# COUNTEREXAMPLES TO $L^P$ BOUNDEDNESS OF WAVE OPERATORS FOR CLASSICAL AND HIGHER ORDER SCHRÖDINGER OPERATORS

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ABSTRACT. We consider the higher order Schrödinger operator $H = (-\Delta)^m + V(x)$ in $n$ dimensions with real-valued potential $V$ when $n > 4m-1$, $m \in \mathbb{N}$. We show that for any $\frac{2n}{n-4m+1} < p \leq \infty$ and $0 \leq \alpha < \frac{n+1}{2} - 2m - \frac{n}{p}$, there exists a real-valued, compactly supported potential $V \in C^\alpha(\mathbb{R}^n)$ for which the wave operators $W^\pm$ are not bounded on $L^p(\mathbb{R}^n)$. As a consequence of our analysis we show that the wave operators for the usual second order Schrödinger operator $-\Delta + V$ are unbounded on $L^p(\mathbb{R}^n)$ for $n > 3$ and $\frac{2n}{n-3} < p \leq \infty$ for insufficiently differentiable potentials $V$, and show a failure of $L^{p'} \to L^p$ dispersive estimates that may be of independent interest.

## 1. Introduction

We consider the higher order Schrödinger equation

$$i\psi_t = (-\Delta)^m \psi + V\psi, \qquad x \in \mathbb{R}^n, \quad m \in \mathbb{N},$$

with a real-valued and decaying potential $V$. We denote the free higher order Schrödinger operator by $H_0 = (-\Delta)^m$ and the perturbed operator by $H = (-\Delta)^m + V(x)$. We study the wave operators, which are defined by

$$W_\pm = s - \lim_{t \to \pm\infty} e^{itH} e^{-itH_0}.$$

Recent work by the first and third authors, [3], showed that for $m > 1$ and $n > 2m$ the wave operators extend to bounded operators on $L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$ for sufficiently smooth small potentials. The case $m = 1$ was established by Yajima in [16]. Here we show that when $n > 4m - 1$ and $\frac{2n}{n-4m+1} < p \leq \infty$ the $L^p$ boundedness of the wave operators may fail even for compactly supported continuous potentials if the potential is not sufficiently smooth. Our main result is the following.

**Theorem 1.1.** *Fix $m \in \mathbb{N}$, let $n > 4m - 1$ and $\frac{2n}{n-4m+1} < p \leq \infty$, for all $0 \leq \alpha < \frac{n+1}{2} - 2m - \frac{n}{p}$ there exists a real-valued compactly supported potential of class $C^\alpha(\mathbb{R}^n)$ for which the wave operators $W_\pm$ are not bounded on $L^p(\mathbb{R}^n)$.*

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The first author was partially supported by the NSF grant DMS-2154031 and Simons Foundation Grant 634269. The second author is partially supported by Simons Foundation Grant 635369. The third author is partially supported by Simons Foundation Grant 511825.

For the convenience of the reader, in Section 3 we prove Theorem 1.1 when $p=\infty$, then adapt the argument in Section 4 to the remaining cases of $\frac{2n}{n-4m+1}<p<\infty$. By considering frequency-localized dispersive estimates, we provide a direct argument for arbitrary integer order $m$ that simplifies the $m=1$ argument for the dispersive bounds in [6] and extends to $m\geq 1$.

For comparison, the results on boundedness in [3] require some smoothness on the potential when $n\geq 4m-1$. Writing $\langle x\rangle$ to denote $(1+|x|^2)^{\frac12}$, $\mathcal{F}(f)$ or $\widehat{f}$ to denote the Fourier transform of $f$ and defining the norm $\|f\|_{H^\delta}=\|\langle\cdot\rangle^\delta\widehat{f}(\cdot)\|_2$, we recall the relevant statements below.

**Theorem 1.2** (Theorem 1.1 in [3])**.** *Let $n>2m$. Assume that the $V$ is a real-valued potential on $\mathbb{R}^n$ and fix $0<\delta\ll 1$. Then $\exists C=C(\delta,n,m)>0$ so that the wave operators extend to bounded operators on $L^p(\mathbb{R}^n)$ for all $1\leq p\leq\infty$, provided that*

*i)* $\big\|\langle\cdot\rangle^{\frac{4m+1-n}{2}+\delta}V(\cdot)\big\|_2<C$ *when* $2m<n<4m-1$,

*ii)* $\big\|\langle\cdot\rangle^{1+\delta}V(\cdot)\big\|_{H^\delta}<C$ *when* $n=4m-1$,

*iii)* $\big\|\mathcal{F}(\langle\cdot\rangle^{\sigma}V(\cdot))\big\|_{L^{\frac{n-1-\delta}{n-2m-\delta}}}<C$ *for some* $\sigma>\frac{2n-4m}{n-1-\delta}+\delta$ *when* $n>4m-1$.

See [3] for other statements that remove the smallness requriement on the potential. The arbitrarily small differentiability assumption when $n=4m-1$ appears to be an artifact of the method in [3]. We note that the norm used when $n>4m-1$ is finite when $\langle x\rangle^\sigma V(x)$ has more than $\frac{n}{n-1}(\frac{n+1}{2}-2m)$ derivatives in $L^2(\mathbb{R}^n)$. We believe this requirement on smoothness of the potential is not sharp, in light of the counterexample constructed here. When $m=1$ the counterexample in [6] and the positive result of the first and third authors in [2] show that $\alpha=\frac{n-3}{2}$ is sharp for $L^1\to L^\infty$ dispersive estimates, at least when $n=5,7$. In effect, our result shows that one cannot expect decay of a class of potentials alone suffice to ensure the boundedness of the wave operators as is the case in lower dimensions, when $2m<n<4m-1$ above. Instead, one also expects a degree of differentiability on the potential is needed to ensure boundedness in high dimensions.

The potentials considered here also suffice to imply, see for example [13, 1, 14], the existence, $L^2$-boundedness, asymptotic completeness, and intertwining identity for the wave operators. In particular, we have the identity

$$f(H)P_{ac}(H)=W_{\pm}f((-\Delta)^m)W_{\pm}^*. \tag{1}$$

Here $P_{ac}(H)$ is the projection onto the absolutely continuous spectral subspace of $H$, and $f$ is any Borel function. One use of (1) is to obtain $L^p$-based mapping properties for the perturbed operator $f(H)P_{ac}(H)$ from the simpler free operator $f((-\Delta)^m)$. The boundedness of the wave operators on $L^p(\mathbb{R}^n)$ for any choice of $p\geq 2$ with the function $f(\cdot)=e^{-it(\cdot)}$ yield dispersive estimates of the form

$$\|e^{-itH}P_{ac}(H)\|_{L^{p'}\to L^p}\lesssim |t|^{-\frac{n}{2m}+\frac{n}{pm}},\qquad \frac1p+\frac1{p'}=1. \tag{2}$$

To establish our results, we appeal to the intertwining identity and show that certain dispersive estimates fail for a specific choice of $f(\cdot)$ for small time when localized to large frequencies.

