

Restriction of Fractional Derivatives of the Fourier Transform

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Abstract

In this paper, we showed that for suitable (β, p, s, ℓ) the β -order fractional derivative with respect to the last coordinate of the Fourier transform of an $L^p(\mathbb{R}^n)$ function is in H^{-s} after restricting to a graph of a function with non-vanishing Gaussian curvature provided that the restriction of the Fourier transform of such function to the surface is in H^ℓ . This is a generalization of the result in [5, Theorem 1.12].

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1 Introduction

1.1 Fourier restriction problem

The desirable properties of the Fourier transform of an $L^p(\mathbb{R}^n)$ function are crucial in classical harmonic analysis with application to Kakeya conjecture, Bochner-Riesz summabilities, and partial differential equations, see [8, 11]. Hausdorff-Young's inequality ensures that the Fourier transform of an $L^p(\mathbb{R}^n)$ function is in $L^{\frac{p}{p-1}}(\mathbb{R}^n)$ whenever $1 \leq p \leq 2$. However, this does not give us any information if the Fourier transform of an $L^p(\mathbb{R}^n)$ function is restricted to a measure-zero subset $\Sigma \subset \mathbb{R}^n$. The Fourier restriction problem, in the most general form, seeks to determine the complete range of (p, q) for which

$$\|\hat{f}|_\Sigma\|_{L^q(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} \quad (1.1)$$

holds for all $f \in L^p(\mathbb{R}^n)$ for a measure-zero set $\Sigma \subset \mathbb{R}^n$ with $n \geq 2$, where $\hat{f}(\iota) = \int_{\mathbb{R}^n} e^{-2\pi i \iota \cdot x} f(x) dx$. Different choices of Σ and n lead to different ranges of (p, q) and it remains an open problem in most cases, and the best result currently in $\mathbb{R}^3$ for parabolas is [10]; and for a higher dimension, see [6]. We also refer to the reader [3] for the survey on Fourier restriction estimates on different Σ . In particular, if Σ is a two-dimensional sphere or a compact parabola around the origin, the range of (p, q) is completely characterized by [2]. It is also worth mentioning that the Tomas-Stein Theorem, established in [7, 9], states that Equation (1.1) holds for $q = 2$ and $1 \leq p \leq \frac{2n+2}{n+3}$ with Σ having non-zero Gaussian curvature at the origin.

One reason for imposing Σ having a point with non-vanishing Gaussian curvature is to rule out “uninteresting” cases. If we consider Σ to be a hyperplane in $\mathbb{R}^n$, then the only possible range of (p, q) is $q = \infty$ and $1 \leq p \leq \infty$. This may suggest that we should consider Σ having non-zero Gaussian curvature at origin (without losing generality). In this paper, we will assume that Σ is a graph of a function h defined on $\mathbb{R}^{n-1}$ with a vanishing gradient at 0 and an invertible Hessian matrix at 0. A precise statement will be given in Section 2.1.

The next step of the Fourier restriction problem is to consider the possibilities of the derivative of the Fourier transform of an $L^p(\mathbb{R}^n)$ function. In [5], various versions of the derivative Fourier restriction problem are discussed, with the core idea being to impose suitable assumptions to obtain $\phi \nabla^k \hat{f} \in H^{-s}(\Sigma)$, or more importantly, $\phi \frac{\partial^\beta \hat{f}}{\partial \xi_n^\beta} \in H^{-s}(\Sigma)$ for a fixed ϕ . Here, $g \in H^{-s}(\Sigma)$ means

$g|_{\Sigma} \in H^{-s}(\Sigma)$, and $H^{-s}(\Sigma)$ is the L^2 -based Bessel potential space as defined in Section 2.1. If $\|\phi \nabla^k \hat{f}\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$, then the trace of $\nabla^k \hat{f} \in H_{loc}^{-s}(\Sigma)$ for $f \in L^p(\mathbb{R}^n)$. Cho, Guo, and Lee [1, Theorem 1.1] studied the trace value of $\hat{f}$ in Sobolev spaces, and G. and Stolyarov [5, Proposition 1.1] used this to obtain a characterization of exponents for which $\|\phi \nabla^k \hat{f}\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ holds. More precisely,

Theorem A ([5, Proposition 1.1]). *Let $p > 1$. The inequality $\|\phi \nabla^k \hat{f}\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}$ holds for all $f \in L^p(\mathbb{R}^n)$ if and only if $k \leq s$, $k < \sigma_p$, and $2k - s \leq \kappa_p$, where $\kappa_p = \frac{n+1}{p} - \frac{n+3}{2}$ and $\sigma_p = \frac{n}{p} - \frac{n+1}{2}$.*

Moreover, G. and Stolyarov also established some necessary conditions and some sufficient conditions for

$$\|\phi \nabla^k \hat{f}\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)} \quad (1.2)$$

for all $f \in L^p(\mathbb{R}^n)$ with $\hat{f}\phi \in H^\ell(\Sigma)$ given $\phi \in C_c^\infty(\Sigma)$, where $C_c^\infty(\Sigma)$ is the space of all smooth functions with support properly contained in Σ .

More precisely, they ([5, Theorem 1.11]) showed that if (1.2) holds, $p > 1$, $s \geq 0$, and $\ell \geq 0$, then the exponents have to satisfy

$$k \leq \min\{s+1, s+\ell\}; \quad k < \sigma_p; \quad 2k - s \leq \kappa_p; \quad (1.3)$$

$$\frac{k\ell}{s+\ell-k} \leq \kappa_p \quad \text{if } k > s. \quad (1.4)$$

However, the sufficient conditions do not match with the necessary conditions due to the constraint $k \in \mathbb{N}$. In order to have (1.2), other than (1.3) and (1.4), they ([5, Theorem 1.12]) additionally assume that $\frac{n+1}{p} - \frac{n+3}{2} \in \mathbb{N}$ and either $2\lceil \frac{\ell-1}{\ell} \kappa_p \rceil \leq \kappa_p$ or $k \leq \lfloor \frac{\kappa_p}{2} \rfloor$ hold, where $\lfloor x \rfloor$ is the smallest integer greater than or equal to x and $\lceil x \rceil$ is the greatest integer less than or equal to x . If $2\lceil \frac{\ell-1}{\ell} \kappa_p \rceil \leq \kappa_p$ and $k \leq \lfloor \frac{\kappa_p}{2} \rfloor$ do not hold, then one has to assume that $s \geq k - \frac{\kappa_p - k}{\kappa_p - \lfloor \kappa_p/2 \rfloor}$. In particular, if $k = 1$ and $s = 0$,

Theorem B ([5, Corollary 1.13]). *Let m be an integer in $[2, \frac{n-1}{2})$ and $\ell \geq \frac{m}{m-1}$. Then,*

$$\|\phi \nabla \hat{f}\|_{L^2(\Sigma)} \lesssim \|f\|_{L^{\frac{2n+2}{n+3+2m}}(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)}$$

for all $f \in \mathcal{S}(\mathbb{R}^n)$ and $\phi \in C_c^\infty(\Sigma)$.

In order to remove the constraint $k \in \mathbb{N}$, we need to consider an alternative for $\frac{\partial}{\partial x_n}$, and the Riesz fractional derivative, defined by $[(-\Delta)^s f]^\wedge(\xi) = c_n |\xi|^{2s} \hat{f}(\xi)$, would be a good choice. The question now becomes:

Question: For what ranges of (β, p, s, ℓ) does the inequality

$$\|\phi (-\Delta)^{\beta/2} \hat{f}\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)}$$

hold true for $f \in \mathcal{S}(\mathbb{R}^n)$?

In this paper, we will focus on the derivative along the transversal direction. We remark that Theorem A holds in this case by replacing k by β with the aid of Lemma 1.

1.2 Results and outline

Our first main theorem is to provide a sufficient condition so that the fractional derivative of $\hat{f}$ is in $L^2(\Sigma)$.

Theorem 1. Suppose that $f \in L^p(\mathbb{R}^n)$ has $\hat{f}|_\Sigma \in H^\ell(\Sigma)$, and $\phi \in C_c^\infty(\Sigma)$. If $p > 1$, $2\beta \leq \kappa_p$, $2\beta < \sigma_p$, $0 \leq \beta \leq 1$, and $\ell \geq \frac{\kappa_p \beta}{\kappa_p - \beta}$, then

1. we have $(-\Delta_{\xi_n})^{\frac{\beta}{2}} \hat{f} = c_n \widehat{|x_n|^\beta f} \in L^2(\Sigma)$ and

$$\|\phi(-\Delta_{\xi_n})^{\frac{\beta}{2}} \hat{f}\|_{L^2(\Sigma)} \lesssim_{n,p,\beta,\ell} \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)},$$

where $-\Delta_{\xi_n}$ denotes the Laplacian with respect to the ξ_n -variable.

2. if $\beta = 1$, we have $\frac{\partial}{\partial \xi_n} \hat{f} \in L^2(\Sigma)$ and

$$\left\| \phi \frac{\partial \hat{f}}{\partial \xi_n} \right\|_{L^2(\Sigma)} \lesssim_{n,p,\beta,\ell} \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)}$$

When $p = 1$, we additionally require $\ell > \frac{\kappa_1 \beta}{\kappa_1 - \beta}$ obtaining the above result.

In particular, we have a continuous analogue of Theorem B when $\beta = 1$. Moreover, this result can be applied to $n = 3$ and $n = 4$, which is not covered by Theorem B. From Theorem 3, the best possible range of β is $[0, \frac{1}{2}]$ when $n = 3$; when $n = 4$, the best range of β is $[0, \frac{3}{4}]$; and when $n = 5$, the best range of β is $[0, 1]$ with $p = 1$, and we also recover the result in [5] that the (usual) derivative of $\hat{f}$ on Σ is in L^2 if $p = 1$ and $\ell > 2$ from the second statement.

The following theorem shows that (1.3) and (1.4) are almost sufficient conditions for (1.2).

Theorem 2. Let $s, \ell, \kappa_p, \sigma_p$, and β be non-negative numbers. Fix $\phi \in C_c^\infty(\Sigma)$. If

$$\beta \leq \min\{s+1, s+\ell\}; \quad \beta < \sigma_p; \quad 2\beta - s \leq \kappa_p; \quad (1.5)$$

$$\frac{\beta \ell}{s + \ell - \beta} \leq \kappa_p \quad \text{if } \beta > s \quad (1.6)$$

hold, $\frac{2\beta}{s+2-\beta} \leq \kappa_p$ whenever $\kappa_p < 2\beta$ and $\beta > s$, and $p > 1$, then

$$\|\phi(-\Delta_{\xi_n})^{\beta/2} \hat{f}\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\mathbb{R}^n)}$$

holds for all $f \in \mathcal{S}(\mathbb{R}^n)$ that $\hat{f}|_\Sigma \in H^\ell(\Sigma)$. In case of $\beta \in \mathbb{N}$, we have

$$\left\| \phi \frac{\partial^\beta \hat{f}}{\partial \xi_n^\beta} \right\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\mathbb{R}^n)}$$

While the results in Theorems 1 and 2 provide bounds on the derivatives of $\hat{f}$, they are largely derived by analysis of the associated difference quotient

$$Tf(\xi, \eta) = \frac{\hat{f}(\xi, h(\xi) + \eta) - \hat{f}(\xi, h(\xi))}{\eta},$$

$\xi \in U \subset \mathbb{R}^{n-1}$, and $\eta \in \mathbb{R} \setminus \{0\}$.

