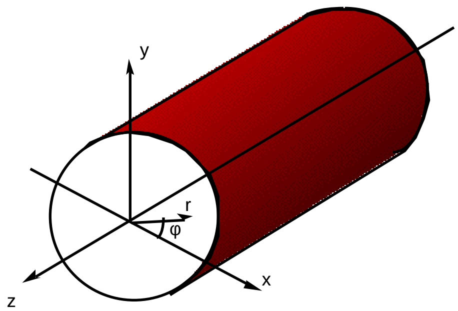
EE-611
Supplementary Notes
Circular Waveguides



TM modes ($H_z=0$)

A circular waveguide with radius of a is given. Assuming that the cylindrical axis coincides with the z axis and transverse coordinates are (ρ,φ). E_z satisfies the Helmholtz Equation

$$\nabla_t^2 E_z(\rho,\varphi)+k_c^2 E_z(\rho,\varphi)=0$$

In cylindrical coordinates

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial E_z(\rho,\varphi)}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 E_z(\rho,\varphi)}{\partial\varphi^2}+k_c^2E_z(\rho,\varphi)=0$$

Assuming separation of variables,

$$E_z(\rho,\varphi)=R(\rho)\Phi(\varphi)$$

Substituting into the above Equation and dividing by $(R\Phi)$

$$\frac{1}{R}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)\right]+\frac{1}{\Phi}\frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2}+k_c^2=0$$

Multiplying by ρ^2,

$$\frac{\rho^2}{R}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)\right]+\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}+\rho^2k_c^2=0$$

The Φ term should be a constant and setting it to $-\nu^2$.

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2}=-\upsilon^2$$

The R term becomes

$$\frac{\rho^2}{R}\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)\right]-\upsilon^2+\rho^2k_c^2=0$$

Dividing by ρ^2.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right)+\left(k_c^2-\frac{\upsilon^2}{\rho^2}\right)R=0 \qquad \text{Bessel Equation}$$

This equation is known as the *Bessel Equations*.

The complete solution to $E_z(\rho,\varphi)$ can be written as

$$E_z(\rho,\varphi)=[AJ_\nu(k_c\rho) + BY_\nu(k_c\rho)][C\sin(\upsilon\varphi)+D\cos(\upsilon\varphi)]$$

Here $J_\nu(k_c\rho)$ is the Bessel Function of the first kind of order ν and argument $(k_c\rho)$ and $Y_\nu(k_c\rho)$ is the Bessel Function of the second kind of order ν and argument $(k_c\rho)$. Since the point $\rho=0$ is a point in the cylindrical waveguide, the field should be finite there. Since $Y_\nu(0)=\infty$, for all orders, B should be equal to zero. Also, the fields should repeat themselves every 2π, i.e., $\nu=n$, an integer. Therefore the final solution can be written as

$$E_z(\rho,\varphi)=A\, J_n(k_c\rho)\, \sin(n\varphi) \qquad \text{or}$$

$$E_z(\rho,\varphi)=A\, J_n(k_c\rho)\, \cos(n\varphi)$$

Applying the boundary conditions that at $\rho=a$, the tangential field should be zero

$$E_z(a,\varphi)=A\, J_n(k_ca)\, \sin(n\varphi) = 0$$

The only way this can be satisfied is if

$$J_n(k_ca)=0 \qquad \text{Eq.A}$$

Similarly for the TE modes, we solve for $H_z(\rho,\varphi)$. The solution is exactly the same as for E_z, but the boundary condition for this case becomes

$$\frac{dJ_n(k_c\rho)}{d\rho}\bigg|_{\rho=a}=0 \qquad \text{(Eq.B)}$$

These two equations lead to the following cut-off frequencies

$$k_ca = p_{nl} \qquad f_{c,nl} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{p_{nl}}{a} \qquad \text{for TM waves}$$

$$k_ca = p'_{nl} \qquad f_{c,nl} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{p'_{nl}}{a} \qquad \text{for TE waves}$$

Here p_{nl} is the l'th zero of the Bessel function of n'th kind and p_{nl}' is the l'th zero of the derivative of the n'th order Bessel function .