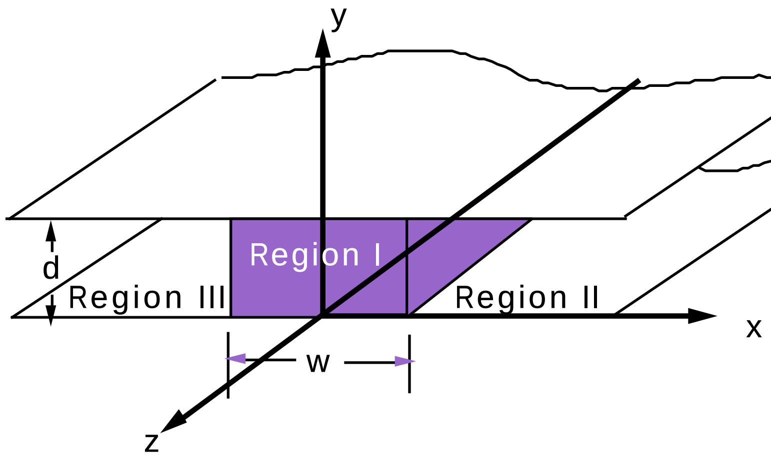
TE Modes in Parallel Plate Waveguide with a Dielectric Insert.

Given a parallel plate waveguide with a dielectric slab of width w inserted between the two plates.



There are three regions as shown in the figure above. We have to solve Helmholz Equation in these three regions, but because of symmetry, it is sufficient to solve the Helmholz Equation between

$0 \le x| \le |w/2|$ Region I and
$|w/2| \le x \le oo$ Region II

Helmholz Equation I Region I is

$${}_t^2 h_z + k_{c1}^2 h_z = 0$$

where

Using Separation of variables

$$k_{c1}^2 = \omega^2 \mu_o \varepsilon_r \varepsilon_o - \beta^2$$

$$\frac{1}{X_1}\frac{\partial^2 X_1}{\partial x^2} + \frac{1}{Y_1}\frac{\partial^2 Y_1}{\partial y^2} + k_{c1}^2 = 0$$

$$(-k_{x1}^2) + (-k_{y1}^2) + k_{c1}^2 = 0$$

$$\frac{\partial^2 X_1}{\partial x^2} = -k_{x1}^2 X_1 \qquad \frac{\partial^2 Y_1}{\partial y^2} = -k_{y1}^2 Y_1$$

$$h_{z1} = [A \sin(k_{x1}x) + B \cos(k_{x1}x)][C \sin(k_{y1}y) + D \cos(k_{y1}y)]$$

In a similar way, the solution in region II is

$$\frac{1}{X_2}\frac{\partial^2 X_2}{\partial x^2} + \frac{1}{Y_2}\frac{\partial^2 Y_2}{\partial y^2} + k_{c2}^2 = 0$$

$$(+\,k_{x2}^2) \quad + \quad (-k_{y2}^2) \quad + \; k_{c2}^2 = 0$$

$$\frac{\partial^2 X_2}{\partial x^2} = k_{x2}^2\, X_2 \qquad \frac{\partial^2 Y_2}{\partial y^2} = -k_{y2}^2\, Y_2$$

$$h_{z2} = [E\, e^{-k_{x2}x} + F\, e^{k_{x2}x}][G \sin(k_{y2}y) + H \cos(k_{y2}y)]$$