The first author proved $L^p \rightarrow L^2$ estimates for T under the assumption that $\hat{f} \equiv 0$ on the sphere in [4]. Here we extend the bound to all cases where the restriction of $\hat{f}$ to Σ is sufficiently smooth.

Corollary 1. Suppose $f \in L^p(\mathbb{R}^n)$ has $\hat{f}|_\Sigma \in H^\ell(\Sigma)$, and $\phi \in C_c^\infty(\Sigma)$. If $1 < p \leq \frac{2n+2}{n+5}$ and $\ell \geq \frac{\kappa_p}{2\kappa_p-1}$, then

$$\|\phi Tf\|_{L_\xi^2 L_\eta^2} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)}.$$

The above estimate is also true for $p = 1$ if $\ell > \frac{\kappa_1}{2\kappa_1-1} = \frac{n-1}{2n-4}$.

The condition $\ell \geq \frac{\kappa_p}{2\kappa_p-1}$ is necessary due to “shifted Knapp” counterexamples identical to those in [5].

In fact, to derive the bounds for the derivatives of $\hat{f}$, we consider an operator using T , namely,

$$S_w f(\xi) = \int_{\mathbb{R}} w(r) \mathcal{F}_{\eta \rightarrow r}(Tf(\xi, \cdot))(-r) dr = \int_{\mathbb{R}} w(-r) \mathcal{F}_{\eta \rightarrow r}(Tf(\xi, \cdot))(r) dr \quad (1.7)$$

The term $-r$ is to eliminate the negative sign obtained from the Fourier transform, and $\mathcal{F}_{\eta \rightarrow r} = \mathcal{F}_\eta$ denotes the one dimensional Fourier transform from η -variable to r -variable.

With S_w , we are able to prove the following theorems.

Theorem 3. Suppose $f \in L^p(\mathbb{R}^n)$ has $\hat{f}|_\Sigma \in H^\ell(\Sigma)$, and $\phi \in C_c^\infty(\Sigma)$. Under the hypothesis in Theorem 1 with $\beta \neq 0$, for all $b \in \mathbb{R}$ we have

$$\sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2\beta-1+ib}w} f\|_{L^2(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)}.$$

Theorem 4. Let s, ℓ, κ_p , and σ_p be non-negative numbers and $\beta > 0$. Fix $\phi \in C_c^\infty(\Sigma)$. Under the same hypothesis in Theorem 2, for all $b \in \mathbb{R}$ we have

$$\sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2\beta-1+ib}w} f\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)} \quad (1.8)$$

whenever $f \in L^p(\mathbb{R}^n)$ with $\hat{f}|_\Sigma \in H^\ell(\Sigma)$.

We remark that the implicit constants depend on $s, \ell, \kappa_p, \sigma_p$, and β , but they are independent of b . Moreover, it suffices to consider $f \in \mathcal{S}(\mathbb{R}^n)$, the space of Schwartz functions on $\mathbb{R}^n$, thanks to a density lemma [5, Lemma 3.10].

Theorems 1 and 2 follow from Theorems 3 and 4 by choosing $w = (\text{sgn}(u))^\beta$ and $w = 1$. This is because one can show that

$$\begin{aligned} S_{i(\text{sgn}(u)u)^{\beta-1}} f(\xi) \\ = 2\pi^{3/2} \beta^{-1} \int_{\mathbb{R}^n} |x_n|^\beta e^{-2\pi i(x' \cdot \xi + x_n h(\xi))} f(x', x_n) dx' dx_n \simeq [(-\Delta_{\xi_n})^{\frac{\beta}{2}} \hat{f}]|_\Sigma(\xi) \end{aligned}$$

and similarly one has $S_{iu^{\beta-1}} f \simeq [x_n^\beta \hat{f}]|_\Sigma(\xi)$ if $\beta \in \mathbb{N}$. From this calculation, we have to exclude the case $\beta = 0$ in Theorem 3, but we can include $\beta = 0$ in Theorem 1 because it follows from the hypothesis.

The organization of the reminder of this paper is as follows. In Section 2, we state the assumptions on h , provide definitions of different types of Sobolev spaces, and compute the kernel of S_w . In Section 3, the proofs of Theorems 3 and 4 are shown. The proof of Corollary 1 is given in Section 4. In Section 5, Corollary 1 for a variation of T is given there.

2 Preliminary

2.1 Notations

In this paper, we assume that Σ is a graph of a function h defined on $\mathbb{R}^{n-1}$. Let U be a neighborhood of the origin in $\mathbb{R}^{n-1}$. We assume that the function $h \in C^\infty(U)$ such that $h(0) = 0$, $\nabla h(0) = 0$, $\det(\frac{\partial^2 h}{\partial \xi^2}(0)) \neq 0$, and the second differential of h in U is sufficiently close to the one at 0.

Since the roles of ξ and η are different, we consider Tf in anisotropic Sobolev (Bessel potential) spaces. To be more precise, if g is a function defined on $\mathbb{R}^{n-1} \times \mathbb{R}$ and $\gamma, \tau \in \mathbb{R}$, then we define

$$\|g(\xi, \eta)\|_{H_\xi^\gamma H_\eta^\tau}^2 := \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} (1 + |x|)^{2\gamma} (1 + |r|)^{2\tau} |\mathcal{F}_{(\xi, \eta) \rightarrow (x, r)} g(x, r)|^2 dx dr,$$

where $\mathcal{F}_{(\xi,\eta)\rightarrow(x,r)}$ denotes the Fourier transform from (ξ,η) -variable to (x,r) -variable. Moreover, we will consider negative homogeneous Sobolev spaces in ξ -variable and inhomogeneous Sobolev spaces in η , and the inner product of such spaces are given by

$$\langle g_1, g_2 \rangle_{\dot{H}_{\xi}^{-\gamma} H_{\eta}^{\tau}} = \int_{\mathbb{R}} \int_U \int_U \mathcal{F}_{\eta \rightarrow r} g_1(\xi, r) \overline{\mathcal{F}_{\eta \rightarrow r} g_2(\zeta, r)} |\xi - \zeta|^{2\gamma-n+1} (1 + |r|)^{2\tau} d\xi d\zeta dr$$

if $\gamma \in (0, \frac{n-1}{2})$. We have used the fact that the negative homogeneous Sobolev space has a norm given by

$$\|\psi\|_{\dot{H}^{-\gamma}(\Sigma)} = C_{n,\gamma} \int_{U \times U} \psi(\xi, h(\xi)) \overline{\psi(\zeta, h(\zeta))} |\xi - \zeta|^{2\gamma-n+1} d\xi d\zeta$$

provided that $\gamma \in (0, \frac{n-1}{2})$.

On the other hand, we also assume that $f \in L^p(\mathbb{R}^n)$ and $\hat{f}|_{\Sigma} \in H^{\ell}(\Sigma)$, where the norm of $H^{\ell}(\Sigma)$ is given by

$$\|\psi\|_{H^{\ell}(\Sigma)}^2 := \int_{\mathbb{R}^{n-1}} |\mathcal{F}_{\xi \rightarrow z}[\psi(\xi, h(\xi))](z)|^2 (1 + |z|)^{2\ell} dz.$$

In this paper, the implicit constant in $\lesssim$ may vary from line to line.

Before we move on to calculating the kernel of $S_w^* S_w$, we shall recall the notations of some useful exponents in this paper:

$$\begin{aligned} \sigma_p &:= \frac{n}{p} - \frac{n+1}{2}; \\ \kappa_p &:= \frac{n+1}{p} - \frac{n+3}{2}. \end{aligned}$$

2.2 Computing the kernel of $S_w^* S_w$

Let us recall S_w , where w is a complex-valued bounded measurable function.

$$S_w f(\xi) = \int_{\mathbb{R}} w(r) \mathcal{F}_{\eta \rightarrow r}(Tf(\xi, \cdot))(-r) dr = \int_{\mathbb{R}} w(-r) \mathcal{F}_{\eta \rightarrow r}(Tf(\xi, \cdot))(r) dr$$

We first compute the Fourier transform of T . To do so, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned} \mathcal{F}_{\eta}(Tf(\xi, \cdot))(-r) &= \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} e^{-\pi^2 \varepsilon \eta^2} e^{2\pi i \eta r} \left(\frac{e^{-2\pi i x_n \eta} - 1}{\eta} \right) e^{-2\pi i (x' \cdot \xi + x_n h(\xi))} f(x', x_n) dx' dx_n d\eta \\ &= \lim_{\varepsilon \rightarrow 0^+} 2\pi i \int_{\mathbb{R}^n} E_{x_n, \varepsilon}(r) e^{-2\pi i (x' \cdot \xi + x_n h(\xi))} f(x', x_n) dx' dx_n, \end{aligned}$$

where

$$E_{x_n, \varepsilon}(r) := \frac{1}{\sqrt{\varepsilon}} \int_{-x_n}^0 e^{-\frac{(-r-y)^2}{\varepsilon}} dy = \int_{-\frac{x_n+r}{\sqrt{\varepsilon}}}^{-\frac{r}{\sqrt{\varepsilon}}} e^{-t^2} dt.$$

Note that as $\varepsilon \rightarrow 0^+$, we have

$$s_r(x_n) = \sqrt{\pi} 1_{x_n}(r) := \lim_{\varepsilon \rightarrow 0^+} E_{x_n, \varepsilon}(r) = \begin{cases} \sqrt{\pi}, & \text{if } 0 < r < x_n; \\ -\sqrt{\pi}, & \text{if } x_n < r < 0; \\ -\operatorname{sgn}(x_n) \frac{\sqrt{\pi}}{2}, & \text{if } r = 0; \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, using Dominated Convergence Theorem, we have

$$\mathcal{F}_{\eta}(Tf(\xi, \cdot))(-r) = -2\pi i \int_{\mathbb{R}^n} s_r(x_n) e^{-2\pi i (x' \cdot \xi + x_n h(\xi))} f(x', x_n) dx' dx_n.$$