We note that our results include the case of $m = 1$, the usual second order Schrödinger operator in dimensions $n > 3$. The results of Theorem 1.1 are, to the best of the authors' knowledge, new even in this case. In particular showing that the wave operators need not be bounded on $\frac{2n}{n-3} < p \leq \infty$ for insufficiently smooth potentials. We also show that the $L^p \to L^{p'}$ dispersive estimates

$$\left\| e^{it(-\Delta+V)} P_{ac}(-\Delta + V) \right\|_{p\to p'} \lesssim |t|^{-\frac{n}{2}(\frac{2}{p}-1)}, \quad \text{when } n > 3 \text{ and } 1 \leq p < \frac{2n}{n+3}$$

fail in this case, see Corollary 4.1 below.

Our work is inspired by recent work by the first and third authors on the $L^p$-continuity of the higher order wave operators in [3] and the counterexample to dispersive estimates in the $m = 1$ case of the second author and Visan in [6]. The recent work on the $L^p$-boundedness of the wave operators for higher order Schrödinger operators was sparked by the work of Feng, Soffer, Wu and Yao on weighted $L^2$-based "local dispersive estimates" for higher order Schrödinger operators considered in [4], which extends the $m = 1$ result of Jensen [8]. In addition, the recent work on the $L^p(\mathbb{R}^3)$ boundedness of the wave operators for the fourth order ($m = 2$) Schrödinger operators by the second and third authors [5], and the extensive works in the case of $m = 1$ most notably that of Yajima, [16, 17, 18]. The $L^p(\mathbb{R})$ boundedness has recently been investigated by Mizutani, Wan and Yao in [12]. The $L^2$ existence and other properties of the higher order wave operators have been studied by many authors, for example by Agmon [1], Kuroda [10, 11], Hörmander [7], and Schechter, [13, 14].

Similar to the usual second order Schrödinger operator, there is a Weyl criterion and $\sigma_{ac}(H) = \sigma_{ac}(H_0) = [0, \infty)$ for sufficiently decaying potentials. In contrast, decay of the potential is not sufficient to ensure the lack of eigenvalues embedded in the continuous spectrum for the higher order operators, [4]. Even perturbing with compactly supported, smooth potentials may induce embedded eigenvalues. For the potentials we consider, we show that the eigenvalues cannot be too large, which allows the use of a limiting absorption principle for the perturbed operator on the interval $[C_V, \infty)$ for a sufficiently large $C_V > 0$, see Lemma 3.2 below.

To prove Theorem 1.1 we show the failure of a dispersive estimate based on detailed analysis of oscillatory integrals involving the resolvent operators. The splitting identity for $z \in \mathbb{C} \setminus [0, \infty)$, (c.f. [4]) allows us to study the resolvent $((-\Delta)^m - \lambda)^{-1}$ in terms of $R_0(z) = (-\Delta - z)^{-1}$, the usual ($2^{nd}$ order) Schrödinger resolvent.

$$\mathcal{R}_0(z)(x, y) := ((-\Delta)^m - z)^{-1}(x, y) = \frac{1}{mz^{1-\frac{1}{m}}} \sum_{\ell=0}^{m-1} \omega_\ell R_0(\omega_\ell z^{\frac{1}{m}})(x, y) \tag{3}$$

here $\omega_\ell = \exp(i2\pi\ell/m)$ are the $m^{th}$ roots of unity. It is convenient to use the change of variables $z = \lambda^{2m}$ with $\lambda$ restricted to the sector in the complex plane with $0 < \arg(\lambda) < \pi/m$,

$$\mathcal{R}_0(\lambda^{2m})(x,y) := ((-\Delta)^m - \lambda^{2m})^{-1}(x,y) = \frac{1}{m\lambda^{2m-2}} \sum_{\ell=0}^{m-1} \omega_\ell R_0(\omega_\ell \lambda^2)(x,y). \tag{4}$$

The kernal of higher order Schrödinger resolvents $\mathcal{R}_0(\lambda^{2m})$ grow like $\lambda^{\frac{n+1}{2}-2m}$ in the spectral variable when $n > 4m-1$, this growth necessitates a control over derivatives of the potential which was controlled in terms of $\mathcal{F}L^r$ norms in [3], similar to the conditions for the second order Schrödinger established by Yajima, [16]. When $m = 1$ this growth may be exploited to cause a failure of $L^1 \to L^\infty$ dispersive estimates in dimensions greater than three without sufficient smoothness of the potential; a counterexample was constructed by the second author and Visan, [6]. We adapt and simplify this argument by considering $L^1 \to L^\infty$ estimates of the operators of the form $H^{\frac{n(m-1)}{2m}} e^{itH} \psi(H/L^{2m}) P_{ac}(H)$, with $\psi(s)$ is a cut-off to frequencies of size $s \approx 1$ and $L$ is a sufficiently large constant. Treating this operator as an element of the functional calculus, the modified Stone's formula is

$$H^{\frac{n(m-1)}{2m}} e^{itH} \psi(H/L^{2m}) P_{ac}(H) f(x) = \frac{1}{2\pi i} \int_0^\infty \lambda^{\frac{n(m-1)}{2m}} e^{it\lambda} \psi(\lambda/L^{2m})\, dE_{ac}(\lambda) f(x).$$

Here the difference of the limiting resolvent operators, $\mathcal{R}_V^{\pm}(\lambda) = \lim_{\epsilon \to 0^+}((-\Delta)^m + V - \lambda \mp i\epsilon)^{-1}$, provides the spectral measure $dE_{ac}(\lambda) = [\mathcal{R}_V^+(\lambda) - \mathcal{R}_V^-(\lambda)]\, d\lambda$. These operators are well-defined between weighted spaces by Agmon's limiting absorption principle, and are well studied in [3]. By appropriately relating the frequency and time, we show a dispersive estimate that the free operator (when $V = 0$) satisfies cannot hold for the perturbed operator as $t \to 0$. Using the intertwining identity, (1), we show the wave operators are not bounded on $L^\infty$. By appropriately rescaling both the powers of $H$ and the time decay, we extend the argument to $p < \infty$.

The presence of an operator $H^{\frac{n(m-1)}{2m}}$ in the dispersive estimate is natural in the following sense: For $m > 1$, the free fundamental solution $e^{itH_0}(x,y)$ has a central peak where $|x-y| \lesssim |t|^{\frac{1}{2m}}$, then experiences a combination of polynomial decay and oscillation (similar to an Airy function) for larger distances $|x-y|$. Differentiating $n(m-1)$ times yields an operator whose kernel is approximately the same size for all values of $|x-y|$, so the $L^1 \to L^\infty$ bound can be achieved at every length scale or when localized to any frequency band.

The paper is organized as follows. In Section 2 we show that the free operator satisfies a family of dispersive estimates. In Section 3 we show that the perturbed operator cannot satisfy the same $L^1 \to L^\infty$ dispersive estimate as the free operator for a specifically constructed potential. As a consequence, the wave operator is not bounded on $L^\infty$. Finally, in Section 4 we show how the argument may be adapted to construct a class of potentials for which the wave operators are unbounded on the larger range of $\frac{2n}{n-4m+1} < p \leq \infty$.

## 2. The free estimate

In this section we establish a family of dispersive estimates that the free operator $H_0 = (-\Delta)^m$ satisfies. In Section 3 we construct a compactly supported potential for which the perturbed operator cannot satisfy the analogous bound.

