle for a function $f\in L^1(\mathbb{R}^2)$ it is also true that $S^{-\frac{3}{4}}f\in L^2(\mathbb{R}^2)$. Complex interpolation then suggests that $S^0 f\in L^q(\mathbb{R}^2)$ for all $q>\frac{5}{4}$. This is a modest improvement over the generic $L^1\mapsto L^{4/3+}$ bound for the ball multiplier. We do not claim that the exponent $q=\frac{5}{4}$ is sharp, and suspect that the range of exponents can be extended further toward 1 by other methods.

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