We are now able to compute the kernel of $S_w^* S_w$. For $\ell \in (0, \frac{n-1}{2})$, we use homogeneous norm, and have

$$\begin{aligned}
& \langle \phi S_w f, \phi S_w g \rangle_{\dot{H}^{-\ell}} \\
& \simeq \int_U \int_U \phi(\xi) S_w f(\xi) \overline{\phi(\zeta) S_w g(\zeta)} |\xi - \zeta|^{2\ell-n+1} d\xi d\zeta \\
& \simeq \int_{U \times U} \int_{\mathbb{R} \times \mathbb{R}} \phi(\xi) w(r) \mathcal{F}_{\eta \rightarrow r}(Tf(\xi, \cdot))(-r) \overline{\phi(\zeta) w(r') \mathcal{F}_{\eta \rightarrow r}(Tg(\zeta, \cdot))(-r')} |\xi - \zeta|^{2\ell-n+1} dr dr' d\xi d\zeta \\
& \simeq \int_{U \times U} \int_{\mathbb{R} \times \mathbb{R}} \phi(\xi) \overline{\phi(\zeta)} w(r) \overline{w(r')} \left(\int_{\mathbb{R}^n} s_r(x_n) e^{-2\pi i(x' \cdot \xi + x_n h(\xi))} f(x', x_n) dx' dx_n \right) \\
& \quad \times \left(\int_{\mathbb{R}^n} s_r(y_n) e^{2\pi i(y' \cdot \zeta + y_n h(\zeta))} \overline{g(y', y_n)} dy' dy_n \right) |\xi - \zeta|^{2\ell-n+1} dr dr' d\xi d\zeta \\
& \simeq \int_{\mathbb{R}^n \times \mathbb{R}^n} \left(\int_{U \times U} e^{-2\pi i(x' \cdot \xi + x_n h(\xi))} e^{2\pi i(y' \cdot \zeta + y_n h(\zeta))} \phi(\xi) \overline{\phi(\zeta)} |\xi - \zeta|^{2\ell-n+1} d\xi d\zeta \right) \\
& \quad \times \left(\int_{\mathbb{R}} w(r) \sqrt{\pi} 1_{x_n}(r) dr \right) \left(\int_{\mathbb{R}} \overline{w(r')} \sqrt{\pi} 1_{y_n}(r') dr' \right) f(x', x_n) \overline{g(y', y_n)} dx dy.
\end{aligned}$$

By denoting $W(x_n) = \int_{\mathbb{R}} w(r) \sqrt{\pi} 1_{x_n}(r) dr$, the kernel of $S_w^* S_w$ (in $\dot{H}^{-\ell}$) can be expressed as

$$K_w(x, y) = \widetilde{\phi}_{-\ell, \Sigma}^2(x, y) W(x_n) \overline{W(y_n)},$$

where

$$\widetilde{\phi}_{-\ell, \Sigma}^2(x, y) = \int_{U \times U} e^{-2\pi i(x' \cdot \xi + x_n h(\xi))} e^{2\pi i(y' \cdot \zeta + y_n h(\zeta))} \phi(\xi) \overline{\phi(\zeta)} |\xi - \zeta|^{2\ell-n+1} d\xi d\zeta.$$

For $\langle \phi S_w f, \phi S_w g \rangle_{L^2(\Sigma)}$, its kernel is

$$K_w(x, y) = \widetilde{\phi}_{0, \Sigma}^2(x, y) W(x_n) \overline{W(y_n)},$$

where

$$\widetilde{\phi}_{0, \Sigma}^2(x, y) = \int_{U \times U} e^{-2\pi i[(x' - y') \cdot \xi + (x_n - y_n) h(\xi)]} |\phi(\xi)|^2 d\xi.$$

2.3 A useful lemma

The following version of Stein-Weiss inequality is based on the proof of Theorem 1.16 in [5].

Lemma 1. *Let $\alpha, \beta \in [0, \frac{n-1}{2}]$, $\gamma \in [0, \frac{n-1}{2}]$, and $B(f, g) := \int_{\mathbb{R}^n \times \mathbb{R}^n} K(x, y) f(x) g(y) dx dy$. Suppose that the kernel K satisfies*

$$|K(x, y)| \lesssim (1 + |x|)^{\alpha-\gamma} (1 + |y|)^{\alpha-\gamma} (1 + |x - y|)^{\beta+\gamma-\frac{n-1}{2}}$$

for all $x, y \in \mathbb{R}^n$. We have

$$|B(f, g)| \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}$$

provided that

- $p > 1$, $\gamma \geq \alpha$, $\sigma_p > \alpha + \beta$, $\kappa_p \geq 2\alpha - \gamma + \beta$; or
- $p = 1$ and $\gamma \geq \alpha$, $\frac{n-1}{2} \geq \alpha + \beta$, $\frac{n-1}{2} \geq 2\alpha - \gamma + \beta$.

We remark that Theorem 1.16 in [5] requires that α and β to be integers because they are the order of partial derivatives.

3 Proof of Theorem 3 and 4

3.1 Proof of Theorem 3

3.1.1 Case $p > 1$

Suppose $\beta > 0$ and $b \in \mathbb{R}$. We start with a lemma.

Lemma 2. *Let $a \in (-1, \infty)$, $0 \leq A < \frac{n-1}{2}$, $\ell \geq 0$, and $w \in L^\infty(\mathbb{R}; \mathbb{C})$ with norm at most 1. We have*

$$\|\phi S_{u^{a+ib}w}f\|_{H^{-A}(\Sigma)}^2 \lesssim \|\phi \hat{f}\|_{H^\ell(\Sigma)} \sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2a+1+bi}w}f\|_{H^{-2A-\ell}(\Sigma)} + \|f\|_{L^p(\mathbb{R}^n)}^2$$

provided that $\kappa_p \geq 2a - A + 2$, $\sigma_p > 2a + 2$, $A \geq a$, and everything on the right-hand side is finite.

Proof. We first observe that for complex-valued $w = w_R + iw_I$, where w_R and w_I are real-valued bounded measurable functions, we have

$$\begin{aligned} S_{u^{a+ib}w}f(\xi) &= \int_{\mathbb{R}} r^{a+ib} [w_R(r) + iw_I(r)] \mathcal{F}_{\eta \rightarrow r}(Tf(\xi, \cdot))(-r) dr \\ &= S_{u^{a+ib}w_R}f(\xi) + i S_{u^{a+ib}w_I}f(\xi). \end{aligned}$$

By taking the norm, one has

$$\sup_{\|w\|_{L^\infty(\mathbb{R}; \mathbb{C})} \leq 1} \|\phi S_{u^{a+ib}w}f\|_{H^{-A}(\Sigma)} \leq 2 \sup_{\|w'\|_{L^\infty(\mathbb{R}; \mathbb{R})} \leq 1} \|\phi S_{u^{a+ib}w'}f\|_{H^{-A}(\Sigma)}.$$

Moreover, $L^\infty(\mathbb{R}; \mathbb{R}) \subset L^\infty(\mathbb{R}; \mathbb{C})$, so we can focus on $w \in L^\infty(\mathbb{R}; \mathbb{R})$ with $\|w\|_{L^\infty(\mathbb{R}; \mathbb{R})} \leq 1$.

Note that one has

$$\begin{aligned} &\langle \phi S_{u^{a+ib}w}f, \phi S_{u^{a+ib}w}f \rangle_{\dot{H}^{-A}(\Sigma)} \\ &\simeq \int_{\mathbb{R}^n \times \mathbb{R}^n} \widetilde{\phi^2}_{-A, \Sigma}(x, y) \left(\frac{|W_{a,b}(x_n)|^2 + |W_{a,b}(y_n)|^2}{2} - \frac{1}{2} R(x_n, y_n) \right) f(x', x_n) \overline{f(y', y_n)} dy dx \\ &=: (I) + (II) - (III), \end{aligned}$$

where

$$W_{a,b}(x_n) := \int_{\mathbb{R}} r^{a+ib} w(r) \sqrt{\pi} 1_{x_n}(r) dr,$$

and $R(x_n, y_n) = |W_{a,b}(x_n) - W_{a,b}(y_n)|^2$.

The terms $|(I)|$ and $|(II)|$ can be controlled similarly, so it suffices to consider $|(I)|$. By writing

$$(I) = \left\langle \phi \mathcal{F} \left(\frac{|W_{a,b}(x_n)|^2}{2} f \right), \phi \hat{g} \right\rangle_{\dot{H}^{-A}(\Sigma)},$$

one has

$$|(I)| \leq \left\| \phi \mathcal{F} \left(\frac{|W_{a,b}(x_n)|^2}{2} f \right) \right\|_{\dot{H}^{-2A-\ell}(\Sigma)} \|\phi \hat{g}\|_{H^\ell(\Sigma)}.$$

If we define

$$\tilde{w}(u) := (u^{-a-1} w(u) W_{a,b}(u) + u^{-a-1-2bi} w(u) W_{a,b}(u)),$$

then $\int_{\mathbb{R}} u^{2a+1+bi} \tilde{w}(u) \sqrt{\pi} 1_{x_n}(u) du = \frac{1}{2} |W_{a,b}(x_n)|^2$ and $\|\tilde{w}\|_{L^\infty(\mathbb{R})} \lesssim 1$ because $|W_{a,b}(u)| \lesssim |u|^{a+1}$ and $\|w\|_{L^\infty} \leq 1$. Therefore,

$$\left\| \phi \mathcal{F} \left(\frac{|W_{a,b}(x_n)|^2}{2} f \right) \right\|_{\dot{H}^{-2A-\ell}(\Sigma)} \simeq \|\phi S_{u^{2a+1+bi}\tilde{w}}f\|_{\dot{H}^{-2A-\ell}(\Sigma)} \lesssim \sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2a+1+bi}w}f\|_{H^{-2A-\ell}(\Sigma)}.$$

The condition $a \in (-1, \infty)$ is used to obtain an estimate for $|W_{a,b}(u)|$, but the relation between p , a , and A is not used.