Here the constant k_{x2} is chosen as imaginary so that the fields die exponentially away from the slab. Also

$$k_{c2}^2 = \omega^2 \mu_o\, \varepsilon_o - \beta^2$$

Calculating the transverse $\mathbf{h_1}$ and $\mathbf{e_1}$ fields in Region I

$$\mathbf{h_1} = -\frac{j\beta}{k_{c1}}\; {}_{t}h_{z1}$$

$$= -\frac{j\beta}{k_{c1}}\left(\frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y}\right) h_{z1}$$

$$\mathbf{e_1} = -Z_{m1}\,(\mathbf{a_z} \times \mathbf{h_1}) = \frac{j\beta}{k_{c1}} Z_{m1}\left(\mathbf{a_z} \times \left(\frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y}\right)\right) h_{z1}$$

$$\mathbf{e_{x1}} = K\, \frac{\partial}{\partial y}\,(X_1)\,[C \sin(k_{y1}y) + D \cos(k_{y1}y)]\; \mathbf{a_x}$$

$$\mathbf{e_{x1}} = K\,(X_1)\, k_{y1}\,[C \cos(k_{y1}y) - D \sin(k_{y1}y)]\; \mathbf{a_x}$$

Applying the boundary conditions on the tangential $\mathbf{e_1}$ ($\mathbf{e_{x1}}$):

from $\quad \mathbf{e_{x1}}(x,0) = 0$

$$= K\,(X_1)\, k_{y1}\,[C \cos(k_{y1}0) - D \sin(k_{y1}0)]\; \mathbf{a_x} = 0 \quad \rightarrow C=0$$

from $\quad \mathbf{e_{x1}}(x, d) = 0$

$$= K\,(X_1)\, k_{y1}\,[-\,D \sin(k_{y1}d)]\; \mathbf{a_x} = 0 \quad \rightarrow k_{y1}d = n\pi$$

$$h_{z1} = [B \cos(k_{x1}x)]\,[\,D \cos(k_{y1}y)]$$

$$h_{z1} = C_1 \cos(k_{x1}x) \cos(k_{y1}y)$$

In a similar way, the fields in Region II can be calculated. Applying the same boundary conditions from $\mathbf{e}_{\mathbf{x2}}(x,0) = 0$

$= K\,(X_2)\,k_{y2}\,[G\cos(k_{y2}0) - H\sin(k_{y2}0)]\,\mathbf{a_x} = 0 \quad \rightarrow G=0$

from $\mathbf{e}_{\mathbf{x2}}(x, d) = 0$

$= K\,(X_2)\,k_{y2}\,[-H\sin(k_{y2}d)]\,\mathbf{a_x} = 0 \quad \rightarrow k_{y2}d = n\pi$

$h_{z2} = [E\,e^{-k_{x2}x}]\,[H\cos(k_{y2}y)]$

$h_{z2} = C_2\,e^{-k_{x2}x}\cos(k_{y2}y)$

Substituting the corresponding k_{y1} and k_{y2} into the respective k_c's

$$k_{c1}^2 = \omega^2\mu_o\varepsilon_r\varepsilon_o - \beta^2 = k_{x1}^2 + \left(\frac{n\pi}{d}\right)^2$$

$$k_{c2}^2 = \omega^2\mu_o\varepsilon_o - \beta^2 = -k_{x2}^2 + \left(\frac{n\pi}{d}\right)^2$$

Eliminating β from the two equations

$$\omega^2\mu_o\varepsilon_o(\varepsilon_r - 1) = k_{x1}^2 + k_{x2}^2 \qquad (1)$$

We now apply the boundary conditions at the boundary x=|w/2|. We calculate the $\mathbf{e_y}$ in both regions:

$$\mathbf{e_1} = -Z_{m1}\,(\mathbf{a_z} \times \mathbf{h_1}) = \frac{j\beta}{k_{c1}^2} Z_{m1}\left(\mathbf{a_z} \times \left(\frac{\partial}{\partial x}\mathbf{a_x} + \frac{\partial}{\partial y}\mathbf{a_y}\right)\right) h_{z1}$$

$$\mathbf{e_{y1}} = \frac{j\beta}{k_{c1}^2} Z_{m1}\,\frac{\partial}{\partial x}(h_{z1})\,\mathbf{a_y}$$

$$\mathbf{e_{y1}} = \frac{j\beta}{k_{c1}^2} Z_{m1}\,\frac{\partial}{\partial x}(C_1\cos(k_{x1}x)\cos(k_{y1}y))\mathbf{a_y}$$

$$\mathbf{e_{y1}} = -\frac{j\beta}{k_{c1}^2} Z_{m1}\,k_{x1}(C_1\sin(k_{x1}x)\cos(k_{y1}y))\mathbf{a_y}$$

