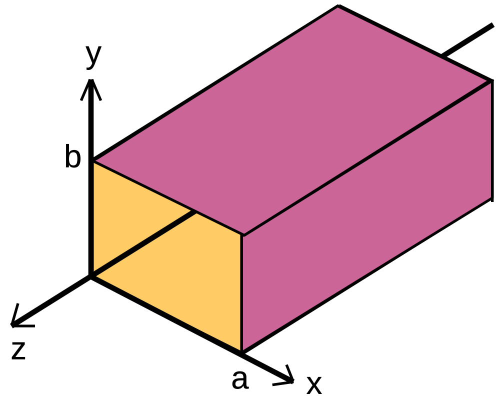
Rectangular Waveguide (TM mode)



Rectangular Waveguide

Helmholtz Equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial y^2} + k_c^2\right) e_z(x,y) = 0$$

Assume (separation of variables)

$$e_z(x,y) = X(x)Y(y)$$

$$k_c^2 = k^2 - \beta^2 = \omega^2\mu\varepsilon - \beta^2$$

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k_c^2 = 0$$

$$-k_x^2 \quad - \quad k_y^2 \qquad + k_c^2 = 0$$

Set each term to a constant

$$\frac{\partial^2 X}{\partial x^2} + k_x^2\, X = 0 \qquad\qquad \frac{\partial^2 Y}{\partial y^2} + k_y^2\, Y = 0$$

Final solution

$$X(x) = A\cos(k_x x) + B\sin(k_x x)$$

$$Y(x) = C\cos(k_y y) + D\sin(k_y y)$$

$$e_z(x,y) = \{A\cos(k_x x) + B\sin(k_x x)\}\{C\cos(k_y y) + D\sin(k_y y)\}$$

Boundary Conditions

at $x=0$ $\quad e_z(0,y) = 0 \quad \rightarrow$

$$A\{C\cos(k_y y) + D\sin(k_y y)\} = 0$$

Therefore: $\qquad A=0$

$(x=a)$ $\quad e_z(a,y) = 0 \quad \rightarrow$

$$B\sin(k_x a)\{C\cos(k_y y) + D\sin(k_y y)\} = 0$$

Therefore : $\qquad (k_x a) = m\pi$

(y=0) $e_z(x,0) = 0 \rightarrow$
$\{A\cos(k_x x) + B\sin(k_x x)\}C=0$

Therefore $C=0$

(y=b) $e_z(x,b) = 0 \rightarrow$

$\{A\cos(k_x x) + B\sin(k_x x)\}D\sin(k_y b)=0$

Therefore $(k_x b) = n\pi$

The propagation constant is

$$\beta_{mn} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

for each integer m and n there is specific propagation constant

$$\beta_{mn} = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

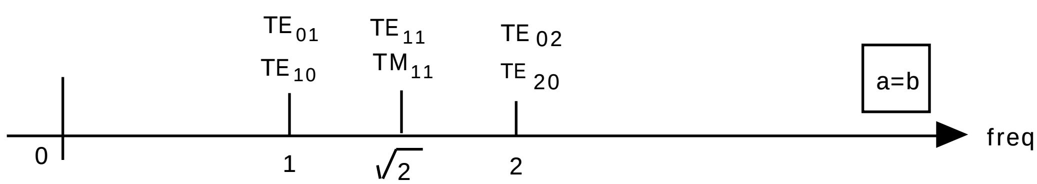
Finally,

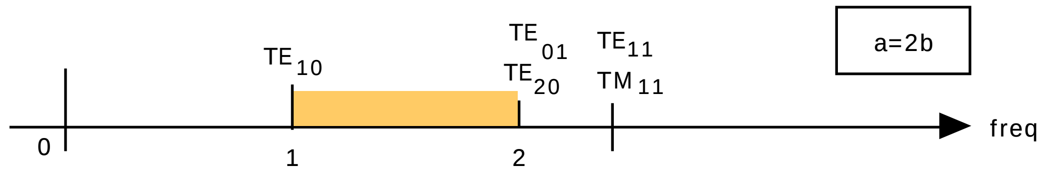
$$E_z(x,y) = B_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)e^{-j\beta z}$$

Note: *Neither m nor n can be zero.*
The other field components can be generated from Maxwell's equations
Similarly, for TE modes

$$\mathbf{H}_z(x,y) = C_{mn}\cos(m\pi x/a)\cos(n\pi y/b)\, e^{-j\beta z}$$

It has the same propagation constant.
Note: *In this case one of the integers, m or n can be zero*