To estimate $|(III)|$, note that $|R(x_n, y_n)| \lesssim |x_n - y_n|^2 \max\{|x_n|^{2a}, |y_n|^{2a}\}$, and rewrite (III) as

$$\begin{aligned} |(III)| &\lesssim \int_{\mathbb{R}} \left(\int_{|x_n| \geq 2|y_n|} + \int_{\frac{|x_n|}{2} \leq |y_n| \leq 2|x_n|} + \int_{|y_n| \geq 2|x_n|} \right) |\widetilde{\phi^2}_{-A, \Sigma}(x, y) R(x_n, y_n)| |f(x', x_n)| |f(y', y_n)| dy dx \\ &=: (III_1) + (III_2) + (III_3). \end{aligned}$$

One can observe that if $|x_n| \geq 2|y_n|$ or $|x_n| \leq 2|y_n|$, then

$$|\widetilde{\phi^2}_{-A, \Sigma}(x, y) R(x_n, y_n)| \lesssim (1 + |x_n|)^{-A} (1 + |y_n|)^{-A} (1 + |x_n - y_n|)^{A+2+2a-\frac{n-1}{2}};$$

while if $\frac{|x_n|}{2} \leq |y_n| \leq 2|x_n|$, then

$$|\widetilde{\phi^2}_{-A, \Sigma}(x, y) R(x_n, y_n)| \lesssim (1 + |x_n|)^{a-A} (1 + |y_n|)^{a-A} (1 + |x_n - y_n|)^{A+2-\frac{n-1}{2}},$$

In order to apply Lemma 1, we require $A \geq 0$, $2 + 2a < \sigma_p$, and $2 + 2a - A \leq \kappa_p$ in the first case; and $A \geq a$, $2 + a < \sigma_p$, and $2a - A + 2 \leq \kappa_p$ in the second case. Therefore, we need $A \geq a$, $2 + 2a < \sigma_p$, and $2a - A + 2 \leq \kappa_p$, and the conditions are satisfied by the hypothesis.

Applying Lemma 1 to (III_1) and (III_3) with $\beta = 0$, $\beta = 2 + 2a$, $\gamma = A$ and to (III_2) with $\beta = a$, $\beta = 2$, $\gamma = A$, for $i = 1, 2, 3$ we have

$$|(III_i)| \lesssim \|f\|_{L^p(\mathbb{R}^n)}^2.$$

The case $A = 0$ follows similarly. $\square$

Let us go back to the proof of Theorem 3. Assume $\beta \in (0, 1]$. By applying Lemma 2 to $A = 0 \geq a = \beta - 1 > -1$, we have

$$\|\phi S_{u^{\beta-1+ib}w} f\|_{L^2(\Sigma)} \lesssim \|\phi \hat{f}\|_{H^\ell(\Sigma)}^{\frac{1}{2}} \sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2\beta-1+ib}w}\|_{H^{-\ell}(\Sigma)}^{\frac{1}{2}} + \|f\|_{L^p(\mathbb{R}^n)} \quad (3.1)$$

with $\kappa_p \geq 2\beta$ and $\sigma_p > 2\beta$.

We also need an *a priori* estimate.

Lemma 3. For $1 < p \leq \frac{2n+2}{n+3}$, we have

$$\sup_{\|w\|_{L^\infty(\mathbb{R}; \mathbb{C})} \leq 1, b \in \mathbb{R}} \|\phi S_{u^{\kappa_p-1+ib}w(u)} f\|_{\dot{H}^{-\kappa_p}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Let us write $W_{p,b}(r) := W_{\kappa_p-1,b}(r) = \int_{\mathbb{R}} u^{\kappa_p-1+ib} w(u) \sqrt{\pi} 1_r(u) du$.

Since $\|w\|_{L^\infty} \leq 1$, we have

$$\left| \int_{\mathbb{R}} u^{\kappa_p-1+ib} w(u) \sqrt{\pi} 1_{x_n}(u) du \right| \lesssim |x_n|^{\kappa_p}.$$

Consider the bilinear form $B(f, g) = \langle \phi S_{u^{\kappa_p-1+ib}w(u)} f, \phi S_{u^{\kappa_p-1+ib}w(u)} g \rangle_{\dot{H}^{-\kappa_p}(\Sigma)}$. Its kernel is bounded by

$$\begin{aligned} |\widetilde{\phi^2}_{-\kappa_p, \Sigma}(x, y) W_{p,b}(x_n) \overline{W_{p,b}(y_n)}| &\lesssim (1 + |x_n|)^{-\kappa_p} (1 + |y_n|)^{-\kappa_p} (1 + |x_n - y_n|)^{\kappa_p - \frac{n-1}{2}} (1 + |x_n|)^{\kappa_p} (1 + |y_n|)^{\kappa_p} \\ &\lesssim (1 + |x_n - y_n|)^{\kappa_p - \frac{n-1}{2}}; \end{aligned}$$

and by the aid of Lemma 1, we have

$$\|\phi S_{u^{\kappa_p-1+ib}w(u)} f\|_{\dot{H}^{-\kappa_p}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)}.$$

When $\kappa_p = 0$ (i.e. when $p = \frac{2n+2}{n+3}$), we use the Hardy-Littlewood-Sobolev inequality instead. $\square$

Using Linderöf Theorem by regarding $(1 + |\xi|)^{m+ib}$ as a weight and applying to the function $\hat{f}$, set $\theta \in [0, 1]$ such that $2\beta - 1 = (1 - \theta)(\kappa_p - 1) + \theta(\beta - 1)$, i.e. $\theta = \frac{\kappa_p - 2\beta}{\kappa_p - \beta} = \frac{\kappa_p - 2\beta}{\kappa_p - \beta}$, we have

$$\begin{aligned}
\|\phi S_{u^{2\beta-1+ib}w}f\|_{\dot{H}^{-\ell}(\Sigma)} &\lesssim \sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{\beta-1+ib}w}f\|_{L^2(\Sigma)}^\theta \sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{\kappa_p-1+ib}w}f\|_{H^{-\kappa_p}(\Sigma)}^{1-\theta} \\
&\lesssim (\|\phi \hat{f}\|_{H^\ell(\Sigma)}^{\frac{1}{2}} \sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2\beta-1+ib}w}f\|_{\dot{H}^{-\ell}(\Sigma)}^{\frac{1}{2}} + \|f\|_{L^p(\mathbb{R}^n)})^\theta \|f\|_{L^p(\mathbb{R}^n)}^{1-\theta} \\
&\lesssim \|\phi \hat{f}\|_{H^\ell(\Sigma)}^\theta \|f\|_{L^p(\mathbb{R}^n)}^{1-\theta} + \sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2\beta-1+ib}w}f\|_{H^{-\ell}(\Sigma)}^\theta \|f\|_{L^p(\mathbb{R}^n)}^{1-\theta} + \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{3.2}$$

Here, $\ell = (1 - \theta)\kappa_p = \frac{\kappa_p\beta}{\kappa_p - \beta}$.

We can obtain the estimate for $\sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{\beta-1+ib}w}f\|_{H^{-\ell}(\Sigma)}$ using (3.2). Let C be the implicit constant in (3.2). If $\sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2\beta-1+ib}w}f\|_{H^{-\ell}(\Sigma)} \leq (2C)^{\frac{1}{1-\theta}} \|f\|_{L^p(\mathbb{R}^n)}$, then we are done. If not, i.e. $\sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2\beta-1+ib}w}f\|_{H^{-\ell}(\Sigma)} > (2C)^{\frac{1}{1-\theta}} \|f\|_{L^p(\mathbb{R}^n)}$, then (3.2) implies

$$\begin{aligned}
\sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2\beta-1+ib}w}f\|_{H^{-\ell}(\Sigma)}^\theta &\leq C(\|\phi \hat{f}\|_{H^\ell(\Sigma)}^\theta \|f\|_{L^p(\mathbb{R}^n)}^{1-\theta} + \|f\|_{L^p(\mathbb{R}^n)})(C\|f\|_{L^p(\mathbb{R}^n)}^{1-\theta})^{-1} \\
&\leq \|\phi \hat{f}\|_{H^\ell(\Sigma)}^\theta + \|f\|_{L^p(\mathbb{R}^n)}^\theta.
\end{aligned}$$

Thus, we have

$$\sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2\beta-1+ib}w}f\|_{H^{-\ell}(\Sigma)} \lesssim \|\phi \hat{f}\|_{H^\ell(\Sigma)} + \|f\|_{L^p(\mathbb{R}^n)}, \tag{3.3}$$

for all $b \in \mathbb{R}$, $\ell \geq \frac{\kappa_p\beta}{\kappa_p - \beta}$ given that $f \in L^p(\mathbb{R}^n)$, $\kappa_p \geq 2\beta$, $p > 1$ and $0 \leq \beta \leq 1$.

Plugging (3.3) into (3.1), we have

$$\begin{aligned}
\sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{\beta-1+ib}w}f\|_{L^2(\Sigma)} &\lesssim \|\phi \hat{f}\|_{H^\ell(\Sigma)}^{\frac{1}{2}} (\|\phi \hat{f}\|_{H^\ell(\Sigma)} + \|f\|_{L^p(\mathbb{R}^n)})^{\frac{1}{2}} + \|f\|_{L^p(\mathbb{R}^n)} \\
&\simeq \|\phi \hat{f}\|_{H^\ell(\Sigma)} + \|f\|_{L^p(\mathbb{R}^n)}.
\end{aligned} \tag{3.4}$$

This finishes the proof of Theorem 3.

3.1.2 Case $p = 1$

When $p = 1$, $\kappa_1 = \frac{n-1}{2}$ and $\frac{\kappa_p\beta}{\kappa_p - \beta} = \frac{(n-1)\beta}{n-1-2\beta}$; thus, the argument in Lemma 3 is no longer true.

Lemma 4. *Let $\varepsilon > 0$. We have*

$$\sup_{\|w\|_{L^\infty(\mathbb{R}; \mathbb{C})} \leq 1, b \in \mathbb{R}} \|\phi S_{u^{\kappa_1-1+ib}w(u)}f\|_{H^{-\kappa_1-\varepsilon}(\Sigma)} \lesssim \|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. Fix $\varepsilon > 0$. Note that for $f \in L^1(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\|\phi S_{u^{\kappa_1-1+ib}w(u)}f\|_{H^{-\kappa_1-\varepsilon}(\Sigma)} \\
&= \left\| \phi(\xi) \int_{\mathbb{R}^n} W_{\kappa_1-1,b}(x_n) e^{-2\pi i(x'\xi + x_n h(\xi))} f(x', x_n) dx' dx_n \right\|_{H^{-\kappa_1-\varepsilon}(\mathbb{R}^{n-1})}.
\end{aligned}$$

We will prove that

$$\left\| \phi(\xi) \int_{\mathbb{R}^n} W_{\kappa_1-1,b}(x_n) e^{-2\pi i(x'\xi + x_n h(\xi))} \mu(dx', dx_n) \right\|_{H^{-\kappa_1-\varepsilon}(\mathbb{R}^{n-1})} \lesssim |\mu(\mathbb{R}^n)|$$

for all Borel measures μ with $|\mu(\mathbb{R}^n)| \leq 1$. Then the lemma follows from taking $d\mu = \frac{f(x)}{\|f\|_{L^1(\mathbb{R}^n)}} dx$.