**Proposition 2.1.**

$$\big\| H_0^{\frac{n(m-1)}{2m}} e^{itH_0} \big\|_{1\to\infty} \lesssim |t|^{-\frac{n}{2}},$$

*and for all* $\sigma \in \mathbb{R}$*,*

$$\big\| H_0^{\frac{n(m-1)+i\sigma}{2m}} e^{itH_0} \big\|_{1\to\infty} \lesssim (1+|\sigma|)^{\frac{n+2}{2}} |t|^{-\frac{n}{2}}.$$

*Proof.* Using the splitting identity (4), along with the fact that $R_0^+(\omega_\ell \lambda^2) = R_0^-(\omega_\ell\lambda^2)$ for $\omega_\ell \notin \mathbb{R}$, that is when $\omega_\ell \neq 1$, we have

$$[\mathcal{R}_0^+ - \mathcal{R}_0^-](\lambda^{2m})(x,y) = \frac{1}{m\lambda^{2m-2}} [R_0^+ - R_0^-](\lambda^2)(x,y). \tag{5}$$

Hence, one need only understand the usual second order free resolvent to understand the estimates in Propostion 2.1. As usual, we estimate the evolution by using the functional calculus, in this case, we'll estimate (ignoring constants)

$$\sup_{x,y\in\mathbb{R}^n} \bigg| \int_0^\infty e^{it\lambda^{2m}} \lambda^{n(m-1)+1+i\sigma} [R_0^+ - R_0^-](\lambda^2)(x,y)\, d\lambda \bigg|. \tag{6}$$

We utilize the "symbol class" representation for the Schrödinger operators used by the second author and Visan in [6] to write

$$[R_0^+ - R_0^-](\lambda^2)(x,y) = \lambda^{n-2}\big(e^{i\lambda|x-y|} A_1(\lambda|x-y|) - e^{-i\lambda|x-y|} A_2(\lambda|x-y|)\big),$$

where $A_1(s), A_2(s) = \widetilde{O}(\langle s\rangle^{\frac{1-n}{2}})$. Here $f(s) = \widetilde{O}(\langle s\rangle^j)$ means $|\partial_s^k f(s)| \lesssim \langle s\rangle^{j-k}$ for each $k = 0,1,2,\dots$

We only consider the case $t > 0$ and when there is a '$-$' sign on the phase; the other cases are similar or easier. Therefore, we only prove that

$$\sup_{x,y\in\mathbb{R}^n, L>0, t>0} t^{n/2} \bigg| \int_0^\infty e^{it\lambda^{2m} - i\lambda|x-y|} \lambda^{nm-1+i\sigma} A_2(\lambda|x-y|) \chi(\lambda/L)\, d\lambda \bigg| \lesssim 1.$$

Only the function $A_2(\lambda|x-y|)$ is present, so we omit the subscript in the rest of the calculation. Changing the variable $\lambda|x-y| \mapsto \lambda$ and appropriately renaming the variables $t \mapsto t|x-y|^{2m}$, $L|x-y| \mapsto L$, it suffices to prove that

$$\sup_{t>0, L>0} t^{n/2} \bigg| \int_0^\infty e^{it\lambda^{2m} - i\lambda} \lambda^{nm-1+i\sigma} A(\lambda) \chi(\lambda/L)\, d\lambda \bigg| \lesssim 1. \tag{7}$$

Note that the phase $\phi(\lambda) = t\lambda^{2m} - \lambda$ has a critical point at $\lambda_0 = (2mt)^{-\frac{1}{2m-1}}$. Let $\psi_1$ be a smooth cutoff for the set $|\lambda| \approx \lambda_0$, $\psi_2$ for the set $|\lambda| \ll \lambda_0$ and $\psi_3$ for the set $|\lambda| \gg \lambda_0$ so that $\psi_1+\psi_2+\psi_3 \equiv 1$. Let

$$I_j := \int_0^\infty e^{it\lambda^{2m}-i\lambda}\lambda^{nm-1+i\sigma}A(\lambda)\psi_j(\lambda)\chi(\lambda/L)\,d\lambda, \quad j=1,2,3.$$

Note that in the support of $\psi_1(\lambda)$, we have $\phi''(\lambda) \approx t\lambda_0^{2m-2}$. Therefore by the corollary on page 334 of [15], we have

$$\begin{aligned}|I_1| &\lesssim (t\lambda_0^{2m-2})^{-1/2}\int \big|\partial_\lambda\big(\lambda^{nm-1+i\sigma}A(\lambda)\psi_1(\lambda)\chi(\lambda/L)\big)\big|d\lambda \\ &\lesssim t^{-1/2}\langle\sigma\rangle\lambda_0^{1-m+nm-1}\langle\lambda_0\rangle^{\frac{1-n}{2}} \lesssim t^{-1/2}\langle\sigma\rangle\lambda_0^{(2m-1)(n-1)/2} \approx \langle\sigma\rangle t^{-n/2}.\end{aligned}$$

Note that on the support of $\psi_2$, where $|\lambda| \ll \lambda_0$, we have $|\phi'| \gtrsim 1$. Therefore we may integrate by parts $N \le nm-1$ times to obtain the bound

$$|I_2| \lesssim \int_0^\infty \Big|\Big[\partial_\lambda\frac{1}{\phi'(\lambda)}\Big]^N\big(\lambda^{nm-1+i\sigma}A(\lambda)\psi_2(\lambda)\chi(\lambda/L)\big)\Big|d\lambda.$$

Noting that the effect of a derivative on each term can be bounded by multiplication by $\langle\sigma\rangle\lambda^{-1}$, we have

$$|I_2| \lesssim \langle\sigma\rangle^N\int_0^{\lambda_0}\lambda^{nm-1-N}d\lambda \lesssim \langle\sigma\rangle^N\lambda_0^{nm-N} \lesssim \langle\sigma\rangle^N t^{-\frac{nm-N}{2m-1}}, \;\; 0 \le N \le nm-1.$$

Picking $N=0$ when $t>1$ and $\frac{n}{2} \le N \le nm-1$ for $0<t<1$ yields stronger bounds than $t^{-n/2}$.

