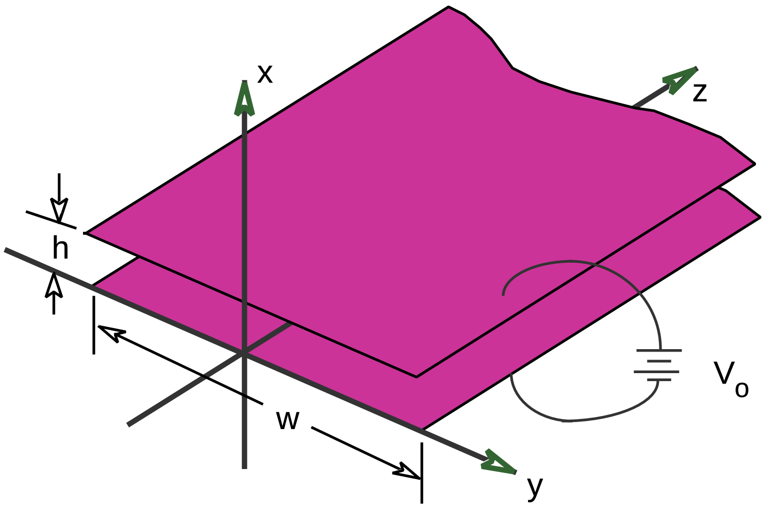
ECECS-473
Electromagnetic Fields

TEM Waves in a Parallel Plate Waveguide

Give a two plates separated by a distance h. The two plates are oriented as shown in the figure. The separation distance (h) is assumed to be lot greater than the width (w) of the plates in the x direction. This implies that the fields do not vary in the y direction



TEM wave solution is obtained from the solution of the Laplace's equation assuming that a potential difference of V_o (similar to a battery connected between the top plate and the lower plate as shown in the figure above) is applied between the plates.

$$\nabla_t^2 \varphi(x,y) = 0$$

$$\frac{\partial^2 \varphi(x,y)}{\partial x^2} + \frac{\partial^2 \varphi(x,y)}{\partial y^2} = 0$$

$$\frac{\partial}{\partial y} = 0$$

$$\frac{\partial^2 \varphi(x,y)}{\partial x^2} = 0$$

$$\frac{d^2 \varphi(x)}{dx^2} = 0$$

$$\varphi(x) = C_1 x + C_2$$

To find the integration constants, we apply the Boundary Conditions:

a) at $x=0$, $\varphi(0) = 0$

$$0 = C_1 0 + C_2 \rightarrow C_2 = 0$$

b) at $x=h$, $\varphi(h) = V_o$

$$V_o = C_1 h \qquad \rightarrow C_1 = \frac{V_o}{h}$$

Finally,

$$\varphi(x) = \frac{V_o}{h} x$$

The resulting Electric Field is calculated form

$$\mathbf{E}(x,y) = -\nabla_t \varphi(x,y) \; e^{-j\beta z}$$

$$\mathbf{E}(x,y) = -\left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y\right)\left[\frac{V_o}{h} x\right] e^{-j\beta z}$$

$$\mathbf{E}(x) = -\frac{V_o}{h} e^{-j\beta z} \; \mathbf{a}_x \quad (\mathrm{V/m})$$

Corresponding Magnetic Field is

$$\mathbf{H}(x,y) = \frac{1}{\eta} \mathbf{a}_z \times \mathbf{E}(x,y)$$

$$\mathbf{H}(x,y) = \frac{1}{\eta} \mathbf{a}_z \times \left[-\frac{V_o}{h} e^{-j\beta z} \mathbf{a}_x\right]$$

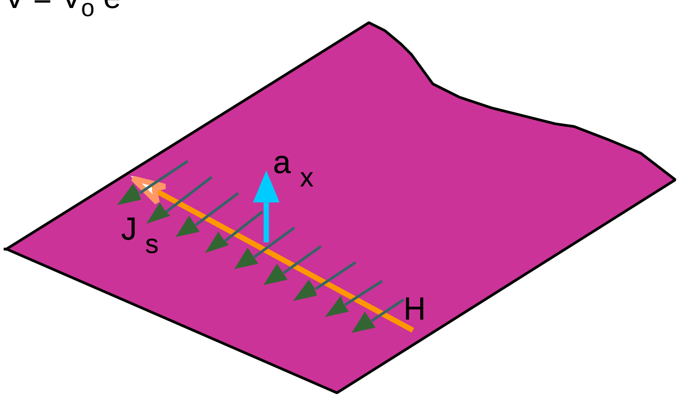
$$\mathbf{H}(x,y) = -\frac{1}{\eta}\left(\frac{V_o}{h}\right) e^{-j\beta z} \mathbf{a}_y \qquad (\mathrm{A/m})$$