In this case, it suffices to prove for the delta measures because the delta measures are extremal points in the unit ball of Borel measures. Let the delta measure be $\delta_{(x', x_n)}$. We have

$$\begin{aligned} & \left\| \phi(\xi) \int_{\mathbb{R}^n} W_{\kappa_1-1, b}(x_n) e^{-2\pi i(x' \xi + x_n h(\xi))} \delta_{(x', x_n)}(dx', dx_n) \right\|_{H^{-\kappa_1-\varepsilon}(\mathbb{R}^{n-1})} \\ &= \left\| \phi(\xi) W_{\kappa_1-1, b}(x_n) e^{-2\pi i(x' \xi + x_n h(\xi))} \right\|_{H^{-\kappa_1-\varepsilon}(\mathbb{R}^{n-1})} \\ &\lesssim (1 + |x_n|)^{\kappa_1} \left\| \phi(\xi) e^{-2\pi i(x' \xi + x_n h(\xi))} \right\|_{H^{-\kappa_1-\varepsilon}(\mathbb{R}^{n-1})} \\ &\simeq (1 + |x_n|)^{\kappa_1} \left(\int_{\mathbb{R}^{n-1}} (1 + |y|)^{-2\kappa_1-2\varepsilon} \left| \mathcal{F}_\xi(\phi(\xi) e^{-2\pi i(x' \xi + x_n h(\xi))})(y) \right|^2 dy \right)^{\frac{1}{2}} \\ &\lesssim (1 + |x_n|)^{\kappa_1} (1 + |x_n|)^{-\kappa_1} \left(\int_{\mathbb{R}^{n-1}} (1 + |y|)^{-2\kappa_1-2\varepsilon} dy \right)^{\frac{1}{2}} \lesssim 1 = |\delta_{(x', x_n)}(\mathbb{R}^n)|. \end{aligned}$$

We have used the Van der Corput Lemma to obtain $(1 + |x_n|)^{-\kappa_1}$ in the last line. $\square$

If we set $\theta = \frac{\kappa_1-2\beta}{\kappa_1-\beta}$, which is in $[0, 1]$ provided that $0 \leq \beta \leq 1$ and $2\beta \leq \kappa_1$, then Equation (3.2) holds if $\ell \geq \frac{\beta\kappa_1}{\kappa_1-\beta} + \frac{\varepsilon\beta}{\kappa_1-\beta} > \frac{\beta\kappa_1}{\kappa_1-\beta}$ and $p = 1$; and this implies (3.3) and (3.4) whenever $\ell > \frac{\beta\kappa_1}{\kappa_1-\beta}$, $0 \leq \beta \leq \min\{1, \kappa_1/2\}$, and $p = 1$.

3.2 Higher power of β

For $\beta > 1$, one can take $A \geq a = \beta - 1$, we have

$$\|\phi S_{u^{\beta-1+ib}w} f\|_{H^{-A}(\Sigma)}^2 \lesssim \|\phi \hat{f}\|_{H^\ell(\Sigma)} \sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{2\beta-1+ib}w} f\|_{H^{-2A-\ell}(\Sigma)} + \|f\|_{L^p(\mathbb{R}^n)}^2.$$

Interpolating this with Lemma 3, we have

$$\|\phi S_{u^{2\beta-1+ib}w} f\|_{H^{-2A-\ell}} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)}$$

provided that $\beta \in [1, \frac{\kappa_p}{2}]$ (which is less than $\frac{n+1}{2}$), $\kappa_p \geq 2\beta - A$, and $\sigma_p > 2\beta$, $\ell \geq \frac{\kappa_p}{\kappa_p-\beta}(\beta - A)$, and $\beta - 1 \leq A < \beta$ hold simultaneously.

Indeed, if we take $A = a = \beta - 1$, we have

Theorem 5. Suppose $f \in L^p(\mathbb{R}^n)$ has $\hat{f}|_\Sigma \in H^\ell(\Sigma)$, and $\phi \in C_c^\infty(\Sigma)$. If $\kappa_p \geq \beta + 1$ and $\sigma_p > 2\beta$, $\ell \geq \frac{\kappa_p}{\kappa_p-\beta}$ and $1 \leq \beta \leq \frac{\kappa_p}{2}$, we have

$$\sup_{\|w\|_{L^\infty} \leq 1, b \in \mathbb{R}} \|\phi S_{u^{\beta-1+ib}w} f\|_{H^{-\beta+1}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)}$$

and

$$\sup_{\|w\|_{L^\infty} \leq 1, b \in \mathbb{R}} \|\phi S_{u^{2\beta-1+ib}w} f\|_{H^{-2\beta+1-\ell}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)}.$$

For $\frac{\kappa_p}{2} \leq \beta \leq \kappa_p$, we would apply Lemma 2 to $a = \frac{\kappa_p}{2} - 1$ and $2A + \ell = \kappa_p$. Therefore, one needs $2A = \kappa_p - \ell \geq \kappa_p - 2 = 2a$, that is, $\ell \leq 2$, and the best possible we have is

$$\|\phi S_{u^{(\kappa_p/2)-1+ib}w} f\|_{H^{-\frac{\kappa_p-\ell}{2}}(\Sigma)} \lesssim \|\phi \hat{f}\|_{H^\ell(\Sigma)} + \|f\|_{L^p(\mathbb{R}^n)}$$

given that $\sigma_p > \kappa_p$ and $\kappa_p \geq \ell$.

By writing

$$\|\phi S_{u^{\beta-1+ib}w}f\|_{H^{-s}(\Sigma)} = \|\phi S_{u^{\frac{\kappa_p}{2}-1+ib}w}f\|_{H^{-\frac{\kappa_p+\ell}{2}}(\Sigma)}^\theta \|\phi S_{u^{\kappa_p-1+ib}w}f\|_{H^{-\kappa_p}(\Sigma)}^{1-\theta}$$

with $\beta = \frac{\kappa_p}{2}\theta + \kappa_p(1-\theta)$ and $s = (\frac{\kappa_p-\ell}{2})\theta + \kappa_p(1-\theta) = \frac{\ell\beta+\beta\kappa_p-\ell\kappa_p}{\kappa_p}$, we have

$$\|\phi S_{u^{\beta-1+ib}w}f\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi\hat{f}\|_{H^\ell(\Sigma)}.$$

Theorem 6. Suppose $f \in L^p(\mathbb{R}^n)$ has $\hat{f}|_\Sigma \in H^\ell(\Sigma)$, and $\phi \in C_c^\infty(\Sigma)$. If $\frac{\kappa_p}{2} \leq \beta \leq \kappa_p$, $\ell \leq \min\{\kappa_p, 2\}$, and $\sigma_p > \kappa_p$, then we have

$$\sup_{\|w\|_{L^\infty} \leq 1, b \in \mathbb{R}} \|\phi S_{u^{\beta-1+ib}w}f\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi\hat{f}\|_{H^\ell(\Sigma)},$$

where $s \geq \frac{\ell\beta+\beta\kappa_p-\ell\kappa_p}{\kappa_p}$.

On the other hand, instead of taking $2A + \ell = \kappa_p$, if we take $A = a = \frac{\kappa_p}{2} - 1$ and set $2A + \ell \geq \kappa_p$ (i.e. $\ell \geq 2$), using the embedding of Sobolev spaces and Lemma 3 we have

$$\begin{aligned} \|\phi S_{u^{(\kappa_p/2)-1+ib}w}f\|_{H^{-\frac{\kappa_p}{2}+1}(\Sigma)}^2 &\lesssim \|\phi\hat{f}\|_{H^\ell(\Sigma)}^2 \sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{\kappa_p-1+ib}w}f\|_{H^{-2A-\ell}(\Sigma)}^2 + \|f\|_{L^p(\mathbb{R}^n)}^2 \\ &\lesssim \|\phi\hat{f}\|_{H^\ell(\Sigma)}^2 \sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{\kappa_p-1+ib}w}f\|_{H^{-\kappa_p}(\Sigma)}^2 + \|f\|_{L^p(\mathbb{R}^n)}^2 \\ &\lesssim \|\phi\hat{f}\|_{H^\ell}^2 + \|f\|_{L^p(\mathbb{R}^n)}^2 \end{aligned}$$

with the conditions $\kappa_p \geq 2$ and $\sigma_p > \kappa_p$. Then, by writing

$$\|\phi S_{u^{\beta-1+ib}w}f\|_{H^{-s}(\Sigma)} = \|\phi S_{u^{\frac{\kappa_p}{2}-1+ib}w}f\|_{H^{1-\frac{\kappa_p}{2}}(\Sigma)}^\theta \|\phi S_{u^{\kappa_p-1+ib}w}f\|_{H^{-\kappa_p}(\Sigma)}^{1-\theta}$$

with $\beta = \frac{\kappa_p}{2}\theta + \kappa_p(1-\theta)$ and $s = (\frac{\kappa_p}{2} - 1)\theta + \kappa_p(1-\theta) = \frac{2\beta+\beta\kappa_p-2\kappa_p}{\kappa_p}$, we have

$$\|\phi S_{u^{\beta-1+ib}w}f\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi\hat{f}\|_{H^\ell(\Sigma)}.$$

Theorem 7. Suppose $f \in L^p(\mathbb{R}^n)$ has $\hat{f}|_\Sigma \in H^\ell(\Sigma)$, and $\phi \in C_c^\infty(\Sigma)$. If $\frac{\kappa_p}{2} \leq \beta \leq \kappa_p$, $\ell \geq 2$, $\kappa_p \geq 2$ and $\sigma_p > \kappa_p$, then we have

$$\sup_{\|w\|_{L^\infty} \leq 1, b \in \mathbb{R}} \|\phi S_{u^{\beta-1+ib}w}f\|_{H^{-s}(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi\hat{f}\|_{H^\ell(\Sigma)},$$

where $s \geq \frac{2\beta+\beta\kappa_p-2\kappa_p}{\kappa_p}$.

3.3 Proof of Theorem 4

To obtain Theorem 4, we assume that (1.5) and (1.6) hold. If $\beta \leq s$, then one can apply Theorem A to conclude that (1.8) holds. In particular, $\beta \leq s$ includes the case $\kappa_p < \beta < \sigma_p$ due to the inequality $2\beta - s \leq \kappa_p$. In the remainder of this section, we only consider the cases $\beta > s$ and $0 \leq \beta \leq \kappa_p$. Note that for fixed ℓ , κ_p , and β , then $s \geq \max\{0, \beta - 1, \beta - \ell, \frac{\ell\beta+\beta\kappa_p-\ell\kappa_p}{\kappa_p}\}$ if $2\beta \leq \kappa_p$; and $s \geq \max\{0, \beta - 1, \beta - \ell, \frac{\ell\beta+\beta\kappa_p-\ell\kappa_p}{\kappa_p}, \frac{2\beta+\beta\kappa_p-2\kappa_p}{\kappa_p}, 2\beta - \kappa_p\}$ if $\kappa_p < 2\beta$.