It remains to consider $I_3$. On the support of $\psi_3$, we have $|\phi'| \gtrsim t\lambda^{2m-1} \gtrsim 1$. Therefore by integration by parts $N$ times we obtain

$$|I_3| \lesssim \langle\sigma\rangle^N\int_{\lambda_0}^\infty \lambda^{nm-1-N}\langle\lambda\rangle^{\frac{1-n}{2}}[t\lambda^{2m-1}]^{-N}d\lambda = \langle\sigma\rangle^N t^{-N}\int_{\lambda_0}^\infty\lambda^{nm-1-2mN}\langle\lambda\rangle^{\frac{1-n}{2}}d\lambda.$$

When $t<1$, $\lambda_0 \gtrsim 1$ and taking $N > \frac{nm+\frac{1}{2}-\frac{n}{2}}{2m}$ implies the stronger bound $\langle\sigma\rangle^N t^{-\frac{n}{2}}t^{\frac{N-\frac{1}{2}}{2m-1}}$. For $t>1$, take $N=0$ on $[\lambda_0, t^{-\frac{1}{2m}}]$, and take $N>n/2$ on $[t^{-\frac{1}{2m}},\infty)$ to obtain

$$\lesssim \int_0^{t^{-\frac{1}{2m}}}\lambda^{nm-1}d\lambda + \langle\sigma\rangle^N t^{-N}\int_{t^{-\frac{1}{2m}}}^\infty \lambda^{nm-1-2mN}d\lambda \lesssim t^{-n/2} + \langle\sigma\rangle^N t^{-N}t^{-\frac{1}{2m}(nm-2mN)} \lesssim \langle\sigma\rangle^N t^{-n/2}.$$

At no point was integration by parts needed more than $\frac{n}{2}+1$ times, so the sum $I_1+I_2+I_3$ is bounded by $\langle\sigma\rangle^{\frac{n+2}{2}}t^{-n/2}$. □

The above bound is all we need in Section 3 to establish the result of Theorem 1.1 when $p=\infty$. For the full range of $p$ considered in Section 4, we need the following corollary.

**Corollary 2.2.** *Let $\psi$ be a smooth cutoff for the set $\{\lambda \in \mathbb{R} : \lambda \approx 1\}$. For all $p \in [1,2]$ and $L>0$ we have*

$$\|H_0^{\frac{n(m-1)}{2m}(\frac{2}{p}-1)}e^{itH_0}\psi(H_0/L)\|_{p\to p'} \lesssim |t|^{-\frac{n}{p}+\frac{n}{2}},$$

*where the implict constant is independent of $p$ and $L$.*

*Proof.* The claim without $\psi$ follows from complex interpolation of the bound in Proposition 2.1 with the $L^2$ conservation law. The claim with $\psi$ follows from this and the fact that $\widehat{\psi} \in L^1$. □

## 3. Failure of the dispersive estimate

In this section we show that the perturbed evolution cannot satisfy the same dispersive bound as the free evolution. For clarity, we concentrate in this section on the failure of the $L^1 \to L^\infty$ dispersive estimate and consequently the unboundedness of the wave operator on $L^\infty$. In Section 4, we show how to adapt this argument to a larger range of $p$.

Let $\psi$ be a smooth cutoff for frequencies $\lambda \approx 1$. Note that the $L^\infty$ boundedness of wave operators together with asymptotic completeness, the intertwining identity (1) and the bound in Corollary 2.2 imply that the bound

$$\big\| H^{\frac{n(m-1)}{2m}} e^{itH} \psi(H/L^{2m}) P_{ac}(H) \big\|_{L^1 \to L^\infty} \lesssim |t|^{-n/2} \tag{8}$$

holds for all $L > 0$ uniformly in $L$.

Let $C^\alpha(B(0,2))$ be the Banach space of real-valued $C^\alpha$ functions supported in $B(0,2)$. We prove that the inequality

$$\sup_{t>0, L>0} t^{n/2} \big\| H^{\frac{n(m-1)}{2m}} e^{itH} \psi(H/L^{2m}) P_{ac}(H) \big\|_{L^1 \to L^\infty} \leq C_V \tag{9}$$

cannot hold for all $V \in C^\alpha(B(0,2))$ if $0 \leq \alpha < \frac{n+1}{2} - 2m$. By showing that this dispersive bound fails for insufficiently smooth potentials $V$ for small times $t \to 0$, we show that the wave operators are not bounded on $L^\infty(\mathbb{R}^n)$.

To do so we iterate the resolvent identity to expand the perturbed resolvent into a Born series

$$\mathcal{R}_V(z) = \sum_{j=0}^{2M-1} \big[ \mathcal{R}_0(z)(-V\mathcal{R}_0(z))^j \big] - (\mathcal{R}_0(z)V)^M \mathcal{R}_V(z) (V\mathcal{R}_0(z))^M. \tag{10}$$

We note that the $j = 0$ term corresponds to the free evolution. In the following subsections we first show that the first term of the Born series, when $j = 1$, for large frequencies and small times does not satisfy the dispersive bound. In the following subsection, we show that the final term in the identity (the "tail") obeys the dispersive bound. Finally, we use these facts to show that the full evolution can't satisfy the dispersive bound.

### 3.1. **The first term of the Born series.**

First we consider the first term of the Born series of the operator in (9). Ignoring the constants, we write it's kernel, $K_{L,t}$, as a difference of kernels

$$K^{\pm}_{L,t}(x,y) = \int_0^\infty \int_{\mathbb{R}^n} e^{it\lambda^{2m}} \lambda^{2m-1} \lambda^{n(m-1)} \mathcal{R}_0^{\pm}(\lambda^{2m})(x,z) V(z) \mathcal{R}_0^{\pm}(\lambda^{2m})(z,y) \psi(\lambda^{2m}/L^{2m}) dz d\lambda$$

Let $f$ be the $L^1$ normalized characteristic function of $B(0,1)$, and let $f_L(x) = (CL)^n f(x(CL))$. Below $C$ will be chosen to be large to guarantee that the relevant values of $|x|$ and $|y|$ are $\ll 1$. In what follows we prove that the dispersive estimate fails by letting $t = L^{-(2m-1)}$ and taking $L \to \infty$. With $t^{\frac{n}{2}} = L^{\frac{n}{2}-nm}$, we define

$$a_{1,L}(V) = L^{\frac{n}{2}-nm} \int_{\mathbb{R}^n \times \mathbb{R}^n} K_{L,L^{-(2m-1)}}(x,y) f_L(x) f_L(y)\, dx dy.$$

Note that $a_{1,L}$ is a linear operator on $C^\alpha(B(0,2))$. By uniform boundedness principle, if the bound in (9) holds for the first term of the Born series and for all $V \in C^\alpha(B(0,2))$, then we have

$$\forall V \in C^\alpha(B(0,2)), \quad \sup_{L>0} |a_{1,L}(V)| \leq C_\alpha \|V\|_{C^\alpha(B(0,2))}. \tag{11}$$

Therefore, it suffices to find a sequence $\{V_L\}_{L>1} \subset C^\alpha(B(0,2))$ so that

$$\lim_{L\to\infty} \frac{|a_{1,L}(V_L)|}{\|V_L\|_{C^\alpha(B(0,2))}} = \infty.$$

Note that for $\lambda|x-z| > 1$, we have

$$R_0^\pm(\lambda^2)(x,z) = e^{\pm i\lambda|x-z|} \lambda^{n-2} \omega^\pm(\lambda|x-z|), \tag{12}$$

where $\omega^\pm(s) = c_\pm s^{\frac{1-n}{2}} + \widetilde{O}(s^{-\frac{1+n}{2}})$, $s > 1$. In our argument below, we have $\lambda \approx L$ large and $|x-z| \approx 1$, which allows us to avoid the logarithmic behavior of even dimensional resolvents when $\lambda|x-z| \ll 1$. Using this in the splitting formula for $\mathcal{R}_0^\pm(\lambda^{2m})$, (4), and noting the exponential decay of the remaining terms in the splitting identity allows them to be absorbed into $\omega^\pm(\lambda|x-z|)$, we have for $\lambda|x-z| > 1$

$$\mathcal{R}_0^\pm(\lambda^{2m})(x,z) = e^{\pm i\lambda|x-z|} \lambda^{n-2m} \omega^\pm(\lambda|x-z|).$$