We can define a voltage wave

$$V = -\int_C \mathbf{E} \cdot d\mathbf{l}$$

$$V = -\int_0^h \left(-\frac{V_o}{h} e^{-j\beta z} \mathbf{a}_x\right) \cdot (dx \, \mathbf{a}_x)$$

$$V = V_o \, e^{-j\beta z}$$



Similarly, we can define a current wave

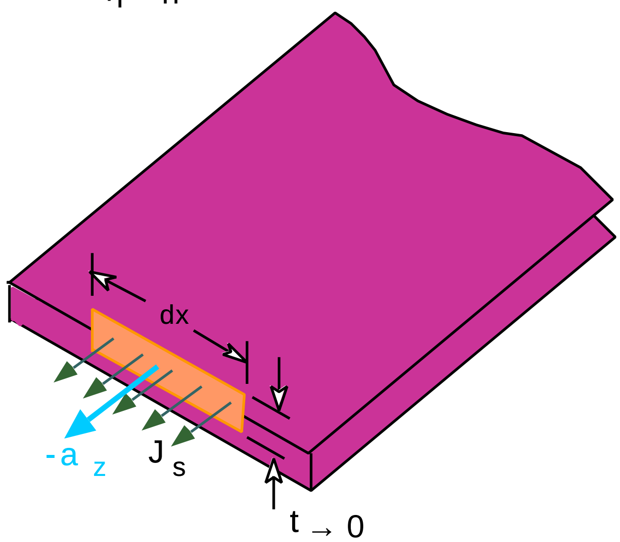
$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H}\big|_{surface} \quad \text{(A/m)}$$

$$\mathbf{J}_s = -\,\mathbf{a}_x \times \left[-\frac{1}{\eta}\left(\frac{V_o}{h}\right) e^{-j\beta z}\, \mathbf{a}_y \right] \qquad n = -\mathbf{a}_x \text{ at the upper plate}$$

$$\mathbf{J}_s = \frac{1}{\eta}\left(\frac{V_o}{h}\right) e^{-j\beta z}\, \mathbf{a}_z \quad \text{(A/m)}$$

$$\mathbf{J}_s = -\,\mathbf{a}_x \times \left[-\frac{1}{\eta}\left(\frac{V_o}{h}\right) e^{-j\beta z}\, \mathbf{a}_y \right] \qquad n = \mathbf{a}_x \text{ at the lower plate}$$

$$\mathbf{J}_s = -\frac{1}{\eta}\left(\frac{V_o}{h}\right) e^{-j\beta z}\, \mathbf{a}_z \quad \text{(A/m)}$$



The current is:

$$I = \int \mathbf{J}_s \cdot d\mathbf{l}$$

$$I = \int_{-w/2}^{w/2} \left[\frac{1}{\eta}\left(\frac{V_o}{h}\right) e^{-j\beta z}\, \mathbf{a}_z \right] \cdot (dx\,\mathbf{a}_z)$$

$$I = \frac{1}{\eta}\left(\frac{V_o}{h}\right) e^{-j\beta z}\, x\Big|_{-w/2}^{w/2} = \left[\frac{1}{\eta}\left(\frac{V_o}{h}\right) w \right] e^{-j\beta z}$$

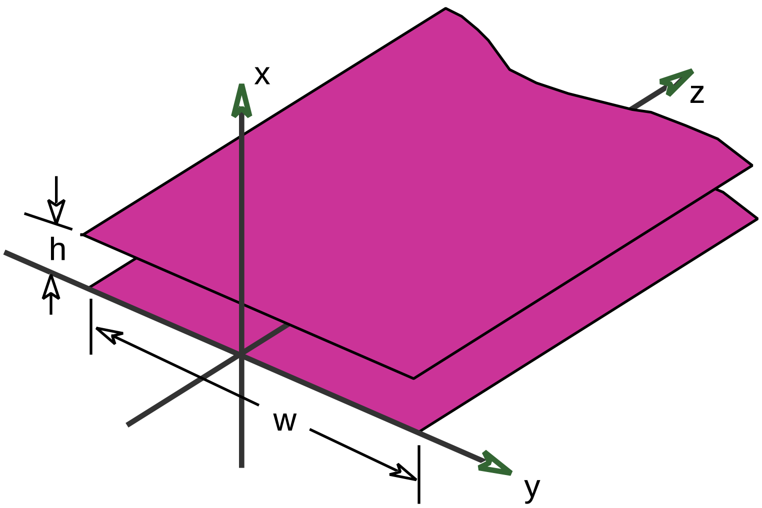
$$I = I_o\, e^{-j\beta z} \quad \text{(Amp)}$$