Similarly

$$\mathbf{e_2} = -Z_{m2}\,(\mathbf{a_z} \times \mathbf{h_2}) = \frac{j\beta}{k_{c2}^2} Z_{m2} \left(\mathbf{a_z} \times \left(\frac{\partial}{\partial x} \mathbf{a_x} + \frac{\partial}{\partial y} \mathbf{a_y} \right) \right) h_{z2}$$

$$\mathbf{e_{y\,2}} = \frac{j\beta}{k_{c2}^2} Z_{m2} \frac{\partial}{\partial x} (h_{z2})\, \mathbf{a_y}$$

$$\mathbf{e_{y\,2}} = \frac{j\beta}{k_{c2}^2} Z_{m2} \frac{\partial}{\partial x} (C_2 \cos(k_{x2} x) \cos(k_{y2} y)) \mathbf{a_y}$$

$$\mathbf{e_{y\,2}} = -\frac{j\beta}{k_{c2}^2} Z_{m2}\, k_{x2} (C_2\, e^{(k_{x2} x)} \cos(k_{y2} y)) \mathbf{a_y}$$

From the continuity of the tangential h_z fields:

$$h_{z1} \left(\left| \frac{w}{2} \right|, y \right) = h_{z2} \left(\left| \frac{w}{2} \right|, y \right)$$

$$C_1 \cos\left(k_{x1} \left| \frac{w}{2} \right| \right) \cos(k_{y1} y) = C_2\, e^{-k_{x2} \left| \frac{w}{2} \right|} \cos(k_{y2} y)$$

From the continuity of the tangential e_y fields:

$$e_{y1} \left(\left| \frac{w}{2} \right|, y \right) = e_{y2} \left(\left| \frac{w}{2} \right|, y \right)$$

$$-\frac{j\beta}{k_{c1}^2} Z_{m1}\, k_{x1} (C_1 \sin(k_{x1} x) \cos(k_{y1} y)) = -\frac{j\beta}{k_{c2}^2} Z_{m2}\, k_{x2} (C_2\, e^{(k_{x2} x)} \cos(k_{y2} y))$$

Dividing the last equations by the previous one, we obtain after simplifications:

$$\frac{C_1 \cos\left(k_{x1} \left| \frac{w}{2} \right| \right) \cos(k_{y1} y) = C_2\, e^{-k_{x2} \left| \frac{w}{2} \right|} \cos(k_{y2} y)}{-\frac{j\beta}{k_{c1}} Z_{m1} (C_1 \sin(k_{x1} x) \cos(k_{y1} y)) = -\frac{j\beta}{k_{c2}} Z_{m2} (C_2\, e^{(k_{x2} x)} \cos(k_{y2} y))}$$

$$\frac{\cos\left(k_{x1} \left| \frac{w}{2} \right| \right) = e^{-k_{x2} \left| \frac{w}{2} \right|}}{\frac{Z_{m1}}{k_{c1}} \sin\left(k_{x1} \left| \frac{w}{2} \right| \right) = \frac{Z_{m2}}{k_{c2}}\, e^{-k_{x2} \left| \frac{w}{2} \right|}} \qquad \text{since } Z_{m1} = \frac{\omega\mu}{\beta} = Z_{m2}$$

$$k_{c2} \tan\left(k_{x1} \left| \frac{w}{2} \right| \right) = k_{c1} \qquad (2)$$

Simultaneous solution of Equations I and 2 will determine the unknowns k_{c1} and k_{c2}.