Here, by a slight abuse of notation, $\omega^\pm$ satisfies the same bounds as $\omega^\pm$ in (12). One can also see Lemmas 3.2 and 6.2 in [3] for more detailed representations of the resolvent. Using this in the formula for $K_{L,t}^\pm(x,y)$, we have

$$\begin{aligned} K_L^\pm(x,y) &:= K^\pm_{L,L^{-(2m-1)}} \\ &= \int_0^\infty \int_{\mathbb{R}^n} e^{i\frac{\lambda^{2m}}{L^{2m-1}} \pm i\lambda R} \lambda^{mn+n-2m-1} \omega^\pm(\lambda r)\omega^\pm(\lambda s) V(z) \psi(\lambda^{2m}/L^{2m})\, dz d\lambda \end{aligned}$$

where $R := r + s := |x-z| + |z-y|$. Letting $\varphi(\lambda) = \lambda^{mn+n-2m-1}\psi(\lambda^{2m})$ be a modified cutoff to the interval $\lambda \approx 1$, we rewrite the kernel above as

$$L^{mn+n-2m-1} \int_0^\infty \int_{\mathbb{R}^n} e^{i\frac{\lambda^{2m}}{L^{2m-1}} \pm i\lambda R} \omega^\pm(\lambda r)\omega^\pm(\lambda s) V(z) \varphi(\lambda/L)\, dz d\lambda.$$

In what follows we have $r, s = 1 + o(1)$ and $R = 2 + o(1)$, this is accomplished by taking $|x|, |y|$ small and $V(z)$ supported on a sufficiently small neighborhood of $|z| = 1$. Changing the variable $\lambda \to L\lambda$ we have

$$K_L^\pm(x,y) = L^{mn+n-2m} \int_{\mathbb{R}^n} \int_0^\infty e^{iL(\lambda^{2m} \pm \lambda R)} \omega^\pm(\lambda L r)\omega^\pm(\lambda L s) V(z) \varphi(\lambda)\, dz d\lambda,$$

Note that the contribution of $+$ sign above is $O(L^{-N}\|V\|_{L^1})$ by nonstationary phase since $\varphi$ is smooth and supported on the set $\lambda \approx 1$ and $|\partial_\lambda^k \omega^\pm(\lambda L r)| \le C_k$ for all $k$. For the $-$ sign we have a critical point at $\lambda_0 = (R/2m)^{\frac{1}{2m-1}} \approx 1$, and hence by stationary phase we have

$$K_L^-(x,y) = C_m L^{mn+n-2m-\frac12} \int_{\mathbb{R}^n} \omega^-(\lambda_0 L r)\omega^-(\lambda_0 L s)\varphi(\lambda_0) e^{ic_m L(\frac{R}{2})^{\frac{2m}{2m-1}}} V(z)dz \\ + O(L^{mn-2m-\frac12}\|V\|_{L^1}),$$

where $c_m = (\frac{1}{m})^{\frac{2m}{2m-1}}(1-2m)$. For the error term, we used the bound $|\omega^\pm(\lambda)| \approx \langle\lambda\rangle^{\frac{1-n}{2}}$. Using the asymptotic expansion above for $\omega^\pm$ and slightly modifying $\varphi$, we can rewrite this as

$$K_L^-(x,y) = C_m L^{mn-2m+\frac12} \int_{\mathbb{R}^n} (rs)^{\frac{1-n}{2}} \varphi(\lambda_0) e^{ic_m L(\frac{R}{2})^{\frac{2m}{2m-1}}} V(z)dz + O(L^{mn-2m-\frac12}\|V\|_{L^1}),$$

Let

$$V_L(z) = \cos\big(c_m L|z|^{\frac{2m}{2m-1}}\big)\rho_\delta(z),$$

where $\delta \ll 1$, and $\rho_\delta$ is a smooth cutoff for the set $|z| \in (1-\delta, 1+\delta)$. We determine $\delta$ later. However, $\delta \ll 1$ is fixed and we later let $L \to \infty$. Therefore, for sufficiently large $L$,

$$\|V_L\|_{C^\alpha(B(0,2))} \approx \delta^{-\alpha} + L^\alpha \approx L^\alpha$$

For $x, y \in$supp$(f_L)$, i.e., $|x|, |y| \le \frac{1}{CL}$, we consider

$$K_L(x,y) = K_L^+(x,y) - K_L^-(x,y) = O(L^{mn-2m-\frac12}\delta) + \\ \frac{C_m}{2i} L^{mn-2m+\frac12} \int_{\mathbb{R}^n} (rs)^{\frac{1-n}{2}} \varphi(\lambda_0) e^{ic_m L[(\frac{R}{2})^{\frac{2m}{2m-1}} - |z|^{\frac{2m}{2m-1}}]} \rho_\delta(z)dz \\ + \frac{C_m}{2i} L^{mn-2m+\frac12} \int_{\mathbb{R}^n} (rs)^{\frac{1-n}{2}} \varphi(\lambda_0) e^{ic_m L[(\frac{R}{2})^{\frac{2m}{2m-1}} + |z|^{\frac{2m}{2m-1}}]} \rho_\delta(z)dz.$$

Note that by nonstationary phase, for all $N \in \mathbb{N}$, the last integral is $O(L^{mn-2m+\frac12-N}\delta^{1-N})$ since $|\nabla^k \rho_\delta| \lesssim \delta^{-k}$. Using this with $N = 1$, the error terms are combined to $O(L^{mn-2m-\frac12})$. On the other hand, since in the support of $f_L$

$$\big|\frac{R}{2} - |z|\big| \le \frac12(|x| + |y|) \le \frac{1}{CL},$$

by choosing $C$ in the definition of $f_L$ sufficiently large depending only on $m$, we see that the phase in the first integral is $o(1)$. Therefore, we have the following lower bound for the first integral:

$$L^{mn-2m+\frac12}\delta.$$

We can conclude that there is a constant $c \in \mathbb{C}$:

$$\Re\big[cK_L^+(x,y) - cK_L^-(x,y)\big] \ge L^{mn-2m+\frac12}\delta,$$

for $\delta \ll 1$ fixed, $L$ sufficiently large, and $|x|,|y| \leq \frac{1}{CL}$. Therefore, we have

$$\frac{|a_{1,L}(V_L)|}{\|V_L\|_{C^\alpha(B(0,2))}} \gtrsim L^{\frac{n+1}{2}-2m-\alpha}\delta \to \infty$$

as $L \to \infty$ unless $\alpha \geq \frac{n+1}{2} - 2m$.

3.2. **The Tail of the Born Series.** Here we establish that, upon sufficient iteration of the Born series, that the tail of the Born series is bounded as $t \to 0$. Specifically, we show the following.