3.3.1 When $0 \leq \ell \leq 2$:

The first subcase is $0 \leq \beta \leq \min\{\frac{\kappa_p\ell}{\kappa_p+\ell}, \frac{\kappa_p}{2}\}$. In this case, we always have $\sigma_p > \kappa_p \geq 2\beta$ and $\ell \geq \frac{\kappa_p\beta}{\kappa_p-\beta}$.

If $\kappa_p \leq 2$, then we have $\min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\} \leq 1$ and (1.8) holds by Theorem 3.

If $\kappa_p \geq 2$ and $0 \leq \ell \leq 1$, then $\min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\} = \frac{\kappa_p \ell}{\kappa_p + \ell} \leq 1$, and Theorem 3 allows us to have (1.8). If $\kappa_p \geq 2$ and $1 < \ell \leq 2$, then $\min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\} \geq 1$ whenever $\kappa_p \geq \frac{\ell}{\ell-1}$. When $2 \leq \kappa_p \leq \frac{\ell}{\ell-1}$, for $0 \leq \beta \leq \min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\} \leq 1$, and we can apply Theorem 3. When $\kappa_p \geq \frac{\ell}{\ell-1}$, we need to consider $0 \leq \beta \leq 1$ and $1 \leq \beta \leq \min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\}$. For $0 \leq \beta \leq 1$, we apply Theorem 3; for $1 \leq \beta \leq \min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\}$, we also have $\kappa_p \geq 2\beta \geq 1 + \beta$ and $\ell \geq \frac{\kappa_p \beta}{\kappa_p - \beta} \geq \frac{\kappa_p}{\kappa_p - \beta}$, thus we can apply Theorem 5.

The second subcase is $\min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\} < \beta \leq \kappa_p$.

If $\min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\} = \frac{\kappa_p}{2}$, then $\ell \geq \kappa_p$. Then for $\beta \in (\frac{\kappa_p}{2}, \kappa_p]$, one just requires $s \geq 2\beta - \kappa_p$ in order to have (1.8). More precisely, if $\frac{\kappa_p}{2} \leq 1$ (i.e. $\kappa_p \leq 2$ only, other situations imply $\kappa_p \geq \ell$), one has $\sup_{\|w\|_{L^\infty} \leq 1} \|\phi S_{u^{\kappa_p/2-1+ib}w} f\|_{L^2(\Sigma)} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^{\kappa_p}(\mathbb{R}^n)}$. By interpolation, we have

$$\|\phi S_{u^{\kappa_p-1+ib}w} f\|_{H^{-s}(\Sigma)} \leq \|\phi S_{u^{\kappa_p/2-1+ib}w} f\|_{L^2(\Sigma)}^\theta \|\phi S_{u^{\kappa_p-1+ib}w} f\|_{H^{-\kappa_p}(\Sigma)}^{1-\theta}$$

with $\beta = \frac{\kappa_p}{2}(\theta) + \kappa_p(1-\theta)$ and $s \geq 0(\theta) + \kappa_p(1-\theta) = 2\beta - \kappa_p$. Thus, we can conclude that (1.8) holds if $\beta \in (\frac{\kappa_p}{2}, \kappa_p]$ and $\min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\} = \frac{\kappa_p}{2}$.

If $\min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\} = \frac{\kappa_p \ell}{\kappa_p + \ell}$, then $\kappa_p \geq \ell$. If $\frac{\kappa_p \ell}{\kappa_p + \ell} \leq 1$ (i.e. when $\kappa_p \leq 2$, or $\kappa_p \geq 2$ and $\ell \in [0, 1]$, or $2 \leq \kappa_p \leq \frac{\ell}{\ell-1}$ and $\ell \in (1, 2]$), then using interpolation as above with $\beta = \frac{\kappa_p \ell}{\kappa_p + \ell}(\theta) + \kappa_p(1-\theta)$ and $s \geq \kappa_p(1-\theta) = \frac{\beta \ell + \beta \kappa_p - \ell \kappa_p}{\kappa_p}$, we can conclude that (1.8) holds. We now consider $\kappa_p \geq \frac{\ell}{\ell-1}$ with $\ell \in (1, 2]$. In this case, $\frac{\kappa_p \ell}{\kappa_p + \ell} \leq \frac{\ell-1}{\ell} \kappa_p \leq \frac{\kappa_p}{2}$. For $\frac{\kappa_p}{2} \leq \beta \leq \kappa_p$, we can directly apply Theorem 6. For $\frac{\kappa_p \ell}{\kappa_p + \ell} \leq \beta \leq \frac{\ell-1}{\ell} \kappa_p$, we have $\ell \geq \frac{\kappa_p}{\kappa_p - \beta}$ and $\max\{0, \beta - 1, \beta - \ell, \frac{\ell\beta + \beta \kappa_p - \ell \kappa_p}{\kappa_p}\} = \beta - 1$, we can apply Theorem 5. For $\frac{\ell-1}{\ell} \kappa_p \leq \beta \leq \frac{\kappa_p}{2}$, we have $\max\{0, \beta - 1, \beta - \ell, \frac{\ell\beta + \beta \kappa_p - \ell \kappa_p}{\kappa_p}\} = \frac{\ell\beta + \beta \kappa_p - \ell \kappa_p}{\kappa_p}$. Therefore, by interpolation as in the case above with $\beta = \frac{\ell-1}{\ell} \kappa_p(\theta) + \frac{\kappa_p}{2}(1-\theta)$ and $s \geq \frac{\ell\beta + \beta \kappa_p - \ell \kappa_p}{\kappa_p}$, we can conclude that (1.8) holds.

3.3.2 When $\ell \geq 2$:

The first subcase is $0 \leq \beta \leq \frac{\kappa_p}{2}$. We have $\sigma_p > \kappa_p \geq 2\beta$ only.

If $\kappa_p \leq 2$, then observe that $\ell \geq 2 \geq \kappa_p \geq \frac{\kappa_p \beta}{\kappa_p - \beta}$. This allows us to apply Theorem 3.

If $\kappa_p \geq 2$, for $0 \leq \beta \leq 1$, we have $\ell \geq 2 \geq \frac{\kappa_p}{\kappa_p - 1} \geq \frac{\kappa_p \beta}{\kappa_p - \beta}$, and (1.8) holds by using Theorem 3. For $1 \leq \beta \leq \frac{\kappa_p}{2}$, we also have $\ell \geq 2 \geq \frac{\kappa_p}{\kappa_p - \beta}$ and $\kappa_p \geq 2\beta \geq \beta + 1$. Therefore, (1.8) follows from Theorem 5.

The second subcase is $\frac{\kappa_p}{2} \leq \beta \leq \kappa_p$.

If $\kappa_p \geq 2$, (1.8) follows from Theorem 7 directly.

If $\kappa_p < 2$, we have $\frac{\kappa_p}{2} < 1$ and $\ell \geq 2 \geq \kappa_p$. In fact, we are in the same situation as in the subcase $\min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\} = \frac{\kappa_p}{2}$ in $0 \leq \ell \leq 2$. Therefore, we can conclude that (1.8) holds.

This finishes the proof of Theorem 4.

Remark. We can see that the conditions on s are sufficient to have (1.8) and sharp in all cases but $\ell \geq 2$ and $1 \leq \frac{\kappa_p}{2} \leq \beta \leq \kappa_p$. It is not sharp in the sense that we additionally assumed $s \geq \frac{2\beta + \beta \kappa_p - 2\kappa_p}{\kappa_p}$, of which the lower bound is strictly larger than $\frac{\ell\beta + \beta \kappa_p - \ell \kappa_p}{\kappa_p}$ if $\ell > 2$. See Figures 3 and 4 for more explanations.

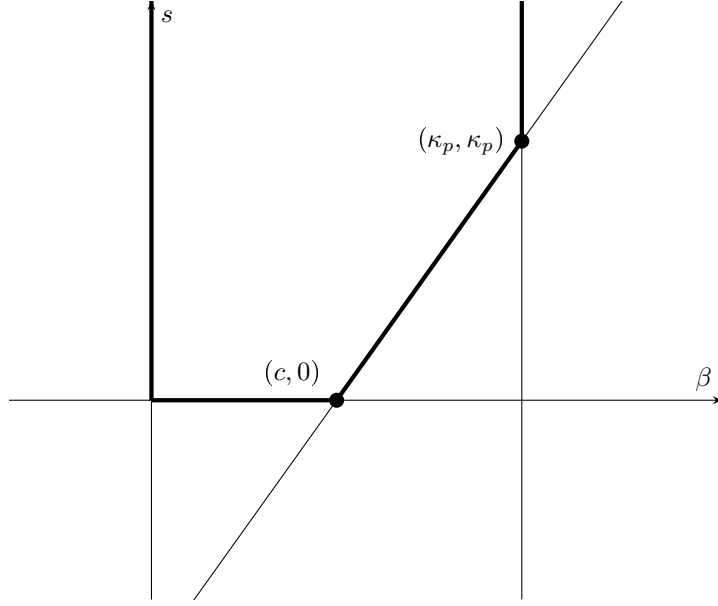


Figure 1: When $c = \min\{\frac{\kappa_p \ell}{\kappa_p + \ell}, \frac{\kappa_p}{2}\} \leq 1$, we directly interpolate the points $(c, 0)$ and (κ_p, κ_p) . The region bounded by the bold lines is the region of all exponents that we can obtain, and it is optimal. The case $\kappa_p \leq 2$ and $\ell \geq 0$, the case $\kappa_p \geq 2$ and $\ell \in [0, 1]$, and the case $\kappa_p \in [2, \frac{\ell}{\ell-1}]$ and $\ell \in (1, 2]$ are in this situation.