Characteristic impedance is defined as:

$$Z_c = \frac{V}{I} = \frac{V_o\, e^{-j\beta z}}{\left[\frac{1}{\eta}\left(\frac{V_o}{h}\right) w \right] e^{-j\beta z}} = \eta\, \frac{h}{w} \quad \text{(ohm)}$$

Parallel Plate Transmission Line (Cont.):

TM Modes in a Parallel plate transmission line



For TM waves, H_z=0 and E_z= finite. e_z is found from the solution of

$$ {}^{2}_{\tau}e_z(x,y) + k_c^2 e_z(x,y) = 0 $$

$$ \beta = \sqrt{\omega^2\mu\varepsilon - k_c^2} $$

Assuming that we can neglect the y variation

$$ \frac{\partial}{\partial y} \rightarrow 0 $$

The wave equation in one dimension becomes

$$ \frac{\partial^2 e_z}{\partial x^2} + k_c^2\, e_z = 0 $$

Solution to this equation is

$$ e_z = A\sin(k_c x) + B\cos(k_c x) \qquad \text{(Eq.1)} $$

We could have written the solutions in the form

$$ e_z = C\, e^{-jk_c x} + D\, e^{+jk_c x} \qquad \text{(Eq.2)} $$

Applying the boundary conditions to the first equation (Eq.1) at x=0 and x=h, Boundary conditions:

(a) at $x=0$, $e_z(0) = 0 \rightarrow B = 0$

(b) at $x=h$, $e_z(h) = 0 \rightarrow \sin(k_c h) = 0 \rightarrow k_c h = n\pi \quad (n=1....\infty)$

$$k_{c,n} = \frac{n\pi}{h}$$

If we had used the second equations (Eq.2) above

(a) at $x=0$, $e_z(0) = 0 \rightarrow C + D = 0 \rightarrow D = -C$

$$e_z = C\, e^{-jk_c x} + D\, e^{+jk_c x} = C\,(e^{-jk_c x} - e^{+jk_c x}) = -2j\, C \sin(k_c x)$$

$$e_z = A \sin(k_c x)$$

(b) at $x=h$, $e_z(h) = 0 \rightarrow \sin(k_c h) = 0 \rightarrow k_c h = n\pi \quad (n=1....\infty)$

$$k_{c,n} = \frac{n\pi}{h}$$

We come up with the same form of solution and the asme k_c.

The electric field is then;

$$E_z(x,z) = e_z\, e^{-j\beta z}$$

$$E_z(x,z) = A \sin\left(\frac{n\pi}{h} x\right) e^{-j\beta z}$$

The propagation constant and the cut-off frequency is

$$\beta = \sqrt{\omega^2 \mu\varepsilon - \left(\frac{n\pi}{h}\right)^2}$$

$$f_{c,n} = \frac{n\pi}{2\pi h\sqrt{\mu\varepsilon}} = \left(\frac{v}{2h}\right) n \quad (\text{Hz}) \qquad \left(v = \frac{1}{\sqrt{\mu\varepsilon}} \text{ speed of light in the medium}\right)$$

Example: Given a parallel plate transmission line with h= 2 mm, and width = w= 100 mm. Find

(a) the characteristic impedance of the line and

(b) Frequency at which the lowest order TM_1 wave is excited.

$h=2$ mm, $w= 100$ mm, $\varepsilon=\varepsilon_o$, $\mu=\mu_o$ $\quad v = \dfrac{1}{\sqrt{\mu_o \varepsilon_o}} = c = 3\cdot 10^8$ m/s

$$Z_o = \eta \frac{w}{h} = 377 * \frac{2}{100} = 7.54\ \Omega$$

$$f_c = \left(\frac{c}{2h}\right) n = \frac{3\cdot 10^8 \text{m/s}}{2\cdot 0.002 \text{ m}} = 7.5\cdot 10^{10} \text{ Hz. For } TM_1 \text{ mode } (n=1)$$