**Proposition 3.1.** *Assume $V \in L^\infty(\mathbb{R}^n)$ is supported on $B(0,2)$, given any choice of $L > C_{V,n,m}$, where $0 < C_{V,n,m} < \infty$ is a constant depending on the size of the potential, spatial dimension and order of the operator, there exists an $M$ so that the tail of the Born series satisfies the following bound*

$$\sup_{x,y\in\mathbb{R}^n} \left| \int_0^\infty e^{it\lambda}\psi(\lambda/L^{2m})\lambda^{\frac{n(m-1)}{2m}}(\mathcal{R}_0^\pm(\lambda)V)^M \mathcal{R}_V^\pm(\lambda)(V\mathcal{R}_0^\pm(\lambda))^M(x,y)\, d\lambda \right| \lesssim 1.$$

We utilize the limiting absorption principle established in [4]. For fixed $m, n$, there are no positive eigenvalues of $H$ on the support of the cut-off provided we take $L$ sufficiently large, depending only on the size of $V$.

**Lemma 3.2.** *Fix $\gamma > \frac{1}{2}$. Assume that $|V(x)| \leq M\langle x\rangle^{-2\gamma}$, $x \in \mathbb{R}^n$. Then, there exists a constant $C = C(n,m,\gamma,M) > 0$ such that any $\lambda > C$ cannot be an eigenvalue of $H$.*

*Proof.* Assume that $\lambda > 0$ is an eigenvalue of $H$, then there exists $0 \neq \Psi \in L^2$ such that

$$((-\Delta)^m + V)\Psi = \lambda\Psi, \quad \Rightarrow \quad ((-\Delta)^m - \lambda - i\epsilon)\Psi = -i\epsilon\Psi - V\Psi.$$

Here $\epsilon > 0$, so that $\lambda + i\epsilon$ is in the resolvent set of $H_0$. Acting the resolvent operator $\mathcal{R}_0(\lambda + i\epsilon)$ on both sides of the above expression yields

$$\Psi = \mathcal{R}_0(\lambda + i\epsilon)[-i\epsilon\Psi - V\Psi].$$

Hence,

$$\|\Psi\|_{L^{2,-\gamma}} \leq \|\mathcal{R}_0(\lambda+i\epsilon)\epsilon\Psi\|_{L^2} + \|\mathcal{R}_0(\lambda+i\epsilon)V\Psi\|_{L^{2,-\gamma}} \lesssim \left\| \frac{\epsilon}{|\xi|^{2m}-\lambda-i\epsilon}\hat{\Psi} \right\|_{L^2_\xi} + \lambda^{\frac{1-2m}{2m}}\|V\Psi\|_{L^{2,\gamma}},$$

where we invoked the limiting absorption principle for the free operator on the second summand. Taking $\epsilon \to 0^+$, the dominated convergence theorem suffices to conclude the first summand vanishes. Using the bound on $V$, we have $\|V\Psi\|_{L^{2,\gamma}} \leq M\|\Psi\|_{L^{2,-\gamma}}$, hence

$$\|\Psi\|_{L^{2,-\gamma}} \lesssim \lambda^{\frac{1-2m}{2m}} M\|\Psi\|_{L^{2,-\gamma}} \lesssim \lambda^{\frac{1-2m}{2m}} M\|\Psi\|_{L^{2,-\gamma}}.$$

From here, we conclude for large enough $\lambda$ that $\|\Psi\|_{L^{2,-\gamma}} = 0$, and have that $\Psi = 0$ a.e. Thus $\lambda$ cannot be an eigenvalue of $H$.

□

The above suffices to allow us to use the limiting absorption principle for the class of potentials we consider. In the statement below $B(s,-s')$ is the space of bounded linear operators mapping $L^{2,s}\to L^{2,-s'}$, and assumes the lack of embedded eigenvalues in the continuous spectrum of $H$.

**Theorem 3.3** (Theorem 3.9 in [4])**.** *For $k=0,1,2,3\dots$, let $|V(x)|\lesssim \langle x\rangle^{-\beta}$ for some $\beta>2+2k$, then for $s,s'>k+\frac{1}{2}$, $\mathcal{R}_V^{(k)}(z)\in B(s,-s')$ is continuous for $z>0$. Furthermore, we have*

$$\big\|\mathcal{R}_V^{(k)}(z)\big\|_{L^{2,s}\to L^{2,-s'}}\lesssim |z|^{\frac{1-2m}{2m}(1+k)}.$$

Note that, in particular, these bounds hold for the free resolvent, for which there are no embedded eigenvalues, as well as for the perturbed operator when $|z|$ is sufficiently large by Lemma 3.2.

*Proof of Proposition 3.1.* The proof here mirrors closely the high energy argument in [3]. Since we need only show boundedness, the argument is straight-forward. We write $M=\ell_1+\ell_2$ where $\ell_1=\lfloor\frac{n}{4m}\rfloor+1$ is the number of iterations of the resolvent required to ensure $(\mathcal{R}_0^{\pm}V)^{\ell_1}$ is locally $L^2$, and $\ell_2$ is selected large enough to ensure that there is sufficient decay in $\lambda$ using the limiting absorption principle in Theorem 3.3. This suffices to ensure the desired integral is bounded.

Following the argument in Propositions 5.3 and 6.5 [3], we have the bound $\sigma>\frac{1}{2}$ and $\ell_1=\lfloor\frac{n}{4m}\rfloor+1$ we have

$$\|(V\mathcal{R}_0^{+})^{\ell_1-1}V\mathcal{R}_0^{\pm}(\lambda^{2m})(\cdot,y)\|_{L^{2,\sigma}}\lesssim \frac{\lambda^{\ell_1(\frac{n+1}{2}-2m)}}{\langle y\rangle^{\frac{n-1}{2}}}, \tag{13}$$

and similarly,

$$\|(\mathcal{R}_0^{+}V)^{\ell_1}(\lambda^{2m})(x,\cdot)\|_{L^{2,\sigma}}\lesssim \frac{\lambda^{\ell_1(\frac{n+1}{2}-2m)}}{\langle x\rangle^{\frac{n-1}{2}}}. \tag{14}$$

Selecting $L$ large enough so there are no eigenvalues on the support of the cut-off allow for iterated use of Theorem 3.3 which yields

$$\|(\mathcal{R}_0^{+}(\lambda^{2m})V)^{\ell_2}\mathcal{R}_V^{+}(\lambda^{2m})(V\mathcal{R}_0^{+}(\lambda^{2m}))^{\ell_2}\|_{L^{2,\sigma}\to L^{2,-\sigma}}\lesssim \lambda^{(2\ell_2+1)(1-2m)}. \tag{15}$$

Combing these, we arrive at the bound

$$\begin{aligned}\Big|\int_0^\infty & e^{it\lambda^{2m}}\psi(\lambda^{2m}/L^{2m})\lambda^{(n+2)m-n-1}(\mathcal{R}_0^{\pm}(\lambda^{2m})V)^M\mathcal{R}_V^{\pm}(\lambda^{2m})(V\mathcal{R}_0^{\pm}(\lambda^{2m}))^M\,d\lambda\Big|\\ &\lesssim \frac{1}{(\langle x\rangle\langle y\rangle)^{\frac{n-1}{2}}}\int_0^\infty \psi(\lambda^{2m}/L^{2m})\lambda^{((n+2)m-n-1+2\ell_1(n+1-4m)+(2\ell_2+1)(1-2m))}\,d\lambda.\end{aligned} \tag{16}$$

Having selected $\ell_1=\lfloor\frac{n}{4m}\rfloor+1$ we selected $\ell_2\in\mathbb{N}$ so that

$$\ell_2>\frac{1+n(m-1)+2\ell_1(n+1-4m)}{4m-2}.$$

Thus, we may bound

$$|(16)| \lesssim \frac{1}{(\langle x\rangle\langle y\rangle)^{\frac{n-1}{2}}} \int_0^\infty \psi(\lambda^{2m}/L^{2m})\lambda^{-2}\, d\lambda \lesssim \int_{\lambda\approx L} \lambda^{-2}\, d\lambda \lesssim L^{-1}.$$

This is uniformly bounded in $x, y \in \mathbb{R}^n$ and $L > 1$.