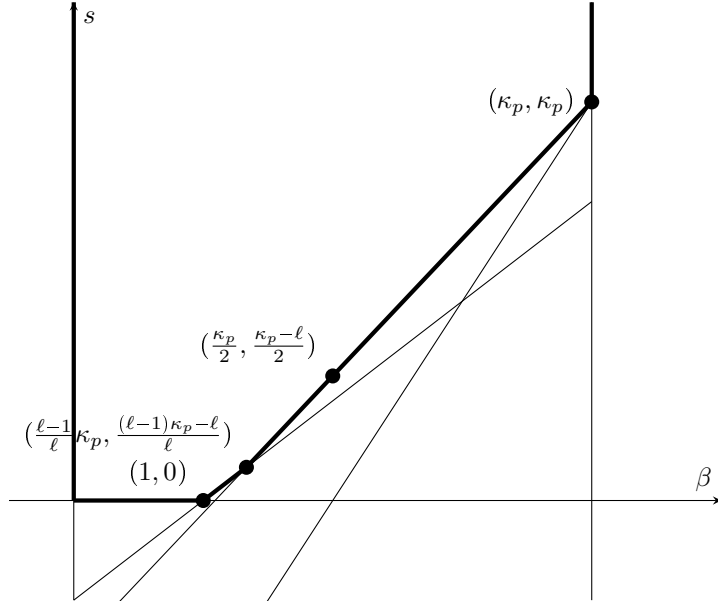


Figure 2: The case $\kappa_p \geq \frac{\ell}{\ell-1}$ and $\ell \in (1, 2]$. The line joining $(1, 0)$ and $(\frac{\ell-1}{\ell} \kappa_p, \frac{(\ell-1)\kappa_p - \ell}{\ell})$ is $s = \beta - 1$, and the line joining $(\frac{\ell-1}{\ell} \kappa_p, \frac{(\ell-1)\kappa_p - \ell}{\ell})$ and (κ_p, κ_p) is $s = \frac{\ell\beta - \ell\kappa_p + \kappa_p\beta}{\kappa_p}$. Theorem 6 allows us to obtain the exponents between $(\frac{\kappa_p}{2}, \frac{(\kappa_p - \ell)}{2})$ and (κ_p, κ_p) , and an interpolation argument allows us to obtain the exponents between $(\frac{\ell-1}{\ell} \kappa_p, \frac{\ell-1}{\ell} \kappa_p - 1)$ and $(\frac{\kappa_p}{2}, \frac{(\kappa_p - \ell)}{2})$. The region bounded by the bold lines is the region of all exponents that we can obtain, and it is optimal.

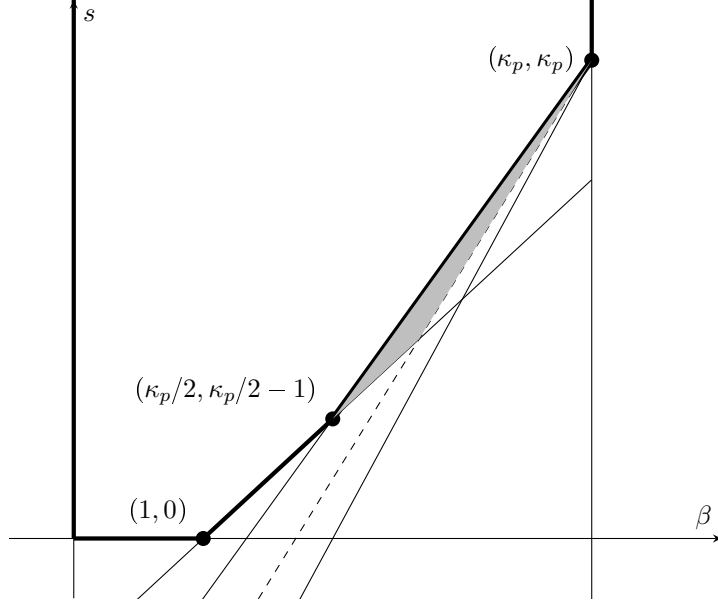


Figure 3: This figure shows a case of $\ell \geq 2$, $\kappa_p \geq 2$, and $\kappa_p \geq \ell$. The line connecting $(1, 0)$ and $(\kappa_p/2, \kappa_p/2 - 1)$ is $s = \beta - 1$, the line joining $(\kappa_p/2, \kappa_p/2 - 1)$ and (κ_p, κ_p) is $s = \frac{2\beta - 2\kappa_p + \kappa_p\beta}{\kappa_p}$, the dashed line is $s = \frac{\ell\beta - \ell\kappa_p + \kappa_p\beta}{\kappa_p}$, and the line below the dashed line is $s = 2\beta - \kappa_p$. The region bounded by the bold lines is the region of all possible exponents that we can obtain, and the shaded region is the region that we cannot obtain using Theorem 7, as we are not able to obtain (1.8) for $(\frac{\ell-1}{\ell}\kappa_p, \frac{\ell-1}{\ell}\kappa_p - 1)$.

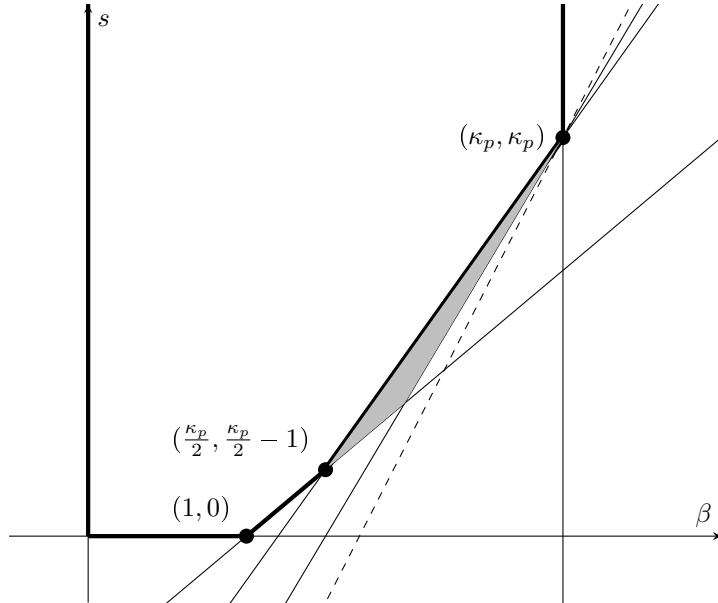


Figure 4: This figure shows a case of $\ell \geq 2$, $\kappa_p \geq 2$, and $\ell \geq \kappa_p$. The line connecting $(1, 0)$ and $(\kappa_p/2, \kappa_p/2 - 1)$ is $s = \beta - 1$, the line joining $(\kappa_p/2, \kappa_p/2 - 1)$ and (κ_p, κ_p) is $s = \frac{2\beta - 2\kappa_p + \kappa_p\beta}{\kappa_p}$, and the dashed line is $s = \frac{\ell\beta - \ell\kappa_p + \kappa_p\beta}{\kappa_p}$, which is below the line $s = 2\beta - \kappa_p$. Although $s = \frac{\ell\beta - \ell\kappa_p + \kappa_p\beta}{\kappa_p}$ does not come into the play, we cannot obtain the exponents in the shaded region because we are not able to obtain (1.8) for $(\kappa_p - 1, \kappa_p - 2)$ due to the limitation of Lemma 2.

4 Proof of Corollary 1

4.1 Kernel of T^*T

Assuming $f, g \in \mathcal{S}(\mathbb{R}^n)$, if $\gamma \in (0, \frac{n-1}{2})$, we have

$$\begin{aligned} & \langle T(f)(\xi, \eta)\phi(\xi), T(g)(\xi, \eta)\phi(\xi) \rangle_{\dot{H}_\xi^{-\gamma} H_\eta^\tau} \\ &= 4\pi^2 C' \int_{\mathbb{R}} \int_U \int_U (1 + |r|)^{2\tau} \left(\int_{\mathbb{R}^n} s_r(x_n) e^{-2\pi i(x' \cdot \xi + x_n h(\xi))} f(x', x_n) dx' dx_n \right) \\ & \quad \times \left(\int_{\mathbb{R}^n} s_r(y_n) e^{2\pi i(y' \cdot \zeta + y_n h(\zeta))} \overline{g(y', y_n)} dy' dy_n \right) \phi(\xi) \overline{\phi(\zeta)} |\xi - \zeta|^{2\gamma - n + 1} dr d\zeta d\xi \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} K_{-\gamma, \tau}(x, y) f(x) g(y) dx dy, \end{aligned}$$

where

$$K_{-\gamma, \tau}(x, y) = 4\pi^2 C' \tau_{min}(x_n, y_n) \widetilde{\phi}_{-\gamma, \Sigma}^2(x, y)$$

and

$$\tau_{min}(x_n, y_n) = \chi_{[0, \infty)}(x_n y_n) \frac{(1 + \min\{|x_n|, |y_n|\})^{1+2\tau} - 1}{1 + 2\tau}.$$

When $\gamma = 0$ and $\tau = 0$, one has

$$\langle T(f)(\xi, \eta)\phi(\xi), T(g)(\xi, \eta)\phi(\xi) \rangle_{L_\xi^2 L_\eta^2} = \int_{\mathbb{R}^n \times \mathbb{R}^n} K_{0,0}(x, y) f(x) g(y) dx dy,$$

where

$$K_{0,0}(x, y) = 4\pi^2 s_{min}(x_n, y_n) \widetilde{\phi}_{0, \Sigma}^2(x, y)$$

and

$$s_{min}(x_n, y_n) = \chi_{[0, \infty)}(x_n y_n) \min\{|x_n|, |y_n|\}.$$

4.2 Main proof

Fix $1 < p \leq \frac{2n+2}{n+5}$. To start with, we write

$$\begin{aligned} & |\langle T(f)\phi, T(g)\phi \rangle_{L_\xi^2 L_\eta^2}| \\ &= \left| \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} K(x, y) f(x', x_n) g(y', y_n) dx' dy' dx_n dy_n \right| \\ &\leq \left| \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \left[K(x, y) - (4\pi^2) \frac{|x_n| + |y_n|}{2} \widetilde{\phi}_{0, \Sigma}^2(x, y) \right] f(x', x_n) g(y', y_n) dx' dy' dx_n dy_n \right| \\ & \quad + (4\pi^2) \left| \int_{\mathbb{R} \times \mathbb{R}} \int_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} \frac{|x_n| + |y_n|}{2} \widetilde{\phi}_{0, \Sigma}^2(x, y) f(x', x_n) g(y', y_n) dx' dy' dx_n dy_n \right| \\ &=: (I) + (II). \end{aligned}$$

We first estimate (I). Note that

$$\left| s_{min}(x_n, y_n) - \frac{|x_n| + |y_n|}{2} \right| \leq C|x_n - y_n|$$

because when $x_n y_n < 0$ we have $|x_n| + |y_n| = |x_n - y_n|$ while $s_{min}(x_n, y_n) = 0$. To proceed, we define the operator

$$\mathcal{T}(g)(x', x_n; y_n) := \int_{\mathbb{R}^{n-1}} \widetilde{\phi}_{0, \Sigma}^2(x, y) g(y') dy' = \mathcal{F}^{-1}(e^{-2\pi i(x_n - y_n)|\xi|^2} |\phi(\xi)|^2 \widehat{g})(x')$$

for $g \in L^1(\mathbb{R}^{n-1})$. Note that the last equality holds because the kernel of $\mathcal{T}$ is of convolution type (*i.e.* $\widetilde{\phi^2}_{0,\Sigma}(x, y) = \widetilde{\phi^2}_{0,\Sigma}(x - y)$). Since $\phi \in C_c^\infty(U)$, one has

$$\|\mathcal{T}(g)(\cdot, x_n; y_n)\|_{L^p(\mathbb{R}^{n-1})} \lesssim_\phi (1 + |x_n - y_n|)^{-(n-1)(\frac{1}{p}-\frac{1}{2})} \|f\|_{L^p(\mathbb{R}^{n-1})}.$$