TE_{10} Mode (Dominant Mode) Fields

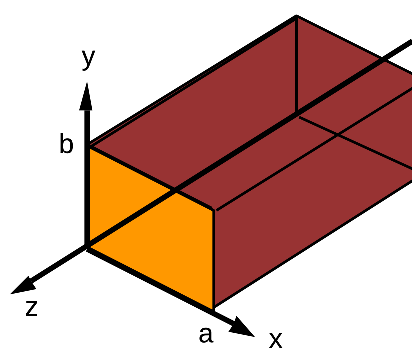
$$H_z = A\cos\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

$$E_y = \frac{-j\omega\mu a}{\pi}A\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

$$H_x = \frac{j\beta a}{\pi}A\sin\left(\frac{\pi x}{a}\right)e^{-j\beta z}$$

$$E_x = E_z = H_y = 0$$

Propagation constant and cut-off wavelength



Rectangular Waveguide

$$k_c = \frac{\pi}{a} \qquad \beta = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2}$$

$$\lambda_c = 2a$$

Total power transmitted

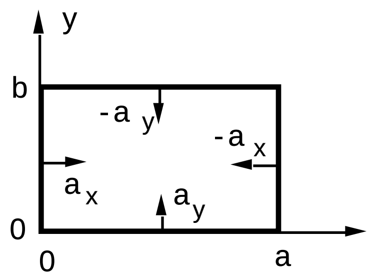
$$P = \frac{1}{2}\mathrm{Re}\int_{x=0}^{a}\int_{y=0}^{b} (\mathbf{E} \times \mathbf{H}^*)\,dy\,dx$$

$$P = \frac{\omega\mu a^3 |A|^2 b}{4\pi^2}\,\mathrm{Re}(\beta)$$

Power loss/unit length due to finite conductivity

$$P_l = \frac{R_s}{2}\int_C (\mathbf{J}_s \cdot \mathbf{J}_s^*)\,dl$$

Current densities on the walls



$$\mathbf{J}_s = \mathbf{n} \times (\mathbf{H_z} + \mathbf{H_x})\big|_{surface}$$

$$\mathbf{J}_s = \mathbf{a}_x \times (A\mathbf{a}_z)\big|_{x=0} + (-\mathbf{a}_x) \times (-A\mathbf{a}_z)\big|_{x=a}$$

$$+ \mathbf{a}_y \times \left[\left\{ A\cos\left(\frac{\pi x}{a}\right)\mathbf{a}_z \right\}_{y=b} + \left\{ \frac{j\beta a}{\pi} A\sin\left(\frac{\pi x}{a}\right)\mathbf{a}_x \right\}_{y=0} \right]$$

$$+ (-\mathbf{a}_y) \times \left[\left\{ A\cos\left(\frac{\pi x}{a}\right)\mathbf{a}_z \right\}_{y=b} + \left\{ \frac{j\beta a}{\pi} A\sin\left(\frac{\pi x}{a}\right)\mathbf{a}_x \right\}_{y=b} \right]$$

Elimination cross products

$$\mathbf{J}_s = -A\mathbf{a}_y\big|_{x=0} - A\mathbf{a}_y\big|_{x=a}$$

$$+ \left\{ A\cos\left(\frac{\pi x}{a}\right)(\mathbf{a}_x) \right\}_{y=0} - \left\{ \frac{j\beta a}{\pi} A\sin\left(\frac{\pi x}{a}\right)\mathbf{a}_z \right\}_{y=0}$$

$$- \left\{ A\cos\left(\frac{\pi x}{a}\right)\mathbf{a}_x \right\}_{y=b} + \left\{ \frac{j\beta a}{\pi} A\sin\left(\frac{\pi x}{a}\right)\mathbf{a}_z \right\}_{y=b}$$

Poser loss is

$$P_l = \frac{R_s}{2}\left\{ 2\int_0^b |A|^2\,dy + 2\int_0^a \left[\cos^2\left(\frac{\pi x}{a}\right) + \frac{\beta^2 a^2}{\pi^2}\sin^2\left(\frac{\pi x}{a}\right) \right] dx \right\}$$

$$P_l = R_s |A|^2 \left(b^2 + \frac{a}{2} + \frac{\beta^2 a^3}{2\pi^2} \right)$$

Attenuation constant is

$$\alpha_c = \frac{P_l}{2P} = \frac{2\pi^2 R_s \left(b^2 + \frac{a}{2} + \frac{\beta^2 a^3}{2\pi^2} \right)}{\omega\mu a^3 b\beta}$$