□

We note that these bounds can be shown to hold for a much larger class of potentials $V$, and one can show that the tail decays in $t$ by utilizing the oscillation in the integral. Such bounds are interesting, but are not needed for our purpose here.

3.3. **Failure of the dispersive bound for the full evolution.** We now prove that the full evolution cannot satisfy the dispersive bound, and consequently that the wave operators are not bounded on $L^\infty$. The proof follows that in [6] for the case of $m = 1$, though the frequency localization allows us to avoid many technical issues such as regularity of the threshold energies or considering larger classes of potentials. Namely we show the following.

**Proposition 3.4.** *Suppose that $n > 3$ and $n > 4m - 1$, $\psi$ is a smooth cut-off to $\lambda \approx 1$ and $0 \le \alpha < \frac{n+1}{2} - 2m$. There cannot exist a bound of the form*

$$\sup_{t>0, L>0} t^{\frac{n}{2}} \|H^{\frac{n(m-1)}{2m}} e^{itH} \psi(H/L^{2m}) P_{ac}(H) f\|_\infty \le C(V)\|f\|_1$$

*with $C(V) < \infty$ for all $V \in C^\alpha(B(0,2))$.*

*Proof.* Assume such a bound can hold. Write $V = \theta W$ with $\theta \in [0,1]$ and $W \in C^\alpha(B(0,2))$. The assumed dispersive bound implies the following holds uniformly in $L > C_V$ where $C_V$ is the constant in the proof of Proposition 3.1 chosen to ensure there are no embedded eigenvalues on the support of the cut-off. Taking $L > C_V$ suffices to ensure the argument for the tail holds for any choice of $0 \le \theta \le 1$.

$$\begin{aligned}
&\sup_{0<t<1, L>C_V} t^{\frac{n}{2}} |\langle H^{\frac{n(m-1)}{2m}} e^{itH} \psi(H/L^{2m}) P_{ac}(H) f, g\rangle| \\
&= \sup_{0<t<1, L>C_V} \frac{m\, t^{\frac{n}{2}}}{\pi} \bigg| \int_0^\infty e^{it\lambda^{2m}} \lambda^{2m-1+n(m-1)} \psi(\lambda^{2m}/L^{2m}) \langle [\mathcal{R}^+_{\theta W}(\lambda^{2m}) - \mathcal{R}^-_{\theta W}(\lambda^{2m})] f, g\rangle\, d\lambda \bigg| \\
&\hspace{25em} \le C(\theta, W)\|f\|_1\|g\|_1.
\end{aligned}$$

We take both $f$ and $g$ to be the function $f_L$ defined above (11). Expanding the perturbed resolvent into a Born series as in (10) allows us to express the evolution as a sum a polynomial in $\theta$ of degree $2M$ and the tail of the Born series. The coefficients of this polynomial depend on $t$, $W$, and $L$, are defined by

$$a_k(W,L) = \frac{m\,t^{\frac{n}{2}}}{\pi i}\int_0^\infty e^{it\lambda^{2m}}\lambda^{2m-1+n(m-1)}\psi(\lambda^{2m}/L^{2m}) \langle[\mathcal{R}_0^+(\lambda^{2m})[W\mathcal{R}_0^+(\lambda^{2m})]^k - \mathcal{R}_0^-(\lambda^{2m})[W\mathcal{R}_0^-(\lambda^{2m})]^k]f_L, f_L\rangle\, d\lambda \tag{17}$$

Proposition 3.1 shows that the tail obeys the desired bound provided $L$ is sufficiently large. It follows that the Born series terms, the polynomial in $\theta$, must also obey the bound as well. Writing

$$P_{L,t}(\theta) = \sum_{k=0}^{2M} a_k(W,L)\theta^k$$

By assumption, $\sup_{0<t<1, L>C_V}|P_{L,t}(\theta)|$ is finite for each $0\leq\theta\leq 1$. Hence the maximum $0\leq j\leq 2M$ of $\sup_{0<t<1,L>C_V}|P_{L,t}(\frac{j}{2M})|$ is bounded. The value of the polynomial at $2M+1$ points suffices to solve for the values of the coefficients, and allows us to conclude that each of the coefficients is bounded. Hence, we conclude that

$$\sup_{0<t<1, L>C_V} t^{\frac{n}{2}}|a_1(W,L)| \leq C(W) < \infty.$$

This, however, is false in light of the estimate in subsection 3.1. Hence, the assumption is false and such a bound cannot hold.

□

The intertwining identity (1) quickly establishes the following corollary, which is the $p=\infty$ statement in Theorem 1.1.

**Corollary 3.5.** *Suppose that $n>3$ and $n>2m$. Then for any $\alpha<\frac{n+1}{2}-2m$, there exists a real-valued compactly supported potential $V$ in $C^\alpha(\mathbb{R}^n)$ for which the wave operators $W^\pm$ do not extend to bounded operators on $L^\infty(\mathbb{R}^n)$.*

## 4. Extension to $L^p$

In this section we complete the proof of Theorem 1.1 to consider the full range of $p$ on which our counterexample to $L^p$ boundedness applies. The argument presented in the previous section can be adapted to show the failure of dispersive bounds from $L^p$ to $L^{p'}$ for $1\leq p\leq 2$, which implies unboundedness of the wave operators on $L^{p'}$. With $\psi$ the same cut-off to frequencies $\lambda\approx 1$, by Corollary 2.2 the free evolution satisfies the bound

$$\left\| H_0^{\frac{n(m-1)}{2m}(\frac{2}{p}-1)} e^{itH_0}\psi(H_0/L^{2m})\right\|_{p\to p'} \lesssim |t|^{\frac{n}{2}(1-\frac{2}{p})}.$$

Here we show that the dispersive bound for the perturbed operator fails by showing that one cannot have a bound of the form

$$\sup_{L>0, t>0} t^{\frac{n}{2}(1-\frac{2}{p})}\left|\left\langle H^{\frac{n(m-1)}{2m}(\frac{2}{p}-1)}e^{itH}\psi(H/L^{2m})f,g\right\rangle\right| \leq C(V) \tag{18}$$

for $f,g$ unit vectors in $L^p$. As in the previous section, we'll select $f=g$ to be functions that concentrate at zero as $L\to\infty$. Namely, we'll use $f_{L,p}(x)=(CL)^{\frac{n}{p}}f(x(CL))$ where $f$ is the $L^p$-normalized characteristice function of $B(0,1)$. The argument proceeds analogously, we show that the first term of the Born series grows too fast as $t\to 0^+$ and the tail remains bounded.