Then, using Hardy-Littlewood-Sobolev inequality with $1 < p \leq \frac{2n+2}{n+5}$, we have

$$\begin{aligned} |(I)| &\lesssim \int_{\mathbb{R} \times \mathbb{R}} |x_n - y_n| \int_{\mathbb{R}^{n-1}} |f(x', x_n)| |\mathcal{T}[g(\cdot, y_n)](x')| dx' dy_n dx_n \\ &\lesssim \int_{\mathbb{R} \times \mathbb{R}} |x_n - y_n| \|f(\cdot, x_n)\|_{L^p(\mathbb{R}^{n-1})} (1 + |x_n - y_n|)^{-(n-1)(\frac{1}{p}-\frac{1}{2})} \|g(\cdot, y_n)\|_{L^p(\mathbb{R}^{n-1})} dx_n dy_n \\ &\lesssim \|f\|_{L^p_x L^p_{x_n}} \|g\|_{L^p_y L^p_{y_n}}. \end{aligned}$$

To estimate (II), it suffices to consider

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|x_n|}{2} \widetilde{\phi^2}_{0,\Sigma}(x, y) f(x) g(y) dx dy = \langle \phi |x_n| f, \phi \hat{g} \rangle_{L^2_\xi}.$$

By duality and (3.3) with $\beta = \frac{1}{2}$, we have

$$\begin{aligned} |\langle \phi |x_n| f, \phi \hat{g} \rangle_{L^2_\xi}| &\lesssim \|\widehat{\phi |x_n| f}\|_{H^{-\ell}(\Sigma)} \|\phi \hat{g}\|_{H^\ell(\Sigma)} \\ &\lesssim (\|\phi \hat{f}\|_{H^\ell(\Sigma)} + \|f\|_{L^p(\mathbb{R}^n)}) (\|\phi \hat{g}\|_{H^\ell(\Sigma)} + \|g\|_{L^p(\mathbb{R}^n)}). \end{aligned}$$

Thus, we can conclude that

$$|\langle T(f)\phi, T(g)\phi \rangle_{L^2_\xi L^2_\eta}| \lesssim (\|\phi \hat{f}\|_{H^\ell(\Sigma)} + \|f\|_{L^p(\mathbb{R}^n)}) (\|\phi \hat{g}\|_{H^\ell(\Sigma)} + \|g\|_{L^p(\mathbb{R}^n)}).$$

One can show the case $p = 1$ similarly.

Remark. The *a priori* bound for Tf is $\|\phi(\xi)(Tf)(\xi, \eta)\|_{\dot{H}^{-\gamma}_{\xi} H^{\tau}_{\eta}} \leq C \|f\|_{L^p(\mathbb{R}^n)}$ with $\frac{1+2\tau}{2} \leq \gamma < \frac{n-1}{2}$, $1 < p < \min\{\frac{2n}{n+2+2\tau}, \frac{2n+2}{n+5-2\gamma+4\tau}\}$ or $p = \frac{2n+2}{n+5-2\gamma+4\tau}$. In particular, if $\gamma = \kappa_p$ and $\tau = \kappa_p - \frac{1}{2}$, then

$$\|\phi(\xi)(Tf)(\xi, \eta)\|_{H^{-\kappa_p}_{\xi} H^{\kappa_p-\frac{1}{2}}_{\eta}} \leq C \|f\|_{L^p(\mathbb{R}^n)}$$

with $p \in (1, \frac{2(n+1)}{n+4}]$ as $\tau \geq 0$.

5 A version of T

Let us consider the operator

$$T_\alpha f(\xi, \eta) = \frac{\hat{f}(\xi, h(\xi) + \eta) - \hat{f}(\xi, h(\xi))}{|\eta|^\alpha}.$$

We can express $T_\alpha f$ in terms of Tf . More precisely, we have

$$T_\alpha f(\xi, \eta) = \frac{\eta}{|\eta|^\alpha} Tf(\xi, \eta).$$

When $\alpha = 1$, it follows immediately from Corollary 1 that

$$\|T_1 f\|_{L^2_\xi L^2_\eta} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)}.$$

The following examples show that it is impossible to have $L^2_\xi L^2_\eta$ bound for $|\alpha - 1| \geq \frac{1}{2}$.

Example. Let $h(\xi) = |\xi|^2$. Consider $f(x) = e^{-|x|^\alpha}$ with $\alpha \geq \frac{3}{2}$. We will show that it is impossible to obtain $L_\xi^2 L_\eta^2$. Note that

$$T_\alpha f(\xi, \eta) = \frac{e^{-|\xi|^2 - (|\xi|^2 + \eta)^2} - e^{-|\xi|^2 - |\xi|^4}}{|\eta|^\alpha} = e^{-|\xi|^2 - |\xi|^4} \frac{e^{-2\eta|\xi|^2 - \eta^2} - 1}{|\eta|^\alpha}$$

When $|\eta| < \frac{|\xi|}{1000}$, then

$$|T_\alpha f(\xi, \eta)| \geq e^{-|\xi|^2 - |\xi|^4} \frac{c|\xi|^2 e^{-c'|\xi|^4}}{|\eta|^{\alpha-1}} \quad (5.1)$$

for some $c, c' > 0$. When we square (5.1) both sides and integrate it with respect to $\eta \in (-\frac{|\xi|}{1000}, \frac{|\xi|}{1000})$, the integral will diverge because $2\alpha - 2 \in [1, \infty)$.

On the other hand, let us consider $\alpha \leq \frac{1}{2}$ with the same function f . If η is sufficiently large so that $e^{-2\eta|\xi|^2 - \eta^2} \leq \frac{1}{2}$, then

$$|T_\alpha f(\xi, \eta)| \geq e^{-|\xi|^2 - |\xi|^4} \frac{1}{2|\eta|^\alpha} \quad (5.2)$$

After squaring and integrating (5.2) with respect to η in that range, this integral will diverge as $2\alpha \leq 1$.

Now we focus on the case $\alpha \in (\frac{1}{2}, \frac{3}{2})$. Let us compute the kernel of $T_\alpha^* T_\alpha$ on $L_\xi^2 L_\tau^2$.

$$\langle T_\alpha(f)(\xi, \eta)\phi(\xi), T_\alpha(g)(\xi, \eta)\phi(\xi) \rangle_{L_\xi^2 L_\eta^2} = \int_{\mathbb{R}^n \times \mathbb{R}^n} K_{0,0}^\alpha(x, y) f(x) g(y) dx dy,$$

where

$$K_{0,0}^\alpha(x, y) = I(x_n, y_n) \widetilde{\phi}_{0,\Sigma}^2(x, y)$$

and

$$I(x_n, y_n) = \int_{\mathbb{R}} \frac{(e^{-2\pi i x_n \eta} - 1)(e^{2\pi i y_n \eta} - 1)}{|\eta|^{2\alpha}} d\eta.$$

By considering the principal value of $I(x_n, y_n)$ and using change of variables,

$$\begin{aligned} I(x_n, y_n) &= \int_{\mathbb{R}} \frac{\cos(2\pi(x_n - y_n)\eta) - \cos(2\pi y_n \eta) - \cos(2\pi x_n \eta) + 1}{|\eta|^{2\alpha}} d\eta \\ &= C_\alpha [|x_n|^{2\alpha-2} x_n + |y_n|^{2\alpha-2} y_n - |x_n - y_n|^{2\alpha-2} (x_n - y_n)], \end{aligned}$$

where $C_\alpha = \int_{\mathbb{R}} \frac{1 - \cos(2\pi t)}{|t|^{2\alpha}} dt < \infty$ as $\alpha \in (\frac{1}{2}, \frac{3}{2})$.

Therefore, we can write

$$\begin{aligned} &\langle T_\alpha(f)(\xi, \eta)\phi(\xi), T_\alpha(g)(\xi, \eta)\phi(\xi) \rangle_{L_\xi^2 L_\eta^2} \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} C_\alpha [|x_n|^{2\alpha-2} x_n + |y_n|^{2\alpha-2} y_n] \widetilde{\phi}_{0,\Sigma}^2(x, y) f(x) g(y) dx dy \\ &\quad - C_\alpha \int_{\mathbb{R}^n \times \mathbb{R}^n} [|x_n - y_n|^{2\alpha-2} (x_n - y_n)] \widetilde{\phi}_{0,\Sigma}^2(x, y) f(x) g(y) dx dy \\ &=: (I) + (II). \end{aligned}$$

Using the Hardy-Littlewood-Sobolev inequality, for $1 \leq p \leq \frac{2n+2}{n+4\alpha+1}$, we have

$$|(II)| \lesssim \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)}.$$

To estimate (I), it suffices to consider the case for x_n . Indeed, we can write

$$\int_{\mathbb{R}^n} |x_n|^{2\alpha-2} x_n f(x) g(y) dy \simeq \langle S_{w(r)r^{2\alpha-2}} f \phi, \hat{g} \phi \rangle_{L^2(\Sigma)},$$

where $w(r) = i(\operatorname{sgn}(r))^{2\alpha-3}$. Using Equation (3.3) with $\beta = \alpha - \frac{1}{2} \in (\frac{1}{2}, 1)$, we have

$$\left| \int_{\mathbb{R}^n} |x_n|^{2\alpha-2} x_n f(x) g(y) dy \right| \lesssim (\|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)}) \|\phi \hat{g}\|_{H^\ell(\Sigma)},$$

for the case $\ell \geq \frac{\kappa_p(2\alpha-1)}{2\kappa_p-2\alpha+1}$ and $1 < p \leq \frac{2n+2}{n+4\alpha+1}$ or the case $\ell > \frac{\kappa_p(2\alpha-1)}{2\kappa_p-2\alpha+1}$ and $p = 1$.

In summary, we have

Theorem 8. *Let $\alpha \in (\frac{1}{2}, \frac{3}{2})$. Suppose $f \in L^p(\mathbb{R}^n)$ has $\hat{f}|_\Sigma \in H^\ell(\Sigma)$, and $\phi \in C_c^\infty(\Sigma)$. If $\ell \geq \frac{\kappa_p(2\alpha-1)}{2\kappa_p-2\alpha+1}$ and $1 < p \leq \frac{2n+2}{n+4\alpha+1}$, then*

$$\|\phi T_\alpha f\|_{L_\xi^2 L_\eta^2} \lesssim \|f\|_{L^p(\mathbb{R}^n)} + \|\phi \hat{f}\|_{H^\ell(\Sigma)}.$$

The estimate is also true for $p = 1$ if $\ell > \frac{\kappa_1(2\alpha-1)}{2\kappa_1-2\alpha+1} = \frac{(n-1)(2\alpha-1)}{2(n-2\alpha)}$.

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