For the first term of the Born series, we note that we now consider the difference of kernels

$$K^{\pm}_{L,p,t}(x,y) = \int_0^\infty \int_{\mathbb{R}^n} e^{it\lambda^{2m}}\lambda^{2m-1}\lambda^{n(m-1)(\frac{2}{p}-1)}\mathcal{R}_0^{\pm}(\lambda^{2m})(x,z)V(z)\mathcal{R}_0^{\pm}(\lambda^{2m})(z,y)\psi(\lambda^{2m}/L^{2m})dz\,d\lambda.$$

The oscillatory integral argument applies ver batim, there are three changes. Due to the different power of $\lambda$, the $\lambda\mapsto\lambda L$ rescaling produces a power of $n(m-1)(\frac{2}{p}-1)$ on $L$. The time factor in front of the integral $t^{\frac{n}{2}(1-\frac{2}{p})}$ becomes $L^{nm-\frac{n}{2}+\frac{n}{p}-\frac{2mn}{p}}$ using the scaling $t=L^{-(2m-1)}$. The final change comes from the fact that the positive functions $f_{L,p}$ are integrated directly, we have $\|f_{L,p}\|_1=(CL)^{n/p-n}\|f\|_1\approx L^{n/p-n}\|f\|_p$ since $f$ is a normalized characteristic function of $B(0,1)$.

Hence we consider the linear operator

$$a_{1,p,L}(V)=L^{nm-\frac{n}{2}+\frac{n}{p}-\frac{2mn}{p}}\int_{\mathbb{R}^n\times\mathbb{R}^n}K_{L,L^{-(2m-1)}}(x,y)f_{L,p}(x)f_{L,p}(y)dxdy.$$

With the rescaling accounted for, the stationary phase arguments apply directly to show that there is a $C_p\in\mathbb{C}$ so that

$$\Re\big[C_pK^+_{L,p}(x,y)-C_pK^-_{L,p}(x,y)\big]\gtrsim L^{\frac{n}{p}+\frac{1-n}{2}-2m}\delta,$$

for $\delta\ll 1$ fixed, and for $L$ sufficiently large and $|x|,|y|\ll\frac{1}{L}$. Therefore, we have

$$\frac{|a_{1,p,L}(V_L)|}{\|V_L\|_{C^\alpha(B(0,2))}}\gtrsim L^{\frac{n}{p}+\frac{1-n}{2}-2m-\alpha}\delta\to\infty$$

as $L\to\infty$ unless $\alpha\geq\frac{n}{p}+\frac{1-n}{2}-2m$, provided $p<\frac{2n}{4m+n-1}$ to ensure the right side of the inequality is positive.

For the tail of the Born series, the extension is straight-forward. By selecting $M$ appropriately large depending on the parameters $j,k$ below, the proof of Proposition 3.1 can be adapted to show

$$\sup_{x,y\in\mathbb{R}^n}\bigg|\int_0^\infty e^{it\lambda}\psi(\lambda/L^{2m})\lambda^k(\mathcal{R}_0^{\pm}(\lambda)V)^M\mathcal{R}_V^{\pm}(\lambda)(V\mathcal{R}_0^{\pm}(\lambda))^M(x,y)\,d\lambda\bigg|\lesssim L^{-j},$$

for any choice of $j,k\in\mathbb{N}$. Again using the $L^p$-normalized function $f_{L,p}$ we have $\|f_{L,p}\|_1\approx L^{n/p-n}\|f\|_p$. Hence, we have

$$\begin{aligned}&\bigg|\int_0^\infty e^{it\lambda}\psi(\lambda/L^{2m})\lambda^k\langle(\mathcal{R}_0^{\pm}(\lambda)V)^M\mathcal{R}_V^{\pm}(\lambda)(V\mathcal{R}_0^{\pm}(\lambda))^M(x,y)f_{L,p},f_{L,p}\rangle\,d\lambda\bigg|\\&\qquad\lesssim\sup_{x,y\in\mathbb{R}^n}\bigg|\int_0^\infty e^{it\lambda}\psi(\lambda/L^{2m})\lambda^k(\mathcal{R}_0^{\pm}(\lambda)V)^M\mathcal{R}_V^{\pm}(\lambda)(V\mathcal{R}_0^{\pm}(\lambda))^M(x,y)\,d\lambda\bigg|\|f_{L,p}\|_1^2\lesssim L^{-1}.\end{aligned}$$

uniformly in $L > C_{n,m,V}$, where $C_{n,m,V}$ is the constant chosen so that there are no embedded eigenvalues on the support of the cut-off.

The polynomial argument now applies with coefficients defined by

$$a_k(W,L,p) = \frac{m\, t^{\frac{n}{2}(\frac{2}{p}-1)}}{\pi i} \int_0^\infty e^{it\lambda^{2m}} \lambda^{2m-1+n(m-1)(\frac{2}{p}-1)} \psi(\lambda^{2m}/L^{2m}) \\ \langle [\mathcal{R}_0^+(\lambda^{2m})[-W\mathcal{R}_0^+(\lambda^{2m})]^k - \mathcal{R}_0^-(\lambda^{2m})[-W\mathcal{R}_0^-(\lambda^{2m})]^k] f_{L,p}, f_{L,p}\rangle \, d\lambda. \tag{19}$$

Hence, we are able to conclude that the $L^p \to L^{p'}$ dispersive bound in (18) fails to hold for all $V \in C^\alpha(B(0,2))$ if $0 \le \alpha < \frac{n}{p} + \frac{1-n}{2} - 2m$. By appealing to the intertwining identity (1), this shows that the wave operators are unbounded on $L^q$ for $\frac{2n}{n-4m+1} < q \le \infty$.

A consequence of these bounds are results that are, to the best of the authors' knowledge, new for the classical $m = 1$ Schrödinger evolution. Namely, with $p'$ the Hölder conjugate of $p$ we have the following.

**Corollary 4.1.** *Suppose that $n > 3$ and $1 \le p < \frac{2n}{n+3}$. Then for any $0 \le \alpha < \frac{n}{p} - \frac{n+3}{2}$ there exists a real-valued compactly supported potential $V$ in $C^\alpha(\mathbb{R}^n)$ for which the dispersive bound*

$$\left\| e^{it(-\Delta+V)} P_{ac}(-\Delta+V) \right\|_{p\to p'} \lesssim |t|^{-\frac{n}{2}(\frac{2}{p}-1)}$$

*fails.*

*In particular, the wave operators $W^\pm$ do not extend to bounded operators on $L^{p'}(\mathbb{R}^n)$. Consequently, we conclude that for all $\frac{2n}{n-3} < q \le \infty$ there exist a compactly supported continuous potential for which the wave operators are unbounded on $L^q(\mathbb{R}^n)$.*

*Proof.* Assuming that the $L^p \to L^{p'}$ bound in the corollary holds and using Theorem 2.1 in [9] for the uniform $L^p$ boundedness of $\psi(H/L)$ we conclude that the bound (18) should hold, which is a contradiction. □

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